Caloric measure and regular Lip(1,1/2) graphs

Steve Hofmann (joint work with S. Bortz, J. M. Martell, and K. Nyström)

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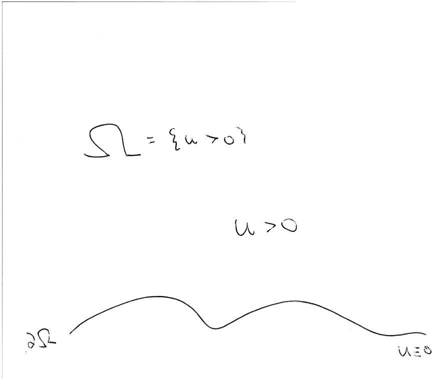
Introduction/History/Context

Classical 1-phase free boundary problem (Alt-Caffarelli, Jerison, Kenig-Toro, others)



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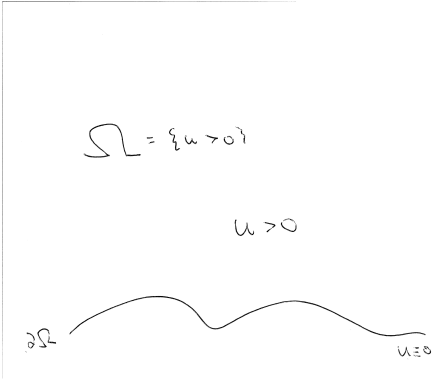

Q: given regularity of $|\nabla u|\big|_{\partial\Omega} = \frac{\partial u}{\partial \nu}$ (with $\nu =$ inward unit normal), what can we deduce re: regularity of the free boundary?

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Remark: The key idea in our argument: "Littlewood-Paley theory on level sets of the Green function". More precisely, we view the level set of the Green function at height $r > 0$, as a substitute for regularizations of the boundary at scale r, for which we can prove local Littlewood-Paley estimates; the latter in turn yield the desired regularity of the boundary.

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What about the parabolic case?

Introduction/History

Let $a(x,t)$ be a Lip(1,1/2) function defined on $\mathbb{R}^{n-1} \times \mathbb{R}$, i.e.,

$$\sup_{(x,t)\neq(y,s)} \frac{|a(x,t) - a(y,s)|}{\|(x-y, t-s)\|} =: \|a\|_{Lip(1,1/2)} < \infty\,,$$

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where $\|(x,t)\| := |x| + |t|^{1/2}$.

We consider Lip(1,1/2) graph domains $\Omega \subset \mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}$ defined by

$$\Omega := \left\{(x_0, x, t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} : x_0 > a(x,t)\right\}.$$

Set

$$\Sigma := \partial\Omega = \left\{(a(x,t), x, t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}\right\}.$$

Define "surface measure" on Σ by $\sigma := \mathcal{H}_p^{n+1}|_\Sigma$, where the parabolic Hausdorff measure $\mathcal{H}_p^{n+1}$ is defined like classical $\mathcal{H}^{n+1}$, but using the parabolic diameter of the sets in the covering (or essentially equivalently, covering by parabolic cubes, i.e., of dimension $r \times r \times ... \times r \times r^2$).

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Let ω denote caloric measure in Ω. As in the elliptic setting, $\omega \in A_\infty(\sigma)$ (in a suitable local, scale invariant way) is equivalent to solvability of $(D)_p$ for the heat equation in Ω.

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Q: For which domains does this happen?

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Motivated by the work of Strichartz on BMO-Sobolev spaces, and by Murray's thesis results under Coifman's direction, J. L. Lewis and M. A. M. Murray proved the following. Let Ω be the domain above a Lip(1,1/2) graph as above.

Theorem (Lewis-Murray 1995)

If Σ is a **Regular** *Lip(1,1/2) ("R-Lip") graph, then $\omega \in A_\infty(\sigma)$; equivalently $(D)_p$ is solvable for some $p < \infty$.*

Definition (R-Lip and GPG)

$a(x,t) \in$ R-Lip if $a(x,t)$ is Lipschitz in x, uniformly in t, and if

$$\mathbb{D}_n a := (\partial_t \circ H^{-1/2}) a \in \text{ BMO}(\mathbb{R}^n).$$

The graph of an R-Lip function is called a *Good Parabolic Graph* ("GPG").

