Maximal and singular operators in codimension 1 and higher

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Work being reported

with I. Parissis (UPV-Ikerbasque)

- *Maximal subspace averages*, arXiv:2107.0210
- *Maximal directional operators along algebraic varieties*, Amer. J. Math. 143, no. 5 (2021), pp. 1463–1503

with N. Accomazzo (U British Columbia), P. Hagelstein (Baylor), I. Parissis, L. Roncal (BCAM-Ikerbasque)

- *Directional square functions*, arXiv:2004.06509, to appear Analysis&PDE

Subspace averages

- $\sigma \in \mathrm{Gr}(d,n)$, the Grassmannian of d-dim. subspaces of $\mathbb{R}^n$
- scale s subspace average of $f \in C(\mathbb{R}^n)$ on σ

$$\langle f \rangle_{s,\sigma}(x) := \int_{sB_n \cap \sigma} f(x-y) \frac{\mathrm{d}y}{s^d}, \qquad x \in \mathbb{R}^n$$

where $B_n \subset \mathbb{R}^n$ is the n-dimensional unit ball

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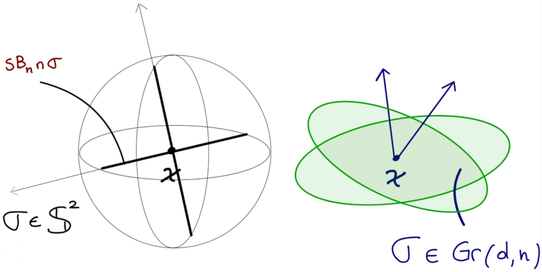

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Sharp $L^p(\mathbb{R}^n)$-norms of the maximal operators single and multiscale

$$\mathrm{M}_{\Sigma,s} f := \sup_{\sigma \in \Sigma} \langle |f| \rangle_{s,\sigma}(\cdot), \qquad \mathrm{M}_{\Sigma} f := \sup_{s \in S} \mathrm{M}_{\Sigma,\{s\}} f$$

in terms of $\#\Sigma$, when $\Sigma \subset \mathrm{Gr}(d,n)$.

- $d = 1, n \geq 2$ *Thin* Nikodym maximal operator
- $1 < d < n-1$: related to (d,n)-Kakeya and Nikodym sets
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Fourier and plate averages

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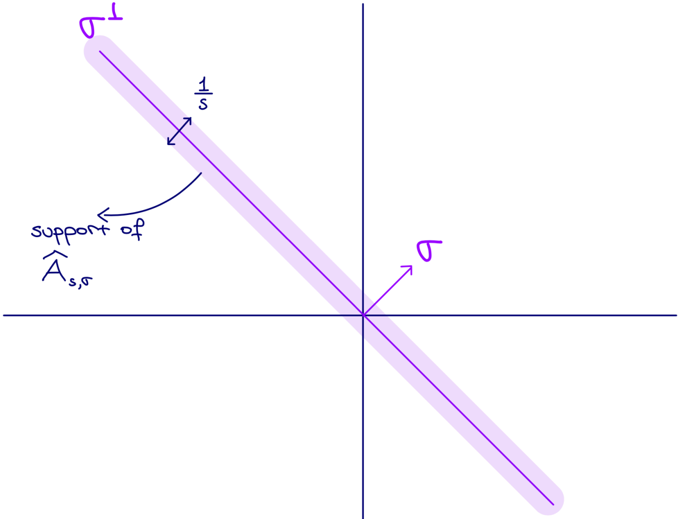

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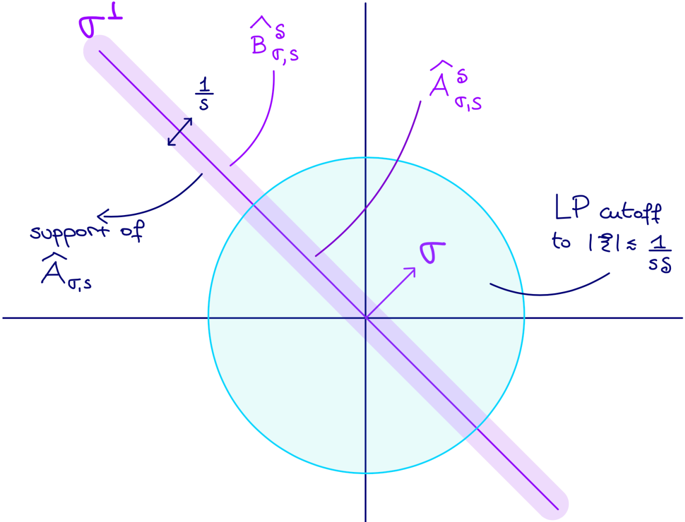

