The LGC method

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General setting

In this talk we will discuss a versatile, robust method for proving L^p bounds for large classes of (sub-)linear and multi-linear operators.

Main Problem (Generic formulation)

Let T be a bilinear (multi/sublinear) operator of the form

$$T(f,g)(x)=\iint_{\mathbb{R}^2}\widehat{f}(\xi)\,\widehat{g}(\eta)\,m(x,\xi,\eta)\,e^{i\xi x}\,e^{i\eta x}\,d\xi\,d\eta\,,\qquad x\in\mathbb{R},$$

with $m(x,\xi,\eta)=e^{i\,\varphi(x,\xi,\eta)}\,\psi(x,\xi,\eta)$ and φ, ψ real, $(\psi>0)$.

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with $m(x,\xi,\eta) = e^{i\,\varphi(x,\xi,\eta)}\,\psi(x,\xi,\eta)$ and φ, ψ real, ($\psi > 0$).

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Assume m satisfies suitable

- *smoothness in the ξ and η variables–but not necessarily in the x variable*
- *boundedness properties in all three variables for the modulus function $\psi(x, \xi, \eta)$.*
- *non-degeneracy conditions*
- *proper non-zero curvature conditions in ξ, η*

Task: Under minimal regularity conditions in x, provide L^p-bounds, $1 < p < \infty$, for T.

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Overview: For the first part of the talk we discuss several important families of operators whose representations involve variable planar curves of the form

$$\gamma(x,t) := \sum_{j=1}^{n} a_j(x)\, t^j$$

with $a_j(\cdot)$ real measurable functions, $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$ with $d \in \{1,2\}$.

General setting

In order to offer a wider and deeper perspective over several key directions in harmonic analysis area, we will review three situations, displayed below in increasing order of complexity:

- The (purely) **non-zero curvature** situation: $\gamma(x,t) := \sum_{j=2}^{n} a_j(x)\, t^j$.
- The (purely) **zero curvature** situation: $\gamma(x,t) := a_1(x)\, t$.
- The **hybrid** situation: $\gamma(x,t) := \sum_{j=1}^{n} a_j(x)\, t^j$.

The LGC method *per se* it was originally designed in order to treat the first item though it can be used in conjunction with modulation invariant techniques in order to also approach the hybrid case: *This method is compatible with zero-curvature techniques.*

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I. Singular and maximal (sub)linear operators

Main Problem (Generic formulation)

Understand the behavior of

- **the linear Hilbert transform along** γ

$$H_\gamma \ :\ S(\mathbb{R}^2) \longrightarrow L^\infty(\mathbb{R}^2)\,, \qquad x=(x_1,x_2)\in\mathbb{R}^2\,,$$

$$H_\gamma(f)(x) := p.v.\int_{\mathbb{R}} f(x_1-t,\, x_2+\gamma(x,t))\,\frac{dt}{t}\,.$$

- **the (sub)linear maximal operator along** γ

$$M_\gamma \ :\ S(\mathbb{R}^2) \longrightarrow L^\infty(\mathbb{R}^2)\,, \qquad x=(x_1,x_2)\in\mathbb{R}^2\,,$$

$$M_\gamma(f)(x) := \sup_{h>0}\frac{1}{2h}\int_{-h}^{h} |f(x_1-t,\, x_2+\gamma(x,t))|\,dt\,.$$

