Jones factorization and Rubio de Francia extrapolation for matrix weights

David V. Cruz-Uribe, OFS

University of Alabama

11th International Conference on
Harmonic Analysis and Partial Differential Equations
El Escorial

Acknowledgments

Joint work with Marcin Bownik, University of Oregon

Hace mucho tiempo en una conferencia muy, muy lejana...

Muckenhoupt A_p weights

For $1 < p < \infty$, $w \in A_p$ if

$$[w]_{A_p} = \sup_Q \fint_Q w\,dx \left(\fint_Q w^{1-p'}\,dx\right)^{p-1} < \infty.$$

When $p = 1$, $w \in A_1$ if

$$[w]_{A_1} = \sup_Q \operatorname*{ess\,sup}_{x \in Q} w(x)^{-1} \fint_Q w\,dx < \infty.$$

Singular integrals

Theorem (The A_2 conjecture)

For $1 < p < \infty$ if T is an SIO and $w \in A_p$, then

$$\left(\int_{\mathbb{R}^n} |Tf|^p w\,dx\right)^{\frac{1}{p}} \leqslant C\left(\int_{\mathbb{R}^n} |f|^p w\,dx\right)^{\frac{1}{p}},$$

where $C = C(p,n,T)[w]_{A_p}^{\max\{1,\frac{1}{p-1}\}}$.

Singular integrals

Theorem (The A_2 conjecture)

For $1 < p < \infty$ if T is an SIO and $w \in A_p$, then

$$\left(\int_{\mathbb{R}^n} |Tf|^p w\, dx \right)^{\frac{1}{p}} \leqslant C \left(\int_{\mathbb{R}^n} |f|^p w\, dx \right)^{\frac{1}{p}},$$

where $C = C(p,n,T)[w]_{A_p}^{\max\{1,\frac{1}{p-1}\}}$.

Reduction to $p = 2$

Theorem (Rubio de Francia extrapolation)

Suppose that for some p_0, $1 \leqslant p_0 < \infty$, and every $w_0 \in A_{p_0}$,

$$\left(\int_{\mathbb{R}^n} |Tf|^{p_0} w_0\, dx\right)^{\frac{1}{p_0}} \leqslant N_{p_0}([w_0]_{p_0}) \left(\int_{\mathbb{R}^n} |f|^{p_0} w_0\, dx\right)^{\frac{1}{p_0}}.$$

Then for every $1 < p < \infty$ and $w \in A_p$,

$$\left(\int_{\mathbb{R}^n} |Tf|^{p} w\, dx\right)^{\frac{1}{p}} \leqslant N_{p}([w]_{p}) \left(\int_{\mathbb{R}^n} |f|^{p} w\, dx\right)^{\frac{1}{p}}.$$

Reduction to $p = 2$

Theorem (Rubio de Francia extrapolation)

Suppose that for some p_0, $1 \leqslant p_0 < \infty$, and every $w_0 \in A_{p_0}$,

$$\left(\int_{\mathbb{R}^n} |Tf|^{p_0} w_0 \, dx \right)^{\frac{1}{p_0}} \leqslant N_{p_0}([w_0]_{p_0}) \left(\int_{\mathbb{R}^n} |f|^{p_0} w_0 \, dx \right)^{\frac{1}{p_0}}.$$

Then for every $1 < p < \infty$ and $w \in A_p$,

$$\left(\int_{\mathbb{R}^n} |Tf|^{p} w \, dx \right)^{\frac{1}{p}} \leqslant N_{p}([w]_{p}) \left(\int_{\mathbb{R}^n} |f|^{p} w \, dx \right)^{\frac{1}{p}}.$$

Reduction to $p = 2$

Theorem (Rubio de Francia extrapolation)

Suppose that for some p_0, $1 \leqslant p_0 < \infty$, and every $w_0 \in A_{p_0}$,

$$\left(\int_{\mathbb{R}^n} |Tf|^{p_0} w_0\,dx\right)^{\frac{1}{p_0}} \leqslant N_{p_0}([w_0]_{p_0})\left(\int_{\mathbb{R}^n} |f|^{p_0} w_0\,dx\right)^{\frac{1}{p_0}}.$$