Here $H = \partial_t - \Delta$ is the usual heat operator in $\mathbb{R}^{n-1} \times \mathbb{R}$, and BMO is parabolic BMO, i.e., defined w.r.t. parabolic cubes in $\mathbb{R}^{n-1} \times \mathbb{R}$.

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Hence $\mathbb{D}_n a \in$ BMO implies Lip$(1/2)$ in t, but is strictly stronger (follows essentially from work of Strichartz on BMO-Sobolev spaces).

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We now know this is true:

Theorem (S. Bortz, J. M. Martell, K. Nyström, S.H.)

Let Ω be a Lip(1,1/2) graph domain as above. Suppose that $\omega \in A_\infty(\sigma)$. Then $a \in$ R-Lip (i.e., $\Sigma \in$ GPG).

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Remark: Can define parabolic uniform rectifiability ("P-UR") analogously to David-Semmes elliptic theory. For a Lip(1,1/2) graph Σ, we have $\Sigma \in$ P-UR iff $\Sigma \in$ GPG. More generally, in work in progress we expect to show (believe we can show) that for a space-time corkscrew domain Ω with parabolic-TSADR boundary Σ, if $\omega \in$ (weak)-$A_\infty(\sigma)$, then $\Sigma \in$ P-UR (uses present result in a crucial way).

We'll sketch the proof of the Theorem. Some notation:

$M_0 := 1 + \|a\|_{Lip(1,1/2)}$.

$X = (x, t) \in \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$.

$\mathbf{X} = (x_0, x, t) = (x_0, X) \in \mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}$.

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(Similarly for Y, Z, $\mathbf{Y}, \mathbf{Z}$ etc.)

For $\mathbf{X} \in \Omega$, we let $\delta(\mathbf{X}) := \mathrm{dist}(\mathbf{X}, \Sigma)$.

Here and below, distances and diameters will always be taken w.r.t. the parabolic distance unless otherwise specified.

The n-dim parabolic cube of "length" r, centered at $X = (x, t) \in \mathbb{R}^n$:

$$Q_r(X) := \{(Y = (y, s) : |x_i - y_i| < r, 1 \leq i \leq n - 1\,,\ |t - s| < r^2\},$$

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$$I_r(\mathbf{X}) := (x_0 - 2M_0\sqrt{n}r, x_0 + 2M_0\sqrt{n}r) \times Q_r(x, t).$$

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For $\mathbf{X} = (x_0, x, t) = (a(x, t), x, t) \in \Sigma$, we define the "surface box"

$$\Delta_r(\mathbf{X}) := \Sigma \cap I_r(\mathbf{X}).$$

(Sketch of) Proof of Theorem:

Fix a surface box $\Delta := \Delta_R(\mathbf{X}^0)$, with $\mathbf{X}^0 = (a(x^0, t^0), x^0, t^0) \in \Sigma$.

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Set $\Delta_* := \Delta_{100R}(\mathbf{X}^0)$.

Let $\mathbf{X}^* = (x_0^*, x^*, t^*)$ be a time forward CS point relative to Δ_*; i.e., $\delta(\mathbf{X}^*) \approx \operatorname{dist}(\mathbf{X}^*, \Delta_*) \approx R$, and $t^* = t^0 + (200R)^2$.

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$G_*(\mathbf{Y}) := R^d G(\mathbf{X}^*, \mathbf{Y})$ is the normalized caloric Green function with pole at $\mathbf{X}^*$.

(Note $d = n + 1$ is the homogeneous dimension of Σ and of parabolic $\mathbb{R}^n$).

$\omega_* = R^d \omega^{\mathbf{X}^*}$ is normalized caloric measure with pole at $\mathbf{X}^*$.

$k_* := d\omega_*/d\sigma$ is the normalized "Poisson kernel".

As an easy consequence of the A_∞ property, we have:

Lemma (1)

Given $\varepsilon > 0$, there is a constant $M_1 = M_1(\varepsilon)$ and a closed set $F_* = F_*(\varepsilon) \subset \Delta_*$ such that

$$(1) \qquad \sigma(\Delta_* \setminus F_*) \leq \varepsilon \sigma(\Delta_*)$$

and such that for all $\mathbf{X} \in F_*$ and all $0 < r \leq 50R$

$$(2) \qquad \frac{1}{M_1} \leq \frac{1}{\sigma(\Delta_r(\mathbf{X}))} \int_{\Delta_r(\mathbf{X})} k_* \, d\sigma \leq M_1 \,.$$

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The lemma follows from doubling and a standard stopping time argument using the fact that k_* belongs to some A_p class, and that $k_*^{-1/(p-1)}$ belongs to $A_{p'}$.