History/motivation (I): $\gamma(x,t) = \gamma(t) = a\,t$

- The simplest fundamental object: the **Hilbert transform**
$$H:\ S(\mathbb{R}) \to S'(\mathbb{R}) \quad Hf(x) := \mathrm{p.v.} \int_R f(x-t)\,\frac{dt}{t}\,.$$
- A celebrated result of M. Riesz (1928) states that *H is a bounded operator from $L^p(\mathbb{R})$ to $L^p(\mathbb{R})$ for any $1 < p < \infty$.*
- Serves as the main prototype for the CZ theory arising from the study of constant coefficient elliptic differential operators.
- The higher dimensional correspondent of the Hilbert transform is given by the **Riesz transform(s)**:
$$R:\ S(\mathbb{R}^d) \to S'(\mathbb{R}^d) \quad Rf(x) := \mathrm{p.v.} \int_{R^d} f(x-t)\,K(t)\,dt\,,$$
where $K(t) = C_d\,\frac{t_j}{|t|^{d+1}}$ with $1 \leq j \leq d$ and $t = (t_j)_{j=1}^d \in \mathbb{R}^d$.
- The analogue of M. Riesz result is due to A. Calderon and A. Zygmund (Acta Math.,1952): *R acts continuously from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ for $1 < p < \infty$.*

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- Model: The heat equation in $\mathbb{R}^{d+1}_+ = \mathbb{R}^d \times R_+$, $d>2$

$$\partial_t u - \triangle u = f\,.$$

- One can reshape u_t and $u_{x_i x_j}$ in an expression of the form $Tf(x,t) := K * f(x,t)$ with K a suitable un-isotropic CZ kernel; one can show that $\|T\|_{p\to p}<\infty$ and hence

$$\|u_t\|_{L^p(\mathbb{R}^{d+1}_+)} + \|u_{x_i x_j}\|_{L^p(\mathbb{R}^{d+1}_+)} \lesssim_p \|f\|_{L^p(\mathbb{R}^{d+1}_+)}, \qquad 1<p<\infty\,.$$

- Via Plancherel+texchnical considerations the L^2-boundedness of $Tf(x,t) := K * f(x,t)$ is essentially equivalent with the L^2-bdd of H_γ along the parabola.

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- Departing from the uniform estimates for the parabola case, E. Stein and S. Wainger initiated a systematic study of the singular oscillatory integral expressions/operators.
- One of their first results (Studia Math., 1970): If $\{\alpha_j\}_{j=1}^n \subset \mathbb{R}_+$ and $\{a_j\}_{j=1}^n \subset \mathbb{R}$, $n\in\mathbb{N}$ one has
$$\left|\int_{\mathbb{R}} e^{i\sum_{j=1}^n a_j\, t^{\alpha_j}}\,\frac{dt}{t}\right| < K(\alpha_1,\ldots,\alpha_n)\,,$$
with K independent of $\{a_j\}_{j=1}^n$. This result is based on Van der Corput estimates.
- Thus, they obtained the $L^2(\mathbb{R}^2)-$boundedness of H_γ for $\gamma(t)=\sum_{j=1}^n a_j\, t^{\alpha_j}$.
- Intense work in '70's (Stein, Wainger, Nagel, Riviere): main breakthrough was the proof of the $L^p(\mathbb{R}^d)$ inequalities $(1<p\neq 2<\infty)$ for H_γ and later for the associated maximal operator M_γ.

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History/motivation (I): $\gamma(x,t) = a(x)\,t$

Conjecture

(Zygmund) *If* $a : \mathbb{R}^2 \to \mathbb{R}$ *is a Lipschitz vector field then the maximal operator*

$$M_{a,\epsilon_0} f(x_1, x_2) := \sup_{0<\epsilon<\epsilon_0} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} |f(x_1 - t, x_2 - a(x_1, x_2)t)|\, dt\,,$$

is bounded on $L^p(\mathbb{R}^2)$ *for any* $1 < p < \infty$ *provided* ϵ_0 *is small enough depending on* $\|a\|_{Lip}$.

