Penn
Department of Mathematics

Testing conditions for multilinear Radon-Brascamp-Lieb inequalities

11th International Conference on Harmonic Analysis and Partial Differential Equations

Philip T. Gressman

6 June 2022

University of Pennsylvania

$\mathrm{SL}_n(\mathbb{R})$ **Invariance in Analysis**

The "Problem" of Coordinate Invariance

The study of L^p-improving properties of Radon-like operators raises many challenges. One is that, particularly in intermediate dimensions, it's not clear what the answers *should be*. There are, however, a lot of symmetries. Can symmetry alone lead us to the right ideas?

The "Sensible Approaches"

Symmetry in analysis can lend itself to multiple valid ways of working:

1. Use quantities invariant under the symmetry.
   - **THINK**: matrices and determinants
2. Look for qualitative features invariant under the symmetry.
   - **THINK**: matrices and null spaces
3. Work in any coordinates whatsoever, then optimize over the choice.
   - **THINK**: Newton diagrams and "height"

The approaches have distinctly different strengths and weaknesses.

Brascamp-Lieb Inequalities

Setup

- V, V_j : Vector spaces of dimension d, d_j, $j = 1, \ldots, m$;
- π_j : Surjective linear maps $V \to V_j$;
- θ_j: constants in $[0, 1]$.

Under what conditions must all nonnegative measurable functions $f_j : V_j \to \mathbb{R}$ satisfy

$$\int_V \prod_{j=1}^{m} (f_j \circ \pi_j)^{\theta_j} \leq \mathrm{BL}(\pi, \theta) \prod_{j=1}^{m} \left(\int_{V_j} f_j \right)^{\theta_j}$$

for finite constant $\mathrm{BL}(\pi, \theta)$?

Hölder's inequality, Young's convolution inequality, Loomis-Whitney inequalities, etc., can all be written in this form.

Sensible Approaches to Brascamp-Lieb

1. Use quantities invariant under the symmetry. (HARD)
2. Look for qualitative features invariant under the symmetry.

Theorem (Bennett, Carbery, Christ, Tao 2005)

$\mathrm{BL}(\pi, \theta) < \infty$ if and only if

$$\dim V' \leq \sum_{j=1}^{m} \theta_j \dim \pi_j(V') \text{ for all } V' \subset V$$

3. Work in any coordinates whatsoever, then optimize over the choice.

Theorem (Lieb 1990)

Gaussians saturate the inequality:

$$\mathrm{BL}(\pi, \theta) = \sup_{\substack{A_j \in \mathrm{GL}(V_j) \\ j=1,\ldots,m}} \frac{\prod_{j=1}^{m} |\det_{V_j} A_j|^{\theta_j}}{\left(\det \sum_{j=1}^{m} \theta_j \pi_j^* A_j^* A_j \pi_j\right)^{\frac{1}{2}}}.$$

Radon-Like Operators and Generalizations of Brascamp-Lieb

Defining Functions and Incidence Relations

(Ω, π, Σ) is a smooth incidence relation on $\mathbb{R}^n \times \mathbb{R}^{n'}$ of codim. k when

- **DOMAIN:** $\Omega \subset \mathbb{R}^n \times \mathbb{R}^{n'}$ is open
- **DEFINING FUNCTION:** $\pi : \Omega \to \mathbb{R}^k$ is smooth
- **COORDINATE-DEPENDENT JACOBIAN:** $||d_x\pi(x,y)||_\omega$ for any n-tuple $\omega := \{\omega_i\}_{i=1}^n$ of vectors in $\mathbb{R}^n$ is given by

$$||d_x\pi(x,y)||_\omega := \left[\frac{1}{k!}\sum_{i_1,\dots,i_k=1}^{n}\left|\det\left[(\omega_{i_1}\cdot\nabla_x)\pi \quad \cdots \quad (\omega_{i_k}\cdot\nabla_x)\pi\right]\right|^2\right]^{\frac{1}{2}}$$