Then for every $1 < p < \infty$ and $w \in A_p$,

$$\left(\int_{\mathbb{R}^n} |Tf|^{p} w\,dx\right)^{\frac{1}{p}} \leqslant N_{p}([w]_{p})\left(\int_{\mathbb{R}^n} |f|^{p} w\,dx\right)^{\frac{1}{p}}.$$

Three important tools

- $w \in A_p$ if and only if $w^{1-p'} \in A_{p'}$
- $M : L^p(w) \to L^p(w)$ and $\|M\|_{L^p(w)} \leqslant C[w]_{A_p}^{\frac{1}{p-1}}$.
- Jones factorization: given $1 < p < \infty$, then $w \in A_p$ if and only if there exist $w_0, w_1 \in A_1$ such that

$$w = w_0 w_1^{1-p}.$$

Three important tools

- $w \in A_p$ if and only if $w^{1-p'} \in A_{p'}$
- $M : L^p(w) \to L^p(w)$ and $\|M\|_{L^p(w)} \leqslant C[w]_{A_p}^{\frac{1}{p-1}}$.
- Jones factorization: given $1 < p < \infty$, then $w \in A_p$ if and only if there exist $w_0,\ w_1 \in A_1$ such that

$$w = w_0 w_1^{1-p}.$$

Three important tools

- $w \in A_p$ if and only if $w^{1-p'} \in A_{p'}$
- $M : L^p(w) \to L^p(w)$ and $\|M\|_{L^p(w)} \leqslant C[w]_{A_p}^{\frac{1}{p-1}}$.
- Jones factorization: given $1 < p < \infty$, then $w \in A_p$ if and only if there exist $w_0, w_1 \in A_1$ such that

$$w = w_0 w_1^{1-p}.$$

Three important tools

- $w \in A_p$ if and only if $w^{1-p'} \in A_{p'}$
- $M : L^p(w) \to L^p(w)$ and $\|M\|_{L^p(w)} \leqslant C[w]_{A_p}^{\frac{1}{p-1}}$.
- Jones factorization: given $1 < p < \infty$, then $w \in A_p$ if and only if there exist $w_0,\ w_1 \in A_1$ such that

$$w = w_0 w_1^{1-p}.$$

The Rubio de Francia iteration algorithm

Given $f : \mathbb{R}^n \to \mathbb{R}$ and $w \in A_p$ define

$$\mathcal{R}f(x) = \sum_{k=0}^{\infty} \frac{M^k f(x)}{2^k \|M\|_{L^p(w)}^k}.$$

- $f(x) \leqslant \mathcal{R}f(x)$
- $\|\mathcal{R}f\|_{L^p(w)} \leqslant 2\|f\|_{L^p(w)}$
- $\mathcal{R}f \in A_1$, $[\mathcal{R}f]_{A_1} \leqslant 2\|M\|_{L^p(w)}$.

The Rubio de Francia iteration algorithm

Given $f : \mathbb{R}^n \to \mathbb{R}$ and $w \in A_p$ define

$$\mathcal{R}f(x) = \sum_{k=0}^{\infty} \frac{M^k f(x)}{2^k \|M\|_{L^p(w)}^k}.$$

- $f(x) \leqslant \mathcal{R}f(x)$
- $\|\mathcal{R}f\|_{L^p(w)} \leqslant 2\|f\|_{L^p(w)}$
- $\mathcal{R}f \in A_1$, $[\mathcal{R}f]_{A_1} \leqslant 2\|M\|_{L^p(w)}$.

The Rubio de Francia iteration algorithm

Given $f : \mathbb{R}^n \to \mathbb{R}$ and $w \in A_p$ define

$$\mathcal{R}f(x) = \sum_{k=0}^{\infty} \frac{M^k f(x)}{2^k \|M\|_{L^p(w)}^k}.$$

- $f(x) \leqslant \mathcal{R}f(x)$
- $\|\mathcal{R}f\|_{L^p(w)} \leqslant 2\|f\|_{L^p(w)}$
- $\mathcal{R}f \in A_1$, $[\mathcal{R}f]_{A_1} \leqslant 2\|M\|_{L^p(w)}$.