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(Stein) *If* $a : \mathbb{R}^2 \to \mathbb{R}$ *is a Lipschitz vector field the Hilbert transform*

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II. Carleson type operators

Main Problem (Generic formulation)

Understand the behavior of **the γ - Carleson operator**

$$C_\gamma \ : \ S(\mathbb{R}) \longrightarrow L^\infty(\mathbb{R}) \, ,$$

$$C_\gamma f(x) := p.v. \int_{\mathbb{R}} f(x-t)\, e^{i\,\gamma(x,t)} \, \frac{dt}{t} \, .$$

History/motivation (II): $\gamma(x,t) = a(x)\,t$

- (1913) Luzin conjectures that if f is square integrable then its Fourier series converges to f almost everywhere.
- (Acta Math., 1966) L. Carleson provides the positive answer.
- Carleson's proof is essentially equivalent with the L^2 boundedness of the Carleson operator $C\gamma f(x) = Cf(x) \equiv \sup_{a\in\mathbb{R}} \left| \int_{\mathbb{R}} f(x-t)\,\frac{e^{iat}}{t}\,dt \right|$.
- Notice that C is **(linear) modulation invariant**, ie,
$$CM_a = C\,,$$
where $M_a f(x) = e^{iax} f(x)$. ($Cf = \sup_{a\in\mathbb{R}} \left|M_a H M_a^* f\right|$).
- (Ann. Math., 1972) C. Fefferman provides an influential, alternative proof to Carleson's result. These two proofs set the foundation of the area of time-frequency analysis.
- R. Hunt (Conf. Proc., 1969) proved L^p,$1<p<\infty$, bounds for C while Sjölin (Ark. Mat., 1971) extended this result to higher dimensions.

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Motivated by the study of singular integral on the Heisenberg group as well as on the previously discussed work with Wainger on the Hilbert transform along curves, E. Stein proposed the following generalization of Carleson's result:

Conjecture (Polynomial Carleson)

(**Stein**, *1995) Let $n \in \mathbb{N}$ and assume $\gamma(x,t) := \sum_{j=1}^{n} a_j(x)\, t^j$ with $\{a_j(\cdot)\}_j$ real measurable functions. Then given any $1 < p < \infty$ the Polynomial Carleson operator C_γ obeys the bound*

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- Relying on Van der Corput estimates and TT^* method, Stein and Wainger, (Math. Res. Lett., 2001) verified the non-zero curvature case $\gamma(x,t) := \sum_{j=2}^{n} a_j(x)\, t^j$. Notice thus that this result does not extend Carleson's Theorem.
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III. Singular and maximal (sub)bilinear operators

Main Problem (Generic formulation)

Understand the behavior of

- **the bilinear Hilbert transform along** γ

$$H_{\gamma}^{\mathcal{B}}\ :\ S(\mathbb{R})\times S(\mathbb{R})\longrightarrow L^{\infty}(\mathbb{R})\,,$$

$$H_{\gamma}^{\mathcal{B}}(f,g)(x):=p.v.\int_{\mathbb{R}}f(x-t)\,g(x+\gamma(x,t))\,\frac{dt}{t}\ ;$$

- **the (sub)bilinear maximal operator along** γ

$$M_{\gamma}^{\mathcal{B}}\ :\ S(\mathbb{R})\times S(\mathbb{R})\longrightarrow L^{\infty}(\mathbb{R})\,,$$

$$M_{\gamma}^{\mathcal{B}}(f,g)(x):=\sup_{h>0}\frac{1}{2h}\int_{-h}^{h}|f(x-t)\,g(x+\gamma(x,t))|\,dt\ .$$

History (III): $\gamma(x,t) = \gamma(t) = a\,t,\ a \in \mathbb{R} \setminus \{0, 1\}$

- **Origin.** A. Calderon's study of the Cauchy integral on Lipschitz curves. $H_\gamma \equiv BHT$.
- **Facts.** $H_\gamma^{\mathcal{B}}$ (with obvious adaptations for $M_g^{\mathcal{B}}$) obeys

$$H_\gamma^{\mathcal{B}}(T_y f, T_y g) = T_y H_\gamma^{\mathcal{B}}(f,g)\,,\ H_\gamma^{\mathcal{B}}(D_\lambda f, D_\lambda g) = D_\lambda H_\gamma^{\mathcal{B}}(f,g)$$

$$H_\gamma^{\mathcal{B}}(M_{ab} f, M_{-b} g) = M_{(a-1)b} H_\gamma^{\mathcal{B}}(f,g)$$