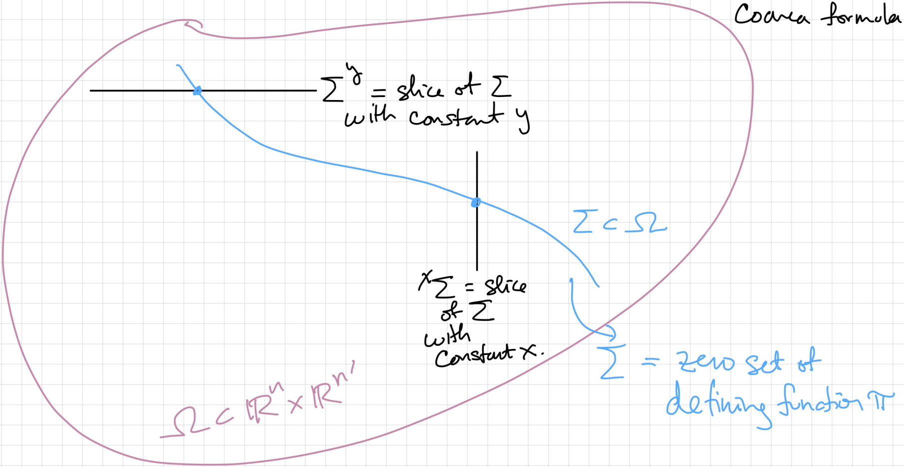
  When ω is omitted, use default coordinates.
- **INCIDENCE RELATION:**

$$\Sigma := \{(x,y) \in \Omega \mid \pi(x,y) = 0, ||d_x\pi(x,y)||, ||d_y\pi(x,y)|| > 0\}.$$

- **INCIDENCE RELATION SLICES:** ${}^x\Sigma$ and Σ^y indicate slices of Σ with fixed x and y, resp.:

$${}^x\Sigma := \left\{y \in \mathbb{R}^{n'} \mid (x,y) \in \Sigma\right\} \text{ and } \Sigma^y := \left\{x \in \mathbb{R}^n \mid (x,y) \in \Sigma\right\}.$$

$\Sigma, {}^x\Sigma, \Sigma^y$ have natural measures induced by Coarea formula.



$\| d_x \pi(x,y) \|_\omega$ quantifies linear independence

of $\omega_1 \cdot \nabla_x \pi, \dots, \omega_n \cdot \nabla_x \pi$

- **NATURAL MEASURE:** On each slice ${}^x\Sigma$ and Σ^y, σ denotes what will be called the coarea measure (also known as the Leray or microcanonical measure elsewhere), given by

$$\int_{{}^x\Sigma} f d\sigma := \int_{{}^x\Sigma} f(y) \frac{d\mathcal{H}^{n'-k}(y)}{||d_y\pi(x,y)||} \text{ and } \int_{\Sigma^y} f d\sigma := \int_{\Sigma^y} f(x) \frac{d\mathcal{H}^{n-k}(x)}{||d_x\pi(x,y)||}$$

  for any Borel-measurable function f on the slices (Borel measurability is assumed for convenience to avoid technical difficulties associated with restricting Lebesgue-measurable functions to submanifolds), where $d\mathcal{H}^s$ is s-dimensional Hausdorff measure.
- **Side Note:** Both $d\mathcal{H}^{n'-k}(y)$ and $||d_y\pi(x,y)||$ are coordinate-dependent, but *the ratio is not*.

Key Quantity to Consider

For fixed $\tau > 0$, the big question is the behavior of the quantity

$$I(x,\tau,\omega) := \int \frac{d\sigma(y)}{||d_x\pi(x,y)||^{\tau}_{\omega}}$$

as ω ranges over all volume-normalized choices of bases.

The Main Theorem

Main Theorem Setup

- **ENSEMBLE OF RADON-LIKE OPERATORS:** For any integer $m \geq 1$ and each $j = 1, \ldots, m$, let $(\Omega_j, \pi_j, \Sigma_j)$ be a smooth incidence relation on $\mathbb{R}^n \times \mathbb{R}^{n_j}$ with codimension $k_j \leq \min\{n, n_j\}$ and let σ_j denote the associated coarea measures on slices. Let $w_j : \Sigma_j \to [0, \infty)$ be continuous, and let T_j be the generalized Radon-like transform given by

$$T_j f(x) := \int_{{}_x\Sigma_j} f_j(y_j) w_j(x, y_j) d\sigma_j(y_j)$$

for all nonnegative Borel-measurable f_j on $\mathbb{R}^{n_j}$.