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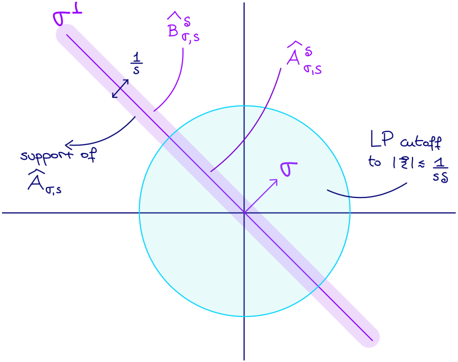

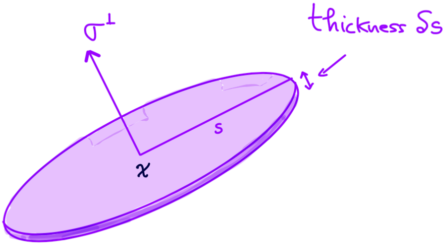

$$A^\delta_{\sigma,s}f(x) \sim \fint_{x+T^\delta_{\sigma,s}} f, \qquad T^\delta_{\sigma,s} := \left\{y\in\mathbb{R}^n : \ |\Pi_\sigma y| \le s, |\Pi_{\sigma^\perp}y| < s\delta\right\}.$$

Fourier methods: the switch lemma

$$\mathrm{d}(\sigma,\tau) := \sup_{v\in\mathbb{S}^{n-1}} |\sigma v - \tau v|, \qquad \sigma,\tau \in \mathrm{Gr}(d,n).$$

Switch lemma. Let $v, w \in \mathrm{Gr}(d,n)$, and $\delta \gtrsim \mathrm{dist}(v,w)$. Then

$$\left|A^{\delta}_{w,s} f(x)\right| \lesssim \sum_{k=0}^{\infty} 2^{-kn} \fint_{x+2^k T^{\delta}_{v,s}} |f| \lesssim \left|\widetilde{A}^{\delta}_{v,s} f(x)\right|.$$

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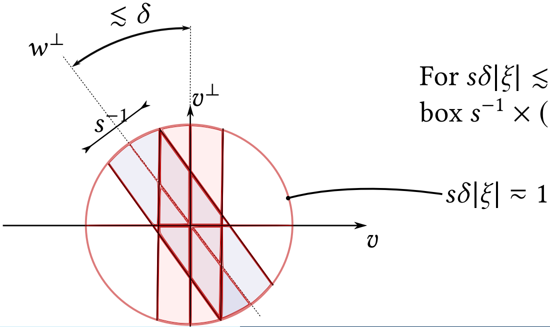
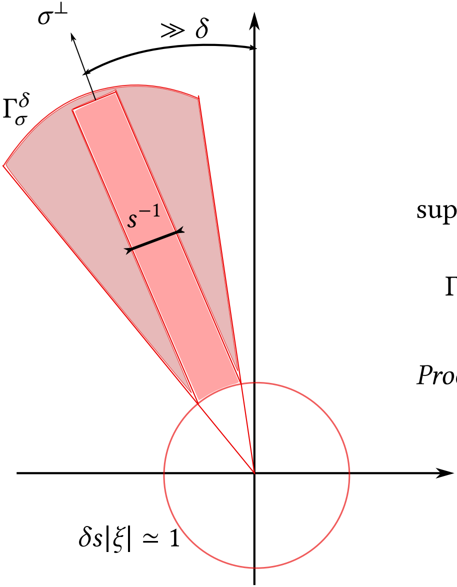


For $s\delta|\xi| \lesssim 1$ both adapted to box $s^{-1} \times (\delta s)^{-1}$ in coordinates $v, v^{\perp}$.

Fourier methods: almost orthogonality of cones



$$\operatorname{supp} \widehat{B^\delta_{\sigma,s}} \subset \Gamma^\delta_\sigma,$$

$$\Gamma^\delta_\sigma := \left\{ \xi \in \mathbb{R}^n \setminus \{0\} : \frac{|\Pi_\sigma \xi|}{|\xi|} < \delta \right\}$$

Proof. $\xi \in \operatorname{supp} \widehat{B^\delta_{\sigma,s}} \implies$

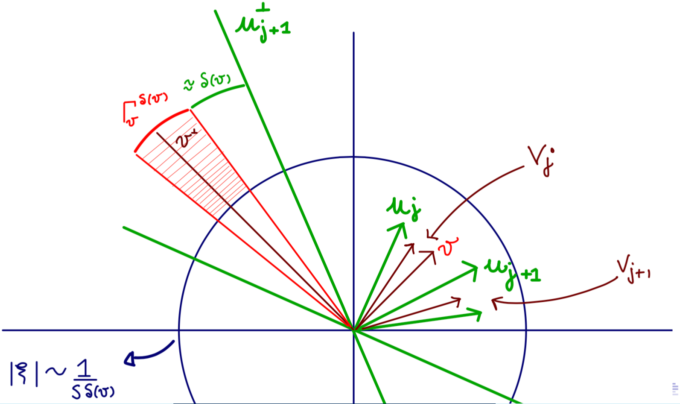
$$|\Pi_\sigma \xi| < s^{-1}, \delta|\xi| > s^{-1}.$$