- **Consequences.** The extra modulation symmetry suggests the use of the wave-packet theory.
- **Theorem** (M. Lacey-C. Thiele (Ann. Math.,1997,1999)).

$$\|H_\gamma^{\mathcal{B}}(f,g)\|_r \le C_\alpha\, \|f\|_p\, \|g\|_q\,,$$

with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, $\frac{2}{3} < r < \infty$ and $1 < p,\, q \le \infty$.
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History/motivation (III): $\gamma(x,t)=\gamma(t)$ non-zero curvature

- X. Li (AJM, 2013) proved that $H_{\gamma}^{\mathcal{B}}: L^2(\mathbb{R})\times L^2(\mathbb{R})\mapsto L^1(\mathbb{R})$ for $\gamma(t)=t^d$ with $d\in\mathbb{N}$, $d\geq 2$ relying on the $\sigma-$uniformity concept inspired by the work of Gowers.
- (AJM, 2015), (RMI, 2017) I proved that $H_{\gamma}^{\mathcal{B}}: L^p(\mathbb{R})\times L^q(\mathbb{R})\mapsto L^r(\mathbb{R})$ with $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$ for $1<p,\,q\leq\infty$ and $1\leq r<\infty$ for γ any smooth "non-flat" curve near zero and infinity - including e.g. the class of Laurent polynomials with no linear term.
- In the context of $M_{\gamma}^{\mathcal{B}}$ similar Hölder range bounds were proved by X. Li and L. Xiao (AJM, 2016) for $\gamma(t)$ polynomial with no linear term and by A. Gaitan and myself (JFAA, 2020) for γ non-flat.

Philosophy

In what follows we would like to outline a method employed by the author in order to analyze the boundedness properties of large classes of (sub-)linear or multi-linear operators:

- phase <u>linearization</u> (L)
- <u>Gabor frame</u> discretization (G)
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- provide a *unified and sharp* method of treating simultaneously both singular and maximal operators
- provide a higher degree of *compatibility* between non-zero and zero curvature techniques by bringing into the picture (linear) wave-packet analysis
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Theorem (L.,2019)

Let $\gamma(x,t):\ \mathbb{R}^2 \to \mathbb{R}$ be such

- $\gamma(\cdot,t)$ *is measurable for every* $t \in \mathbb{R}$;
- $\gamma(x,\cdot)$ *is "non-flat" in the variable* t *for a.e.* $x \in \mathbb{R}$ *and piecewise* C^2*-smooth;*
- γ *satisfy suitable nondegeneracy condition.*

[In particular $\gamma(x,t) = \sum_{j=1}^{d} a_j(x)\, t^{\alpha_j}$ *with* $\alpha_j \in \mathbb{R} \setminus \{1\}$, $d \in \mathbb{N}$, *and* $\{a_j\}_j$ *real measurable functions obeys all of the above.]*

Then, for any $1 < p < \infty$, *one has*

$$\|H_\gamma f\|_p,\ \|M_\gamma f\|_p,\ \|C_\gamma f\|_p \lesssim_p \|f\|_p\,.$$

The Hybrid Bilinear Hilbert transform

Definition: For $a \in \mathbb{R} \setminus \{-1\}$, $b \in \mathbb{R}$ and $\alpha \in \mathbb{R}_+ \setminus \{1,2\}$, we define

- the bilinear Hilbert transform along $\gamma(t) := at + bt^\alpha$

$$H^\alpha_{a,b}(f,g)(x) := \int_{\mathbb{R}} f(x-t)\, g(x + a\,t + b\,t^\alpha)\, \frac{dt}{t}\,,$$

- the subbilinear maximal function along $\gamma(t) := at + bt^\alpha$

$$M^\alpha_{a,b}(f,g)(x) := \sup_{\varepsilon>0} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} |f(x-t)\, g(x + a\,t + b\,t^\alpha)|\, dt\,.$$

γ is referred as a hybrid curve and satisfies the following ($\alpha > 1$)

- as $|t|$ approaches the 0-singularity $\gamma(t)$ tends asymptotically to a line of the form $a\,t$ for some $a \in \mathbb{R} \setminus \{-1\}$,
- and as $|t|$ approaches the ∞-singularity $\gamma(t)$ tends asymptotically to a $b\,t^\alpha$ for some $b \in \mathbb{R}$.