- **CRITICAL SCALING LINE:** Suppose $p_1, \ldots, p_m \in [1, \infty)$ and $q_1, \ldots, q_m \in (0, \infty)$ satisfy the scaling condition

$$n = \sum_{j=1}^{m} \frac{k_j q_j}{p_j}.$$

This is the line that $L^\infty \times \cdots \times L^\infty \to L^\infty$ lies on; one can show that estimates never hold when $=$ is replaced by $<$ (i.e., q_j's could never be larger for given p_j).

Main Theorem Necessity

Let $||T||$ be the smallest positive constant (supposing one exists) such that for all nonnegative Borel measurable functions $f_j \in L^{p_j}(\mathbb{R}^{n_j})$,

$$\int_{\mathbb{R}^n} \prod_{j=1}^{m} |T_j f_j(x)|^{q_j} dx \leq ||T|| \prod_{j=1}^{m} ||f_j||_{L^{p_j}(\mathbb{R}^{n_j})}^{q_j}.$$

There exists a constant C depending only on n and n_j, k_j, p_j, q_j for $j = 1, \ldots, m$ such that for any $x \in \mathbb{R}^n$ and any vectors $\omega_1, \ldots, \omega_n$ with $|\det(\omega_1, \ldots, \omega_n)| = 1$ (where $\det(\omega_1, \ldots, \omega_n)$ is the determinant of the matrix whose columns are coordinates of $\omega_1, \ldots, \omega_n$ in the standard coordinate system),

$$\prod_{j \,:\, p_j = 1} \sup_{y_j \in {}^x\Sigma_j} \frac{|w_j(x, y_j)|^{q_j}}{||d_x \pi_j(x, y_j)||_\omega^{q_j}} \prod_{j \,:\, p_j > 1} \left[\int_{{}^x\Sigma_j} \frac{|w_j(x, y_j)|^{p_j'} d\sigma_j(y_j)}{||d_x \pi_j(x, y_j)||_\omega^{p_j' - 1}} \right]^{\frac{q_j}{p_j'}} \leq C||T||$$

where for each j, p_j and p_j' are Hölder dual exponents.

Main Theorem Sufficiency

Conversely, suppose $[[T]]$ is defined to be the supremum of

$$\prod_{j\,:\,p_j=1} \sup_{y_j \in {}^x\Sigma_j} \frac{|w_j(x,y_j)|^{q_j}}{||d_x\pi_j(x,y_j)||_\omega^{q_j}} \prod_{j\,:\,p_j>1} \left[\int_{{}^x\Sigma_j} \frac{|w_j(x,y_j)|^{p_j'} d\sigma_j(y_j)}{||d_x\pi_j(x,y_j)||_\omega^{p_j'-1}} \right]^{\frac{q_j}{p_j'}}$$

over all $x \in \mathbb{R}^n$ and all $\{\omega_i\}_{i=1}^n$ with $|\det(\omega_1,\ldots,\omega_n)| = 1$. If $[[T]] < \infty$ and each $\pi_j(x,y_j)$ is a polynomial function of x with bounded degree as a function of y_j,

$$\int_{\mathbb{R}^n} \prod_{j=1}^m |T_j f_j(x)|^{q_j} dx \leq ||T|| \prod_{j=1}^m ||f_j||_{L^{p_j}(\mathbb{R}^{n_j})}^{q_j}.$$

holds for nonnegative f_j with a finite value of $||T||$ satisfying $||T|| \leq C'[[T]] \prod_{j=1}^m (\deg \pi_j)^{q_j/p_j}$ for some C' depending only on n, and n_j, k_j, p_j, q_j, where $\deg \pi_j := \sup_{y_j} \deg \pi_j^1(\cdot,y_j) \cdots \deg \pi_j^k(\cdot,y_j)$.