Almost orthogonality in the plane

THEOREM D. (Reproof of Alfonseca-Soria-Vargas) Let

$$U = \{u_1, \ldots, u_{N+1}\}, V = V_1 \cup \cdots \cup V_N \subset \mathbb{S}^1, \text{ each } V_j \text{ bordered by } u_j, u_{j+1}.$$

Then $\|\mathrm{M}_V\|_{L^2(\mathbb{R}^2)} \leq C\,\|\mathrm{M}_U\|_{L^2(\mathbb{R}^2)} + \max_{1\leq j\leq N} \left\|\mathrm{M}_{V_j}\right\|_{L^2(\mathbb{R}^2)}$.



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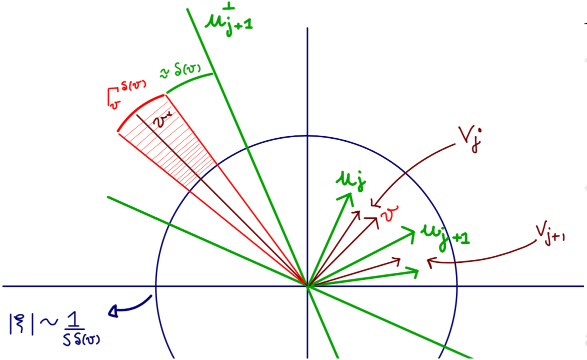

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- Switch lemma:

$$v \in V_j \implies A^{\delta(v)}_{s,v} f \sim A_{s,u_j} f$$

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are disjoint over j.

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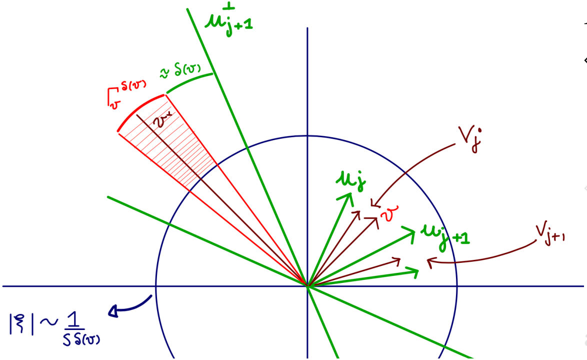

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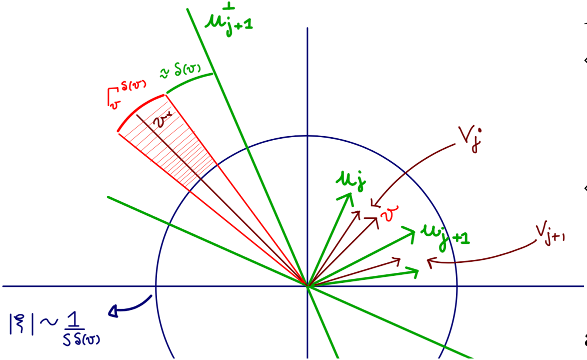

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Algebraic almost orthogonality

THEOREM E. Let $p \in \mathbb{R}[x_1, \dots, x_n]$ be a degree D polynomial. Then

$$\|M_V\|_{L^2(\mathbb{R}^n)} \lesssim \sup_{\substack{U \subset \mathbb{S}^{n-1} \cap Z(p) \\ \#U \leq \#V}} \|M_U\|_{L^2(\mathbb{R}^n)} + D^{\frac{(n-2)}{2}} \sup_{\mathrm{C} \in \mathrm{C}(p)} \|M_{V \cap \mathrm{C}}\|_{L^2(\mathbb{R}^n)} .$$

◇ Precursors: Alfonseca-Soria-Vargas, Parcet-Rogers,...

◇ Gain: the set $Z := \mathbb{S}^{n-1} \cap Z(p)$ is a $(n-2)$-dimensional algebraic variety of controlled degree and M_U is better behaved when $U \subset Z$.