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Theorem (A. Gaitan, V. Lie, 2022)

Let $a \in \mathbb{R} \setminus \{-1\}$, $b \in \mathbb{R}$ *and* $\alpha \in \mathbb{R}_+ \setminus \{1,2\}$. *For any* $p,\, q \geq 1$ *and* $r > \frac{2}{3}$ *such that* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, *one has*

$$\|H^{\alpha}_{a,b}(f,g)\|_{L^r}\,, \|M^{\alpha}_{a,b}(f,g)\|_{L^r} \lesssim_{a,b,\alpha,p,q} \|f\|_{L^p}\, \|g\|_{L^q}\,.$$

Observation

The restriction $\alpha \neq 2$ *is not a mere technicality but rather a fundamental, philosophical obstruction appearing as a result of a hidden quadratic modulation resonance. Consequently, one needs a fundamental new idea - see the discussion on the Polynomial Carleson operator and the Bilinear Hilbert Carleson operator.*

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Generalized Polynomial Carleson-type operator

Theorem

Consider the variable planar curve $\gamma(x,t)=a_0(x)t+\sum_{k=1}^{d} a_k(x)t^{\alpha_k}$ *for* $d\in\mathbb{N}$ *and let* $\{\alpha_k\}_{k=1}^{d}\subset(0,\infty)\setminus\{2\}$ *and* $\{a_k(\cdot)\}_{k=0}^{d}$ *be real measurable functions. Define the* $\gamma-$*Carleson operator as*

$$C_\gamma(f)(x) := \int_{\mathbb{R}} f(x-t)\, e^{i\,\gamma(x,t)}\, \frac{dt}{t}\,.$$

Then C_γ *are bounded (sub)linear operators from* L^p *to* L^p, $1<p<\infty$.

A curved model for the triangular Hilbert transform

Theorem (M. Christ, P. Durcik, J. Roos, 2021.)

Consider the bilinear singular integral operator given by

$$T(f,g)(x,y) := \int_{\mathbb{R}} f(x+t,y)\, g(x,\, y+t^2)\, \frac{dt}{t}, \quad \text{with} f,\, g \in S(\mathbb{R}^2)\,.$$

Then T extends to a bounded operator from $L^p \times L^q \to L^r$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, $1 < p,\, q < \infty$ and $r \in [1,2)$.

Proposition

Fix $m \in \mathbb{N}$ and $\phi_m(\xi) := \phi(\frac{\xi}{2^m})$ with $\phi \in C_0^\infty(\mathbb{R})$ and $\operatorname{supp} \phi \subseteq \{\xi \,|\, \frac{1}{4} < |\xi| < 4\}$

$$T_m(f,g)(x,y) := \int_{\mathbb{R}} (f *_1 \check{\phi}_m)(x+t,y)\, (g *_2 \check{\phi}_m)(x, y+t^2)\, \rho(t)\, dt.$$

Then there exists $\varepsilon > 0$ such that

$$\|T_m(f,g)\|_{L^1(\mathbb{R}^2)} \lesssim 2^{-\varepsilon m}\, \|f\|_{L^2(\mathbb{R}^2)}\, \|g\|_{L^2(\mathbb{R}^2)}\,.$$

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The non-resonant Bilinear Hilbert Carleson operator

Theorem (jointly with C. Benea, F. Bernicot and M. Vitturi, 2021)