Example: Brascamp-Lieb Inequalities Recovered

- If $\pi_j(x)$ are surjective linear maps from $\mathbb{R}^n$ to $\mathbb{R}^{n_j}$, we can define $\pi_j(x, y_j) := y_j - \pi_j(x)$ on $\mathbb{R}^n \times \mathbb{R}^{n_j}$. Slices ${}^x\Sigma_j$ are zero-dimensional: $y_j = \pi_j(x)$ and the multilinear object T simply becomes

$$\int \prod_{j=1}^{m} f_j(\pi_j(x))dx.$$

- The measures $d\sigma_j$ are simply counting measure and the estimate for $||T||$ is comparable to the exact value.
- **Side Note:** There are some interesting and nontrivial implications for discrete Brascamp-Lieb inequalities which happen when the incidence relation is changed to equal $|y_j - \pi_j(x)|^2 - 1 = 0$.

Example: Spherical Averages

For each $n \geq 2$, $\exists\ C_n < \infty$ such that every $f \in L^{(n+1)/n}(\mathbb{R}^n)$ satisfies

$$\left[\int_{\mathbb{R}^n} \left| \int_{\mathbb{S}^{n-1}} f(x+z) d\mathcal{H}^{n-1}(z) \right|^{n+1} dx \right] \leq C_n ||f||_{L^{(n+1)/n}(\mathbb{R}^n)}.$$

The spherical averaging operator can be expressed via defining function

$$\pi(x,y) := 1 - |x-y|^2 \quad , \quad ||d_x \pi(x,y)||^2_\omega = 4 \sum_{i=1}^n |\omega_i \cdot (x-y)|^2$$

The endpoint $L^{(n+1)/n} \rightarrow L^{n+1}$ inequality is a consequence of the bound

$$\int_{\mathbb{S}^{n-1}} \frac{d\mathcal{H}^{n-1}(z)}{\left(\sum_{i=1}^n |\omega_i \cdot z|^2 \right)^{n/2}} \leq C$$

for all $|\det(w_1, \ldots, w_n)| = 1$.

Sample Computation: Circular Averages ($n = 2$)

$$\int_{\mathbb{S}^1} \frac{d\mathcal{H}^1(z)}{\left(|z \cdot \omega_1|^2 + |z \cdot \omega_2|^2\right)^{s/2}},$$

Reductions allow one to assume that ω_1 and ω_2 are orthogonal; after rotation, the integral is comparable to

$$\int_{-1}^{1} \frac{dt}{(\alpha^2 + \alpha^{-2} t^2)^{s/2}}$$

for arbitrary $\alpha > 0$. **Criterion is not finiteness but uniform bddness.**

- If $\alpha > 1$, then there is no difficulty bounding the integral uniformly.
- If $\alpha < 1$, then the integrand is $\gtrsim \alpha^{-s}$ on an interval of size $\approx \alpha^2$ around $t = 0$. Uniform bounds can only hold when $s \leq 2$.
- When $s = 2$, change variables $t \mapsto \alpha^2 t$: the Jacobian gives α^2 in the numerator which exactly cancels the α's in the denominator.
- "Flat spots" on the circle would arrange it so that the set where the integrand is $\gtrsim \alpha^{-s}$ is much larger than α^2 and reduce the range of s on which uniformity holds.

Proof of the Main Theorem

Visibility Lemma (Guth 2010, Carbery-Valdimarsson 2013)