◇ Also holds for $M_{V,1}$. Leads via polynomial partitioning to

THEOREM B. (DP-Parissis, 2021)

$$\sup_{\substack{V \subset \mathbb{S}^2 \\ \#V = N^2}} \|M_{V,1} f\|_{L^2(\mathbb{R}^3)} \lesssim_k N^{\frac{1}{2}} \log^{[k]} N$$

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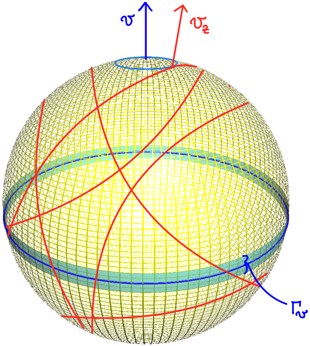
Proof: algebraic almost-orthogonality

Let $P \in \mathbb{R}[x_1, x_2, x_3]$ degree $\lesssim D$; $\mathrm{C} \in \mathcal{C}(P)$ cc. of $\mathbb{S}^2 \setminus Z(P)$.



- $v \in V \cap \mathrm{C}$, $\delta(v) := \frac{\mathrm{dist}(v,Z(P))}{2} = \frac{\mathrm{dist}(v,v_z)}{2}$
- $\Gamma_v = \{\xi \in \mathbb{R}^d : |\xi \cdot v| < \delta(v)|\xi|\}$
- $\left[B^{\delta}_{v,s} f\right]^{\wedge}(\xi) \neq 0 \implies \xi \in \Gamma_{v,\delta}$
- (switch) $\left|A^{\delta}_{v,s} f\right| \lesssim \left|\widetilde{A}^{\delta}_{v_z,s} f\right|$
- $\Gamma_{\mathrm{C}} = \bigcup_{v \in V \cap \mathrm{C}} \Gamma_v$, $\widehat{T_{\mathrm{C}} f} = \widehat{f} 1_{\mathrm{C}}$.
- if $\xi \in \Gamma_{\mathrm{C}}$ then $\xi^{\perp} \cap \mathrm{C} \neq \emptyset$
- $\Theta = \sup_{\xi \neq 0} \#\{\mathrm{C} \in \mathcal{C}(P) : \xi \in \Gamma_{\mathrm{C}}\}$.

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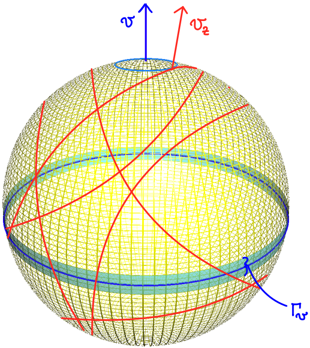

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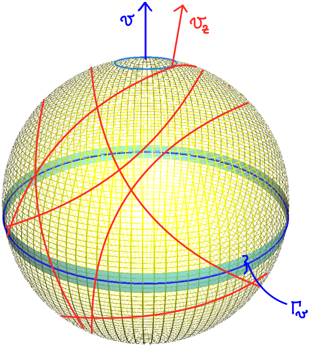

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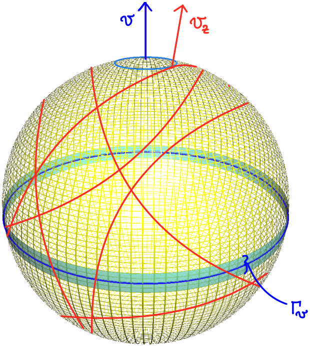

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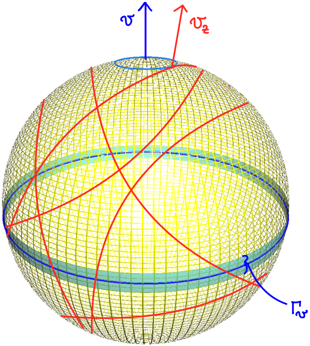

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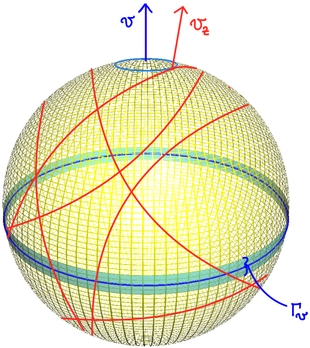

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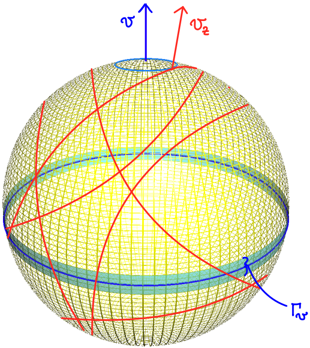

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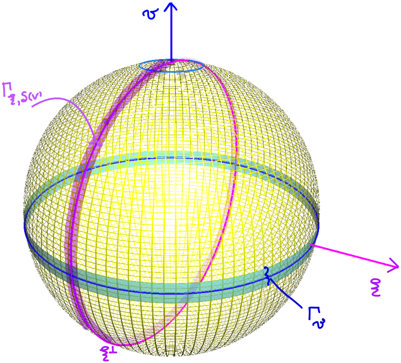