Let $a \in (0, \infty) \setminus \{1, 2\}$ *and assume* p, q, r *are Hölder indices, i.e.* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, *with* $1 < p, q \leq \infty$ *and* $\frac{2}{3} < r < \infty$. *Then the non-resonant Bilinear–Hilbert Carleson operator defined as*

$$BC^a(f, g)(x) := \sup_{\lambda \in \mathbb{R}} \left| \int f(x - t)\, g(x + t)\, e^{i\lambda t^a} \frac{dt}{t} \right|$$

extends continuously from $L^p(\mathbb{R}) \times L^q(\mathbb{R})$ *into* $L^r(\mathbb{R})$.

Relevance - key properties

The operator $T = BC^a$ has a **hybrid** nature:

- commutes with translations and dilations

$$T(\tau_b f,\, \tau_b g) = \tau_b T(f,g) \;\text{ and }\; T(D_\alpha f,\, D_\alpha g) = D_\alpha T(f,g)\,;$$

- is invariant under the action of modulation

$$|T(M_c f,\, M_c g)| = |T(f,g)|\,;$$

- is a maximal singular integral operator whose kernel representation involves a maximal (generalized) modulation in the presence of curvature.

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Further Motivation

(I) Power decay property for multilinear oscillatory integrals

Theorem (M. Christ, X. Li, T. Tao and C. Thiele, Duke Math. J., 2005)

Let $P = P(x,t)$ be a real polynomial of degree at most $d \in \mathbb{N}$ and, for $f, g \in C_0^1(\mathbb{R})$, define

$$B^P(f,g)(x) := \int_{\mathbb{R}} e^{iP(x,t)} f(x-t) g(x+t) \frac{dt}{t}.$$

Then, for $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, with $1 < p, q \leq \infty$ and $\frac{2}{3} < r < \infty$, there exists $C = C(d) > 0$ such that uniformly for all real valued polynomials of degree at most d

$$\|B^P(f,g)\|_{L^r} \leq C \|f\|_{L^p} \|g\|_{L^q}.$$

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Further Motivation - (II) Convergence of the bilinear Fourier Series

Question

What meaning can one assign to the equality

$$f(x)g(x) = \sum_{n_1,n_2\in\mathbb{Z}} \hat{f}(n_1)\hat{g}(n_2)e^{in_1x}e^{in_2x},$$

when $f,g:\mathbb{R}\to\mathcal{C}$ *are both suitable* 1*-periodic functions with* $(f,g)\in L^p(\mathbb{T})\times L^q(\mathbb{T})$ *for some* $1<p,q\le\infty$*?*

Decide the summability along (convex) regions $\mathcal{P}$, that is

$$\lim_{N\to\infty}\sum_{n\in N\mathcal{P}} \hat{f}(n_1)\hat{g}(n_2)e^{2\pi i(n_1x+n_2x)}\ ?$$

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- If one takes $\mathcal{P} = \mathcal{P}_1 := \{(\xi,\eta) \,:\, 0 \leq \xi \leq \eta \leq 1\}$ then the last question is equivalent with understanding the boundedness properties of the Bi-Carleson operator

$$Bi\text{-}C(f,g)(x) := \sup_N \left| \iint_{\xi<\eta<N} \hat{f}(\xi)\hat{g}(\eta)e^{ix(\xi+\eta)}\, d\xi\, d\eta \right|.$$

- It is a known result due to C. Muscalu, T. Tao and C. Thiele, (GAFA, 2006) that $Bi\text{-}C(f,g)$ maps $L^p(\mathbb{R}) \times L^q(\mathbb{R}) \to L^r(\mathbb{R})$ continuously, whenever $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} < \frac{3}{2}$ and $p, q > 1$.
- If one takes $\mathcal{P} = \mathcal{P}_2 := \{(\xi,\eta) \,:\, 0 \leq \xi \leq \eta + 1 \leq 1\}$ then the question on the pointwise behavior is equivalent precisely with understanding the boundedness properties of the Bilinear Hilbert Carleson operator

$$\widetilde{BC}^1(f,g)(x) := \sup_{N>0} \left| \iint_{\xi-\eta\leq N} \hat{f}(\xi)\hat{g}(\eta)e^{ix(\xi+\eta)}\, d\xi\, d\eta \right|.$$