Given any finitely-supported, nonnegative, integer-valued function $M(Q)$ defined on the lattice of unit cubes $\Lambda_1 \subset \mathbb{R}^n$, $\exists$ an algebraic hypersurface Z of degree at most $C_n(\sum_Q M(Q))^{1/n}$ such that

$$\overline{\text{Vis}}[Z \cap Q] \geq M(Q) \text{ for all } Q \in \Lambda_1,$$

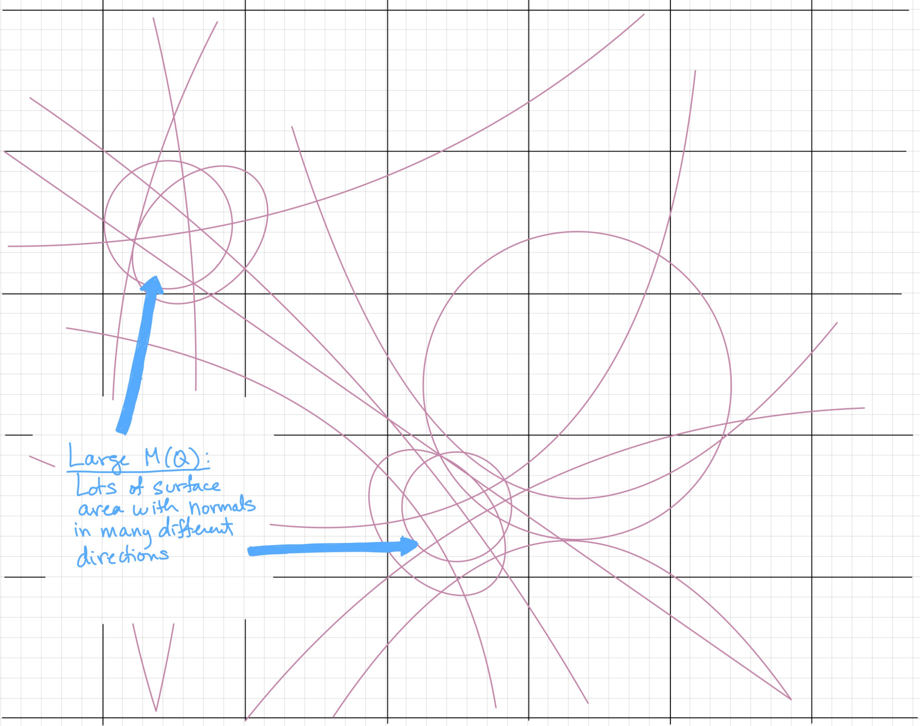
where $\overline{\text{Vis}}[Z \cap Q]$ is the mollified visibility, i.e., the reciprocal of the Euclidean volume of the convex set of vectors u for which $||u|| \leq 1$ and

$$\frac{1}{|B(Z,\epsilon)|} \int_{B(Z,\epsilon)} \int_{Z' \cap Q} |u \cdot \widehat{n}(z')| d\mathcal{H}^{n-1}(z') dZ' \leq 1.$$

Here $\widehat{n}(z)$ is the unit normal to Z' at the point z'.

Point

Inside each Q, Z has at least a prescribed amount of surface area with surface normals in very different directions. The bound on the deg Z is really what matters and is approximately sharp.


Large M(Q):
Lots of surface area with normals in many different directions

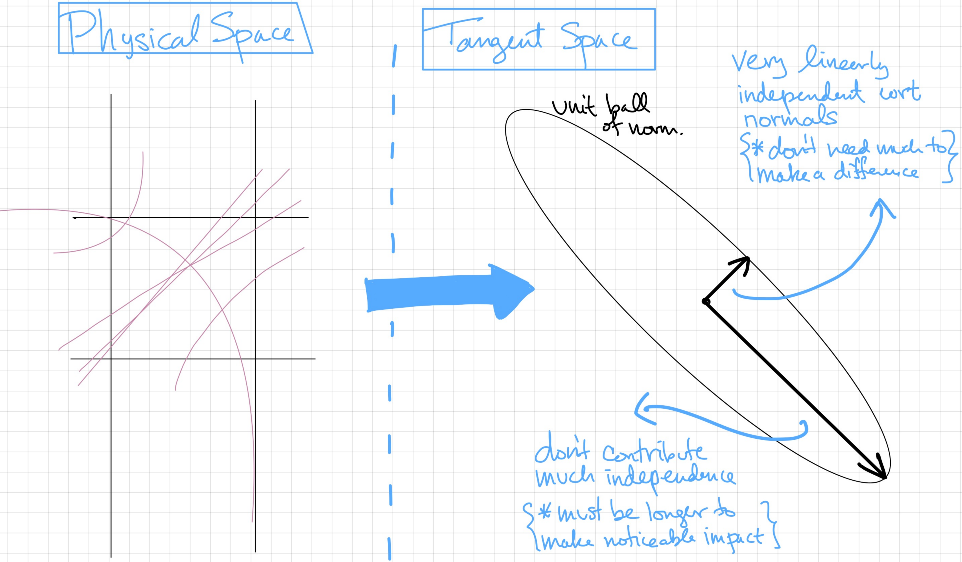
Theorem (Zhang 2018)