By parabolic "CFMS estimates", due to FGS in caloric case, the lemma implies that

$$(3) \qquad G_*(\mathbf{Y}) \approx \delta(\mathbf{Y})$$

in a very ample sawtooth (relative to F_*). Also by an estimate of Caffarelli and Salsa, for some small enough constant η depending only on n and M_0,

$$(4) \quad \partial_{x_0} G_* \approx \frac{G_*}{\delta}, \quad \text{in } \Omega' := I_{50R}(\mathbf{X}^0) \cap \{\mathbf{Y} \in \Omega : \delta(\mathbf{Y}) < \eta R\}.$$

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Thus, the level set $\Sigma_r := \{\mathbf{Y} : G_*(\mathbf{Y}) = r\}$ is a graph, at least inside of Ω', i.e.,

$$\Sigma_r = \{(a_r(y,s), y, s)\} =: \{A_r(y,s)\}$$

inside Ω'.

Also, by (3) and (4), in a slightly restricted sawtooth Ω_* (relative to $F_* \cap \Delta_{50R}(\mathbf{X}^0)$, with shortened "ceiling" depending on η).

(5) $$\partial_{x_0} G_* \approx \frac{G_*}{\delta} \approx 1\,.$$

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In addition, in Ω^*, since $G_* \approx \delta$,

$$\delta(A_r(y,s)) \approx r \,=\, G_*(A_r(y,s))\,. \tag{6}$$

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To this end, we will view $a_r, r > 0$ as an extension of a, substituting for $P_r a$ (where $P_r =$ nice parabolic approx. identity).

The first main step is to prove Littlewood-Paley estimates for a_r in

$$\Omega_0 := \{(r, x, t) \in \mathbb{R}^{n+1}_+ : A_r(x, t) \in \Omega_*\}.$$

Observe that Ω_0 is essentially a sawtooth in $\mathbb{R}^{n+1}_+$ (localized at scale R), relative to $F = \pi(F_*)$, where $\pi(x_0, x, t) := (0, x, t)$.

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We differentiate the level set equation $G_*(a_r(x,t),x,t) = r$ implicitly, and use (5), to get, e.g.,

$$|\partial_t a_r(x,t)| = \left|\frac{\partial_t G_*(A_r(x,t))}{\partial_{x_0} G_*(A_r(x,t))}\right| \approx |\partial_t G_*(A_r(x,t))|\,. \tag{7}$$

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$$(7) \qquad |\partial_t a_r(x,t)| = \left|\frac{\partial_t G_*(A_r(x,t))}{\partial_{x_0} G_*(A_r(x,t))}\right| \approx |\partial_t G_*(A_r(x,t))|\,.$$

Similarly we get

$$(8) \qquad |\nabla_{x,r} a_r(x,t)| \lesssim 1\,, \quad |\nabla^2_{x,r} a_r(x,t)| \lesssim |(\nabla^2 G_*)(A_r(x,t))|$$

etc.

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$$r|\partial_t a_r(x,t)| + r|\nabla^2_{x,r} a_r(x,t)| \lesssim 1$$

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We then obtain

Lemma (2)

$$\iint_{\Omega_0} \left(|r\partial_t a_r(X)|^2 + |r\nabla^2_{x,r} a_r(X)|^2\right) \frac{dr}{r} dX \lesssim R^d .$$

and

$$\iint_{\Omega_0} |r^2 \nabla_{x,r} \partial_t a_r(X)|^2 \frac{dr}{r} dX \lesssim R^d .$$

$(d = n+1)$.

Sketch of proof of Lemma (2):

Use bounds for derivatives of a_r in terms of derivatives of G_* (i.e., (7), (8), etc.), along with Moser type local boundedness and (6) (i.e., $\delta(A_r) \approx r$) to reduce matters to analogous local square function bounds for G_* in Ω_*.

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In turn, to prove the latter, we use Lemma (1) plus CFMS. This allows us to replace δ by a normalized Green function in Ω^*, so that we can then integrate by parts to obtain Carleson measure estimates for $1_{\Omega_*}(\mathbf{X})|\partial_t G_*(\mathbf{X})|^2 \delta(\mathbf{X}) d\mathbf{X}$, etc.