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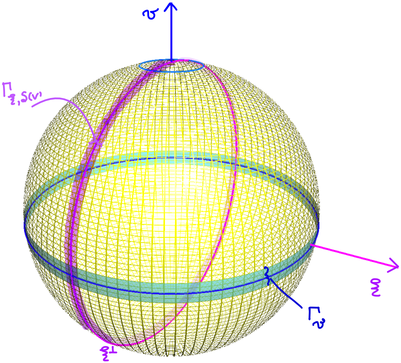

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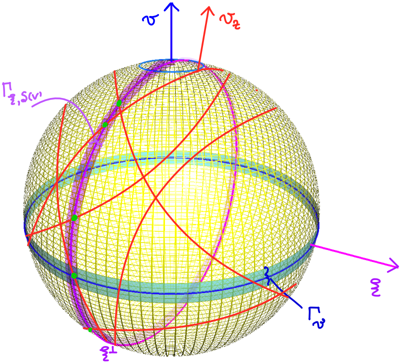

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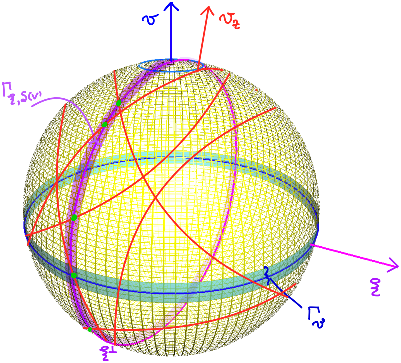

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The case of 1-varieties on $\mathbb{S}^2$: better behaved?

Theorem (DP-Parissis, AJM2020)

If $V \subset Z$ where $Z \subset \mathbb{S}^2$ is a real 1-dim. algebraic variety of degree D

$$\|M_{V,1} f\|_{L^2(\mathbb{R}^3)} \lesssim D^{\frac{1}{2}} (\log \# V)^{\frac{3}{2}} \|f\|_{L^2(\mathbb{R}^3)}$$

- If Z is a single parametrizable polynomial curve this result is rather easy and originally due to Córdoba (variational estimate) (see also Christ, Duoandikoetxea, Rubio de Francia).
- General 1-dim. variety: issues caused by existence of up to $O(D^2)$ connected components of Z. Dependence $D^{\frac{1}{2}}$ is crucial
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M_V along polynomial directions

Theorem A (FDP-IP, 2018)

Let $Z_m \subset \mathbb{R}^n$ be a m-dimensional algebraic variety. Then

$$\left\|\mathrm{M}_{V,1}\right\|_{L^2(\mathbb{R}^n)} \lessapprox N^{\frac{m-1}{2}}$$

uniformly over $V \subset Z_m$ with $\#V \leq N^m$. Sharp up to $\lessapprox$.

REM. Estimate uniform over degree D *transverse complete intersections*

$$Z_m = \left\{x \in \mathbb{R}^n : P_1(x) = \ldots = P_{n-m}(x) = 0\right\}, \quad \operatorname{rank} \mathrm{D}\vec{P} \equiv n-m, \quad \max \deg P_j \leq D.$$

Example: degree $\leq D$ polynomial graph over m variables.

Past results (not mentioning the structured cases)

- $Z_1 = \mathbb{S}^1$: Strömberg ('78) uniformly distr. V,... Katz (2000) general case.
- $m = 1$: Córdoba ('83), Barrionuevo ('95)
- $Z_2 = \mathbb{S}^2$: Demeter ('12)

L^2-sharp bounds for singular $M_{\Sigma,1}$ - all d

Theorem A. (DP-Parissis2021)

For all $N > 0$, $1 \le d < n$, $\Sigma \subset \mathrm{Gr}(d,n)$, $\#\Sigma \le N^{d(n-d)}$

$$\left\| \mathrm{M}_{\Sigma,1} \right\|_{L^2(\mathbb{R}^n)} \lesssim_{d,n} N^{\frac{d(n-d-1)}{2}} (\log N)^{\frac{n-d}{2}}$$

Bound is sharp in terms of N.