$$\int_{\Sigma_\pi}\int_{Z'_1}\cdots\int_{Z'_{n-k}} \chi_{[-1,1]^{(n-k)k}}(z'_1 - x, \dots, z'_{n-k} - x)$$
$$\cdot \left| \frac{d\pi(x)}{||d\pi(x)||} \wedge \widehat{n}(z'_1) \wedge \cdots \wedge \widehat{n}(z'_{n-k}) \right|$$
$$d\mathcal{H}^{n-1}(z'_{n-k}) \cdots d\mathcal{H}^{n-1}(z'_1) d\mathcal{H}^{n-k}(x)$$
$$\leq 2^{n(n-k)} \deg \pi \prod_{j=1}^{n-k} \deg Z'_j.$$

Point

Carrying out the inner integrals measures the size of $d\pi$ in some **unnormalized**, pointwise-varying system $\{\omega_1^x, \dots, \omega_n^x\}$ with volume like $\overline{\text{Vis}}[Z \cap Q]$ on Q. This system doesn't depend on π and

$$\int_{\Sigma_\pi} ||d\pi(x)||_{\omega^x} d\sigma(x) \lesssim (\deg \pi) \left(\sum_Q M(Q) \right)^{\frac{n-k}{n}}$$



In the end, the norm matters in the proof because of its quantitative properties, but the variety generating it does not.

Zhang's Theorem is the discrete version of the following:

Lemma (Visibility Lemma, Continuous Version)

For any Borel measurable, nonnegative integrable function ψ on the box $B_R := [-R, R)^n$, there exist Borel measurable $\mathbb{R}^n$-valued functions $\omega_1^x, \ldots, \omega_n^x$ on B_R such that $|\det\{\omega_i^x\}_{i=1}^n| = 1$ at all points and a nonnegative Borel-measurable function $\widetilde{\psi}$ on B_R equal to ψ a.e. such that every polynomial map $\pi : \mathbb{R}^n \to \mathbb{R}^k$ with $1 \leq k \leq n$ satisfies

$$\int_{\Sigma_\pi \cap B_R} \left[\widetilde{\psi}(x)\right]^{\frac{n-k}{n}} ||d\pi(x)||_{\omega^x} d\sigma(x) \leq C_n(\deg \pi) \left[\int_{B_R} \psi(x) dx\right]^{\frac{n-k}{n}}$$

The passage from discrete to continuous is mostly predictable:

- Rescale the unit scale result to all dyadic scales and approximate.
- A minor but critical technical point: there's no real control of the ω_x^i.
- Also note that because we're integrating over sets of measure zero, it's not trivial that $\widetilde{\psi}$ can disagree with ψ on a set of measure zero.

Practically speaking, this lemma plays the role analogous to that of the change of variables formula in Christ's continuum incidence methodology.

What does this lemma mean?

Let $f_1(x), \ldots, f_n(x)$ be (arbitrary) coordinate functions that map the box B_R to some box B' and let $\psi(x) := |\det \partial f / \partial x|$.

- We can think of $\psi^{-1/n}(x)\nabla f_1(x), \ldots, \psi^{-1/n}(x)\nabla f_n(x)$ as "unit-size" covectors with respect to some exotic norm. The normalization w.r.t. ψ guarantees that the basis is unit volume.
- Now take $\omega_x^1, \ldots, \omega_x^n$ to be the dual basis of vectors.
- The change of variables formula implies that

$$\int_{\Sigma_\pi \cap B_R} [\psi(x)]^{\frac{n-k}{n}} ||d\pi(x)||_{\omega^x} d\sigma(x) \leq C_n K \left[\int_{B_R} \psi(x) dx \right]^{\frac{n-k}{n}}$$

  where K counts the maximum number of times that a k-dimensional "coordinate affine subspace" meets Σ_π transversely.