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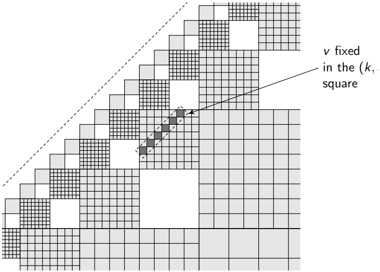
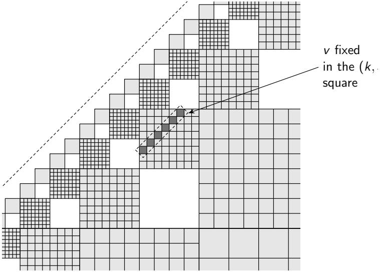


Two resolutions analysis:

- the high resolution - LGC methodology - extracts the cancellation encoded in the oscillatory term at a fixed scale
- the low resolution will be more similar to a *BHT* analysis, allowing to put together the various scales.

Not an absolutely (m-decaying) summable model! The control over the time-frequency correlations involves number theory via Weyl/exponential sums.



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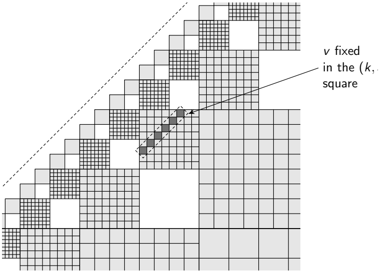

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Trilinear Hilbert transform - zero curvature

Open Problem

THT *Define the Trilinear Hilbert Transform*

$$THT(f,g,h)(x) := \int_{\mathbb{R}} f(x+t)\,g(x+2t)\,h(x+3t)\,\frac{dt}{t}\,.$$

Does this operator obey any Hölder L^p-type bounds?

Observation

In addition to translation, dilation and linear modulation symmetries, the trilinear Hilbert transform also obeys a **quadratic** *modulation invariance relation:*

$$THT(M^2_{3a}f,\, M^2_{-3a}g,\, M^2_a h)(x) = M^2_a\, THT(f,\, g,\, h)(x) \quad \forall a,\, x \in \mathbb{R}\,.$$

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Curved trilinear Hilbert transform (ongoing project with Bingyang Hu)

Open Problem

CTHT *Define the Curved Trilinear Hilbert Transform*

$$CTHT(f,g,h)(x) := \int_{\mathbb{R}} f(x+t)\, g(x+t^2)\, h(x+t^3)\, \frac{dt}{t}\,.$$

Does this operator obey any Hölder L^p-type bounds?

Hybrid trilinear Hilbert transform

Open Problem

HTHT *Define the Hybrid Trilinear Hilbert Transform*

$$HTHT(f,g,h)(x) := \int_{\mathbb{R}} f(x-t)\,g(x+t)\,h(x+t^2)\,\frac{dt}{t}\,.$$

Does this operator obey any Hölder L^p-type bounds?

THANK YOU!

History/motivation (I): $\gamma(x, t) = a(x)\, t$

- This originates in Lebesgue's theory of integration; he showed that for any (locally) integrable function over the real line and for almost every point, the value of the integrable function is the limit of infinitesimal averages taken about the point.
- Natural question: what about similar differentiability results in higher dimensions, say for functions on $\mathbb{R}^2$?
- This problem is much more subtle: reason - the existence of "pathological" objects such as Besicovitch sets.
- Indeed, the geometry of the sets over which we take the averages is critical for the well-posedness of this problem.
- In light of these challenging aspects of higher-dimensional differentiation problem, an alternative line of inquiry is offered by studying the problem of differentiation for averages along (variable) one-dimensional sets (curves) in $\mathbb{R}^2$.