◇ Lower bound:

$$\Sigma = \text{direct sums } v \oplus \omega,\ \omega \in \mathrm{Gr}(d-1,n) \text{ fixed},\ v \text{ equispaced on } \mathbb{S}^{n-d}.$$

◇ Proof ingredients

- Fix d, induct on $n \ge d+1$
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Spatial methods: multiscale, codimension 1, sharp

THEOREM C. Let $n = d + 1$ and $\Sigma \subset \mathrm{Gr}(d, d+1)$. Then

$$\|\mathrm{M}_\Sigma : L^2(\mathbb{R}^n) \to L^{2,\infty}(\mathbb{R}^n)\| \lesssim (\log \#\Sigma)^{\frac{1}{2}}, \qquad \|\mathrm{M}_\Sigma\|_{L^2(\mathbb{R}^n)} \lesssim \log \#\Sigma.$$

All bounds are best possible in terms of $\#\Sigma$.

◇ $d = 1, n = 2$ a theorem of N. Katz (Duke, 2000)

◇ Methods.

- $TT^\star$ type argument $\rightsquigarrow$ directional Carleson embedding
- continuity/slicing estimate for $Q \cap P$, Q, P d-plates in $\mathbb{R}^n$

◇ Same abstract theorem yields a Rubio de Francia estimate

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All bounds are best possible in terms of $\#\Sigma$.

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- $TT^\star$ type argument $\rightsquigarrow$ directional Carleson embedding
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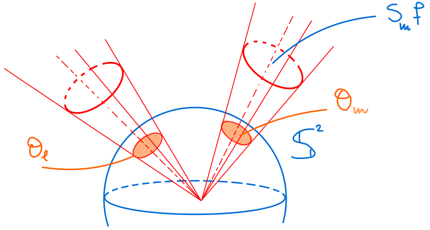
Directional Rubio de Francia square functions

$$S_m f(x) = \int_{\mathbb{R}^{d+1}} \widehat{f}(\xi)\Phi_{\theta_m}\left(\frac{\xi}{|\xi|}\right) e^{ix\cdot\xi}\, d\xi$$

Φ_{θ_m} smooth cutoff to cap $\theta_m \subset \mathbb{S}^d$

$\theta_1, \dots, \theta_N$ bdd overlap caps - arb size



Theorem J. If $S_m, \ell = 1, \dots, N$ bdd overlap smooth conical projections,

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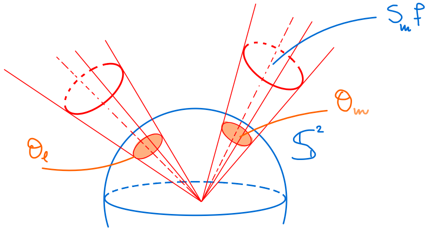

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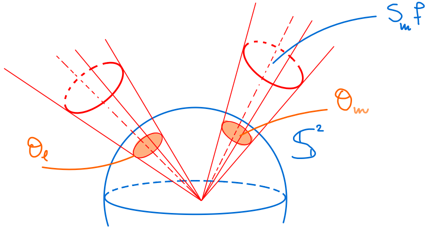

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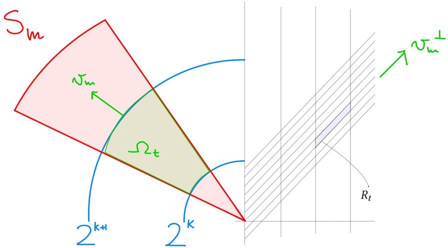

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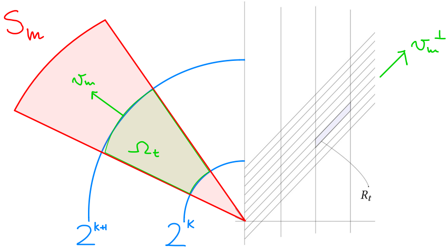

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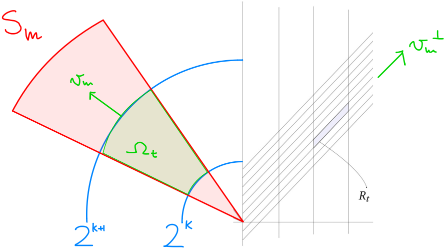

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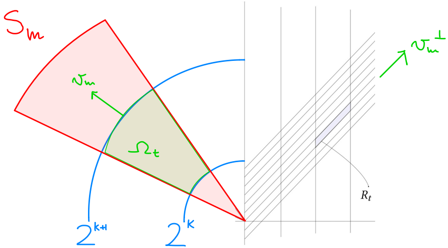

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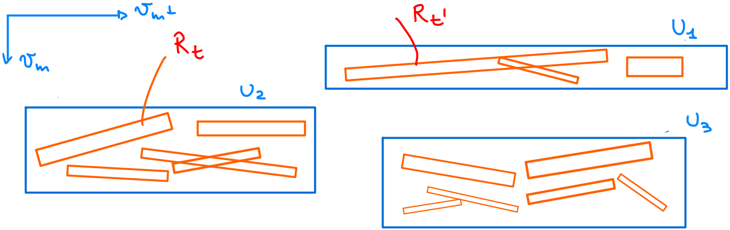