Speculation

Is this a Monge-Ampère problem in disguise? Are there non-algebraic analogues of this lemma? Extending these ideas to the smooth category would require answers here.

Now combine Hölder's inequality, the Fubini-type identity

$$\int_{B_R} [\widetilde{\psi}(x)]^{\frac{n-k_j}{n}} \int_{{}^x\Sigma_j} f_j(y_j) ||d_x\pi_j(x, y_j)||_{\omega^x} d\sigma_j(y_j)\, dx = \int_{\mathbb{R}^{n_j}} f_j(y_j) \int_{\Sigma_j^{y_j} \cap B_R} [\widetilde{\psi}(x)]^{\frac{n-k_j}{n}} ||d_x\pi_j(x, y_j)||_{\omega^x} d\sigma_j(x)\, dy_j,$$

and a judicious choice of ψ to conclude that

Theorem (The Quantity)

Fix exponents $r_1, \ldots, r_m \geq 0$ satisfying $k_1 r_1 + \cdots + k_m r_m = n$, and for each nonnegative integrable Borel function f_j on $\mathbb{R}^{n_j}$, $j = 1, \ldots, m$, let

$$Q(f_1, \ldots, f_m)(x) := \inf_{\substack{\{\omega_i\}_{i=1}^n \\ |\det\{\omega_i\}_{i=1}| = 1}} \prod_{j=1}^m \left(\int_{{}^x\Sigma_j} f_j(y_j) ||d_x\pi_j(x, y_j)||_\omega d\sigma_j(y_j) \right)^{r_j}.$$

There exists a constant C_n depending only on n such that

$$\int_{[-R,R)^n} Q(f_1, \ldots, f_m)(x) dx \leq \prod_{j=1}^m \left[C_n (\deg \pi_j) \int_{\mathbb{R}^{n_j}} f_j \right]^{r_j}.$$

Important

We are **not** going to apply Brascamp-Lieb inequalities at any point. These are actually already encoded in the strange quantity Q.

The integrals appearing in Q have undesired factors of $||d_x\pi_j(x, y_j)||_\omega$. We can eliminate these by making a bound of Q from below by Hölder:

$$\int_{^x\Sigma} f(y)w(x,y)d\sigma(y) \le \left[\int_{^x\Sigma} |f(y)|^p ||d_x\pi(x,y)||_\omega d\sigma(y)\right]^{\frac{1}{p}} \cdot \left[\int_{^x\Sigma} \frac{|w(x,y)|^{p'} d\sigma(y)}{||d_x\pi(x,y)||_\omega^{p'-1}}\right]^{\frac{1}{p'}}.$$

It follows that

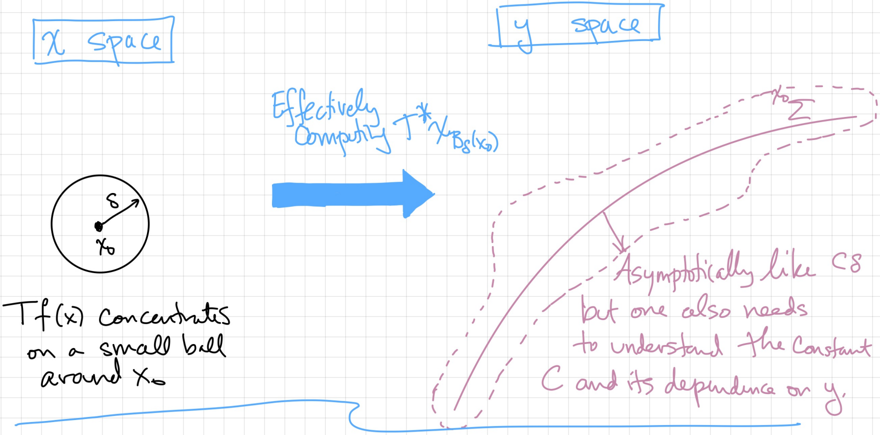
$$\int_{\mathbb{R}^n} \prod_{j=1}^m |T_j f_j(x)|^{q_j} dx \le ||T|| \prod_{j=1}^m ||f_j||_{L^{p_j}(\mathbb{R}^{n_j})}^{q_j},$$