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- Bourgain (1989) proved that for every real analytic function a there exists $\epsilon_0 > 0$ such that M_{a,ϵ_0} is bounded on L^2 (L^p).
- The analogue for H_{a,ϵ_0} was proved by Stein and Street (2012).
- A breakthrough in terms of the methods/regularity assumptions is due to Lacey and Li (2006). They showed - up to a conditional bound on a Kakeya type maximal operator - that if a is $C^{1+\epsilon}$ then H_{a,ϵ_0} is bounded on L^2.
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- If $\gamma(x,t) = P(x_1)\,\tilde{\gamma}(t)$ with P polynomial and $\tilde{\gamma}$ smooth and obeying various non-vanishing curv. cond. then H_γ is $L^2(\mathbb{R}^2)$ bdd. (Bennett, 2002; Chen and Zhu, 2012; Li and Yu, 2018);
- If $\gamma(x,t) = a(x_1)\,t^\alpha$ with $a(x_1)$ only *measurable* and $1 \neq \alpha > 0$, H_γ and M_γ are $L^p(\mathbb{R}^2)$-bdd. for $p > 1$ (Guo, Hickman, L. and Roos, 2016);

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- If $\gamma(x,t) = a(x_1)\,t^\alpha$ with $a(x_1)$ only *measurable* and $1 \neq \alpha > 0$, H_γ and M_γ are $L^p(\mathbb{R}^2)$-bdd. for $p > 1$ (Guo, Hickman, L. and Roos, 2016);

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- The non-zero curvature case:
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  - If one assumes supplementary regularity in x, y (convexity and doubling hypothesis uniformly in (x,y)) then M_{Γ}, H_{Γ} are $L^p(\mathbb{R}^2)$-bounded for $p > 1$ (Seeger and Wainger, 2003).
  - If u measurable then M_{Γ} is $L^p(\mathbb{R}^2)-$bounded for $p > 2$ (Marletta and Ricci, 1998);
  - If u is Lipschitz then the above holds for $1 < p \leq 2$ (Guo, Hickman, L. and Roos, 2016);
  - Same paper, u is Lipschitz then $H_{\Gamma} \circ P_k^2$ is bounded on L^p;
  - Using the annulus estimate above and a square function argument, Di Plinio, Guo, Thiele and Zorin-Kranich, (2017) completed the global result for H_{Γ}.

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Further Motivation - (II) Convergence of the bilinear Fourier Series

- The problem of almost-everywhere pointwise summability along discs is widely open

$$f(x)g(x) = \lim_{N\to\infty} \sum_{n_1^2+n_2^2\leq N^2} \hat{f}(n_1)\hat{g}(n_2)e^{2\pi i(n_1 x+n_2 x)}\,.$$

- The problem of summability along rectangles of arbitrary eccentricity holds both pointwise and in norm (consequence of the classical Carleson-Hunt):

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- What about the summability along polygonal regions $\mathcal{P}$, i.e.

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Further Motivation -(I) Power decay property for multilinear oscillatory integrals

Proposition

Let P be a real nondegenerate polynomial of degree d and $\eta \in C_0^1(\mathbb{R})$ supported away from zero. Then there exists $\epsilon > 0$ such that for any $u \in \mathbb{N}$

$$\left\| \int_{\mathbb{R}} e^{iP(2^u x, 2^u t)} f(x-t) g(x+t) \eta(t) dt \right\|_{L_x^r} \leq C 2^{-\epsilon u} \|f\|_{L^p} \|g\|_{L^q} .$$

The bilinear Hilbert–Carleson operator BC^a – together with its polynomial bilinear Hilbert–Carleson generalization BC_d – extends the behavior of B^P to the much wilder classes of phases $P(x,t) = \sum_{j=3}^{d} a_j(x) t^j$ with $d \in \mathbb{N}$ and $\{a_j\}_j$ arbitrary real measurable functions.

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