Return to Proof of the Theorem:

By the (parabolic) John-Stromberg Lemma, it suffices to show that there are constants N, θ such that for each parabolic cube $Q_R \subset \mathbb{R}^n$,

$$\inf_C |\{X \in Q_R : |\mathbb{D}_n a(X) - C| > N\}| \leq \theta |Q_R|$$

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Recall that $F = \pi(F_*)$. We take M_1 large (hence ε small) in Lemma (1), and we have

$$|Q_R \setminus F| \leq \sigma(\Delta_* \setminus F_*) \leq \varepsilon\sigma(\Delta_*) \lesssim_{n,M_0} \varepsilon |Q_R|.$$

We then have

$$|\{X \in Q_R : |\mathbb{D}_n a(X) - C| > N\}| \leq N^{-2} \int_{F\cap Q_R} |\mathbb{D}_n a(X) - C|^2 + C\varepsilon |Q_R| \,.$$

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Taking first ε small enough, and then N large enough, and making a (standard) localization argument, we reduce to proving

$$|Q_R|^{-1} \int_{F\cap Q_R} |\mathbb{D}_n^R a(X)|^2 \lesssim 1 \tag{9}$$

with the implicit constant depending on allowable parameters, where $\mathbb{D}_n^R$ is a localized (at scale R) version of $\mathbb{D}_n$.

We proceed to the proof of (9). For notational convenience, for $r > 0$ set

$$a(r, x, t) := a_r(x, t)$$

and if $r = 0$ set $a(0, x, t) := a(x, t)$.

We proceed to the proof of (9). For notational convenience, for $r > 0$ set

$$a(r, x, t) := a_r(x, t)$$

and if $r = 0$ set $a(0, x, t) := a(x, t)$.

Let $h(x, t) \approx \operatorname{dist}((x, t), F)$ be a regularized distance function, so the graph $\{(h(x, t), x, t)\}$ is the (lower part of the) boundary of a (regularized) sawtooth (call it Ω_1) in $\mathbb{R}^{n+1}_+$, relative to F.

We can construct h so that $\Omega_1 \subset \Omega_0$, and so that $h(x, t)$ *can be taken to be R-Lip*, thus the graph $\{(h(x, t), x, t)\} \in$ GPG.

For future reference, we record an estimate for $a(r, Y)$ along the graph of h.

Note that $a(r, Y)$ is monotone increasing in r (since G_* is monotone increasing in x_0). Using this fact and the definition of the level sets, one can show (it's non-trivial, but not too hard) that

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$$0 \leq a(h(Y), Y) - a(0, Y) \lesssim h(Y)\,. \tag{10}$$

Set

$$a^h(x,t) := a(h(x,t),x,t)\,.$$

The next step is to show that we may replace a by a^h in (9) (i.e., in the L^2 estimate for $\mathbb{D}_n^R a$ on $F \cap Q_R$).

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The kernel of $\mathbb{D}_n^R$ (localized version of $\mathbb{D}_n$), call it K^R, satisfies

$$|K^R(X)| \lesssim \|X\|^{-d-1}\, 1_{\{\|\cdot\|\lesssim R\}}(X)\,.$$

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$$|K^R(X)| \lesssim \|X\|^{-d-1}\, 1_{\{\|\cdot\| \lesssim R\}}(X)\,.$$

Also, $h(Y) \approx \operatorname{dist}(Y,F) = 0$ on F by definition.

Hence,

$$
\begin{aligned}
&|\mathbb{D}_n^R a(X) - \mathbb{D}_n^R a^h(X)| \\
&\lesssim \int_{F^c\cap\{\|X-Y\|\lesssim R\}} \|X - Y\|^{-d-1} |a(0, Y) - a(h(Y), Y)|\, dY \\
&\qquad\qquad \lesssim \int_{F^c\cap\{\|X-Y\|\lesssim R\}} \|X - Y\|^{-d-1} \operatorname{dist}(Y, F)\, dY\,,
\end{aligned}
$$

where the last step uses (10) and the definition of h.