$$||T|| \lesssim \prod_{j\,:\,p_j=1} \sup_{y_j \in {}^x\Sigma_j} \frac{|w_j(x,y_j)|^{q_j}}{||d_x\pi_j(x,y_j)||_\omega^{q_j}} \prod_{j\,:\,p_j>1} \left[\int_{^x\Sigma_j} \frac{|w_j(x,y_j)|^{p_j'} d\sigma_j(y_j)}{||d_x\pi_j(x,y_j)||_\omega^{p_j'-1}}\right]^{\frac{q_j}{p_j'}}$$

over all choices of $\{\omega_i\}_{i=1}^n$.

The Reverse Direction

Enhanced Knapp Example Geometry



Trick: Renormalize $\pi(x,y)$ so that Jacobian $D_x\pi(x,y)$ has orthonormal rows. Then the purple region in y space corresponds to the set $|\pi(x_0,y)| \le \delta$. This captures varying thickness.

The δ-neighborhood of ${}^{x_0}\Sigma_j$ is roughly where $\pi_j(x_0, y) < \delta$. To be more precise, we renormalize π_j so that the rows of the Jacobian $D_x\pi_j(x_0, y)$ are orthonormal on ${}^{x_0}\Sigma_j$. This can be accomplished by replacing π_j by $\overline{\pi}_j := (D_x\pi_j(D_x\pi_j)^T)^{-1/2}\pi_j$. The test functions used are

$$f_{j,\delta}(y) := \delta^{-\frac{k}{p_j}} \chi_{||\overline{\pi}_j(x_0,y)||<\delta} \left[\frac{w_j(x_0, y)}{||d_x\pi_j(x_0, y)||} \right]^{p_j'-1}$$

$$T_j f_{j,\delta}(y) \sim \delta^{-\frac{k}{p_j}} \int_{{}^{x_0}\Sigma_j} \frac{|w_j(x_0, y)|^{p_j'} d\sigma_j(y)}{||d_x\pi_j(x_0, y)||^{p_j'-1}}$$

for $x \in B_\delta(x_0)$. Also

$$||f_{j,\delta}||_{p_j}^p = \delta^{-k} \int_{|u|<\delta} \int_{\overline{\pi}_j(x_0,y)=u} \left[\frac{w_j(x_0, y)}{||d_x\pi_j(x_0, y)||} \right]^{p_j(p_j'-1)} \frac{d\mathcal{H}^{n_j-k}(y)}{||d_y\overline{\pi}_j(x_0, y)||} du$$

$$\sim c_k \int_{\overline{\pi}_j(x_0,y)=0} \left[\frac{w_j(x_0, y)}{||d_x\pi_j(x_0, y)||} \right]^{p_j'} \frac{||d_x\pi_j(x_0, y)|| d\mathcal{H}^{n_j-k}(y)}{||d_y\pi_j(x_0, y)||}$$

Work in any coordinates you like modulo linear change of variables, then optimize over the choice.

What’s Next

Some Natural Questions

- This brings us close but not all the way to positively understanding what curvature is "supposed" to be, particularly in the case of the nicest operators in intermediate dimensions.
- Algebra versus Analysis
  - To what extent can the class of polynomials be replaced by something broader? (Nash functions; turns out to be important)
  - Approximating smooth operators by algebraic ones is possible because we have good, explicit dependence of constants on degree.
- How do you compute the key quantity anyway? (Restricted strong type is a comfortable next step.) **Geometrically, what is being quantified?**
- Bounds for inequalities off the scaling line? Weighted inequalities may be able to help.

Thank You