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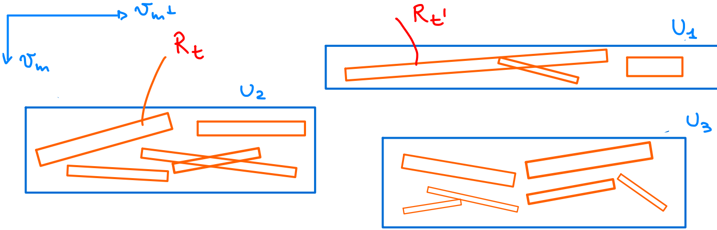

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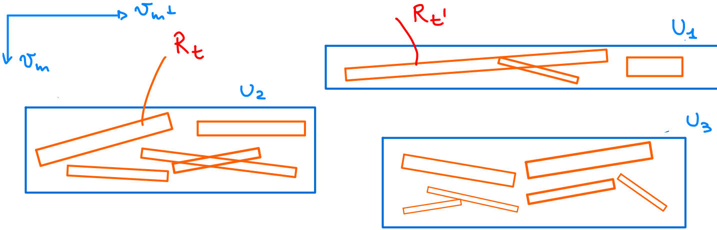

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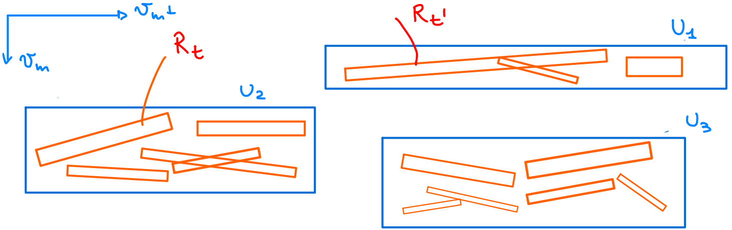

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Theorem (Accomazzo+2020, APDE, to appear, $n = 2$) (DP-Parissis, 2021 $n \geq 2$)

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Maximal function: $\mathrm{T}(m)$ all $s \times \delta$-plates R_t along $v_m^\perp$, $A_t(f) = \int_{\{x \in R_t : R(x) = R_t\}} |f|$

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$T_N f$ = Fourier restriction of f to a regular N-gon

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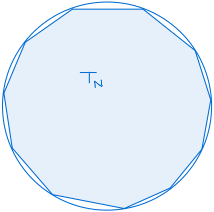

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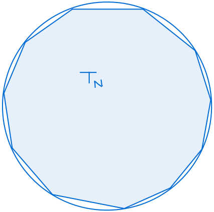

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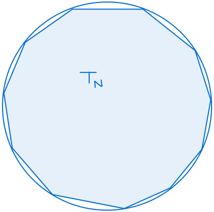

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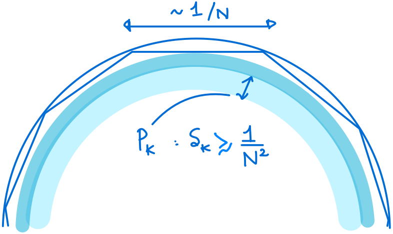

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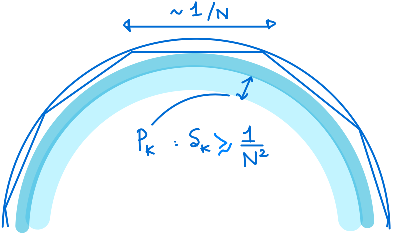

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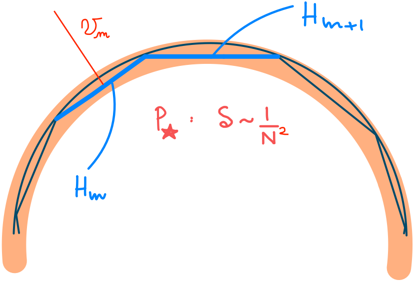

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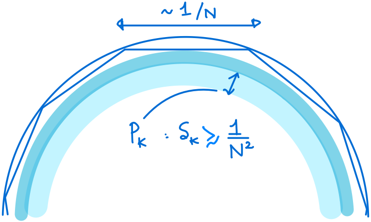

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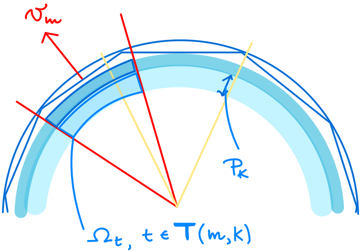