Hence,

$$|\mathbb{D}_n^R a(X) - \mathbb{D}_n^R a^h(X)|$$

$$\lesssim \int_{F^c \cap \{\|X-Y\| \lesssim R\}} \|X - Y\|^{-d-1} \, |a(0,Y) - a(h(Y),Y)| \, dY$$

$$\lesssim \int_{F^c \cap \{\|X-Y\| \lesssim R\}} \|X - Y\|^{-d-1} \, \mathrm{dist}(Y,F) \, dY \,,$$

where the last step uses (10) and the definition of h.

The latter expression is a parabolic Marcinkiewicz integral, so

$$\int_{F \cap Q_R} |\mathbb{D}_n^R a(X) - \mathbb{D}_n^R a^h(X)|^2 \, dX \lesssim R^d \approx |Q_R|$$

as desired.

Last step is to prove

$$\int_{Q_R} |\mathbb{D}_n^R a^h(X)|^2 \, dX \lesssim |Q_R| \, .$$

Equivalently, it remains to prove

$$\left| \int_{\mathbb{R}^n} \mathbb{D}_n^R a^h(X) \, f(X) \, dX \right| \lesssim |Q_R|^{1/2} \|f\|_{L^2} \, . \tag{11}$$

for $f \in L^2(\mathbb{R}^n)$ with support in Q_R.

Recall that Ω_1 (the regularized sawtooth domain above the graph of h) is contained in Ω_0, hence in particular we have

$$(r + h(X), X) \in \Omega_1 \subset \Omega_0\,, \quad X \in Q_{5R}\,, \quad 0 < r \lesssim R\,.$$

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Let P_r be a nice parabolic approximate identity on $\mathbb{R}^n$. For $\gamma > 0$ small,

$$|P_{\gamma r} h(X) - h(X)| \lesssim \gamma r \ll r\,,$$

since $h \in Lip(1, 1/2)$.

Hence

$$r + P_{\gamma r} h(X) \approx r + h(X) \geq r$$

so $(r + P_{\gamma r} h(X), X)$ also lies inside a slightly fattened version of Ω_1, call it Ω_2, for which we still have $\Omega_2 \subset \Omega_0$.

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Recall that the regularized distance function h is in R-Lip, so we have Carleson measure estimates for the measures

$$|\nabla_{x,r} P_r h(X)|^2 \, r \, dr \, dX \,, \quad |r \partial_t P_r h(X)|^2 \, r \, dr \, dX$$

etc. (*Remark: this is why we want to replace h by* $P_{\gamma r} h$).

Combining these observations, setting

$$\widetilde{a}(r,X) := a(r + P_{\gamma r} h(X), X)\,,$$

and using Lemma (2) (local square function bounds in Ω_0 for derivatives of $a(r,X)$), we obtain

Lemma (3)

$$\iint_{\Omega_R} \left(|r\partial_t\, \widetilde{a}(r,X)|^2 + |r\partial_r^2\, \widetilde{a}(r,X)|^2\right) \frac{dr}{r} dX \lesssim |Q_R|\,.$$

and

$$\iint_{\Omega_R} |r^2 \partial_r \partial_t\, \widetilde{a}(r,X)|^2 \,\frac{dr}{r} dX \lesssim |Q_R|\,,$$

where Ω_R is a Carleson box in $\mathbb{R}^{n+1}_+$ with base Q_{5R}.

Note that at $r = 0$, we have $\widetilde{a}(0, X) = a^h(X)$.

To prove (11), we then write, with $f \in L^2(\mathbb{R}^n)$ with support in Q_R

$$\int_{\mathbb{R}^n} \mathbb{D}_n^R a^h f = \text{ boundary term (with } r = R)$$
$$- \int_0^R \int_{\mathbb{R}^n} \partial_r \left[\mathbb{D}_n^R \big(\widetilde{a}(r, \cdot) \big)(X) \, P_r f(X) \right] dr \, dX \, .$$

The boundary term is fairly routine to handle.

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When both r-derivatives land on $P_r f$, we use that the kernel of $\mathbb{D}_n^R$ is of the form $K^R(X) = \partial_t V_R(X)$, where V_R is a smoothly truncated (at scale R) version of the parabolic fractional integral kernel. Hence

$$\mathbb{D}_n^R \widetilde{a} = K^R * \widetilde{a} = V_R * (\partial_t \widetilde{a})$$

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and we can move the convolution kernel V_R onto $\partial_r^2 P_r f$.

After the derivatives are appropriately balanced, we then use Cauchy-Schwarz, Lemma (2), and the fact that square function bounds hold for derivatives of $P_r f$, to obtain (11).

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