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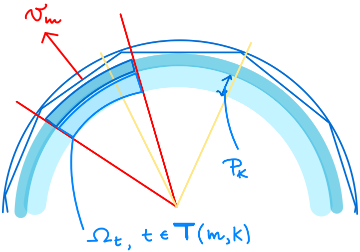
3. (SFE) $\left\| \left(\sum_{m,k} |S_m P_k f|^2 \right)^{\frac{1}{2}} \right\|_4 \lesssim (\log N)^{\frac{1}{4}} \|f\|_{4,1}$ (short side of R_t fixed)

Radial multipliers term

$$T_N f = T_0 f + \left(\sum_{1 \lesssim 2^k \lesssim N^2} P_k f \right) + T_N P_\star f$$



1. (L^4-reverse) $\|P_k f\|_4 \lesssim \left\| \left(\sum_{m=1}^{N} |S_m P_k f|^2 \right)^{\frac12} \right\|_4$
2. (triv.) $\left\| \sum_{1 \lesssim 2^k \lesssim N^2} P_k f \right\|_4 \lesssim (\log N)^{\frac34} \left\| \left(\sum_{m,k} |S_m P_k f|^2 \right)^{\frac12} \right\|_4$
3. (SFE) $\left\| \left(\sum_{m,k} |S_m P_k f|^2 \right)^{\frac12} \right\|_4 \lesssim (\log N)^{\frac14} \|f\|_{4,1}$ (short side of R_t fixed)

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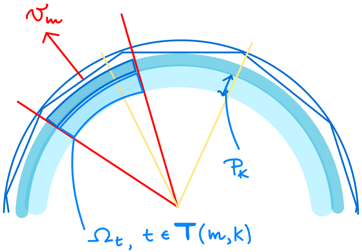

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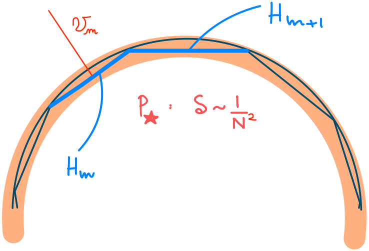

$$\|T_N P_\star f\|_p$$

$$\lesssim (\log N)^{\alpha(p)} \left\| \left(\sum_{m=1}^{N} |S_m T_N P_\star f|^2 \right)^{\frac{1}{2}} \right\|_p = (\log N)^{\alpha(p)} \left\| \left(\sum_{m=1}^{N} |H_m H_{m+1} S_m P_\star f|^2 \right)^{\frac{1}{2}} \right\|_p$$

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Outer layer term

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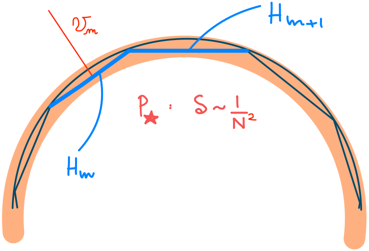

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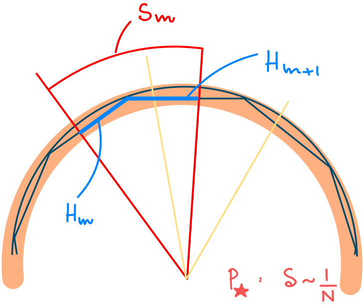

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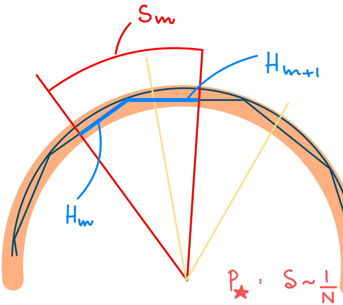

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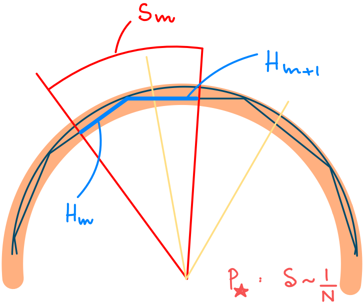

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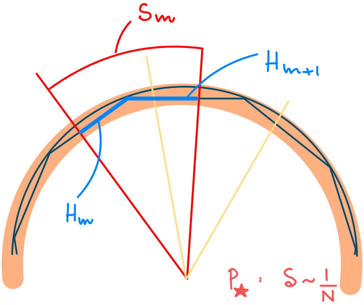

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THANK YOU FOR YOUR ATTENTION!

Work being reported

with Ioannis Parissis

- *Maximal subspace averages*, arXiv:2107.0210
- *Maximal directional operators along algebraic varieties*, Amer. J. Math. 143, no. 5 (2021), pp. 1463–1503

with N. Accomazzo, P. Hagelstein, I. Parissis, L. Roncal

- *Directional square functions*, arXiv:2004.06509, to appear Analysis&PDE