The Rubio de Francia iteration algorithm

Given $f : \mathbb{R}^n \to \mathbb{R}$ and $w \in A_p$ define

$$\mathcal{R}f(x) = \sum_{k=0}^{\infty} \frac{M^k f(x)}{2^k \|M\|_{L^p(w)}^k}.$$

- $f(x) \leqslant \mathcal{R}f(x)$
- $\|\mathcal{R}f\|_{L^p(w)} \leqslant 2\|f\|_{L^p(w)}$
- $\mathcal{R}f \in A_1$, $[\mathcal{R}f]_{A_1} \leqslant 2\|M\|_{L^p(w)}$.

An old and ongoing problem

Extend the theory of
Muckenhoupt A_p weights
to matrix weights

Some notation

For $1 \leqslant p < \infty$, define $L^p(\mathbb{R}^n, \mathbb{R}^d)$ to be vector-valued functions

$$\mathbf{f} = (f_1, \ldots, f_d)$$

such that

$$\|\mathbf{f}\|_p = \left(\int_{\mathbb{R}^n} |\mathbf{f}(x)|^p \, dx \right)^{1/p} < \infty.$$

Matrix weights

Let $\mathcal{S}_d$ be $d \times d$, self-adjoint, positive semi-definite matrices.

A matrix weight is a measurable function

$$W : \mathbb{R}^n \to \mathcal{S}_d.$$

$$|W(x)|_{\mathrm{op}} = \sup_{|\xi|=1} |W(x)\xi|.$$

Matrix A_p

For $1 < p < \infty$, $W \in A_p$ if

$$[W]_{\mathbf{A}_p} = \sup_Q \fint_Q \left(\fint_Q |W(x)^{\frac{1}{p}} W(y)^{-\frac{1}{p}}|_{\text{op}}^{p'} \, dy \right)^{\frac{p}{p'}} dx < \infty.$$

$W \in A_1$ if

$$[W]_{\mathbf{A}_1} = \operatorname*{ess\,sup}_{x \in \mathbb{R}^d} \sup_{Q \ni x} \fint_Q |W(y) W^{-1}(x)|_{\text{op}} \, dy < \infty.$$

$$\|\mathbf{f}\|_{L^p(W)} = \left(\int_{\mathbb{R}^n} |W^{\frac{1}{p}}(x)\mathbf{f}(x)|^p \, dx \right)^{\frac{1}{p}} < \infty.$$

Matrix A_p

For $1 < p < \infty$, $W \in A_p$ if

$$[W]_{\mathbf{A}_p} = \sup_Q \fint_Q \left(\fint_Q |W(x)^{\frac{1}{p}} W(y)^{-\frac{1}{p}}|_{\text{op}}^{p'} \, dy \right)^{\frac{p}{p'}} dx < \infty.$$

$W \in A_1$ if

$$[W]_{\mathbf{A}_1} = \operatorname*{ess\,sup}_{x \in \mathbb{R}^d} \sup_{Q \ni x} \fint_Q |W(y) W^{-1}(x)|_{\text{op}} \, dy < \infty.$$

$$\|\mathbf{f}\|_{L^p(W)} = \left(\int_{\mathbb{R}^n} |W^{\frac{1}{p}}(x) \mathbf{f}(x)|^p \, dx \right)^{\frac{1}{p}} < \infty.$$

Matrix A_p

For $1 < p < \infty$, $W \in A_p$ if

$$[W]_{\mathbf{A}_p} = \sup_Q \fint_Q \left(\fint_Q |W(x)^{\frac{1}{p}} W(y)^{-\frac{1}{p}}|_{\mathrm{op}}^{p'} \, dy \right)^{\frac{p}{p'}} dx < \infty.$$

$W \in A_1$ if

$$[W]_{\mathbf{A}_1} = \operatorname*{ess\,sup}_{x \in \mathbb{R}^d} \sup_{Q \ni x} \fint_Q |W(y) W^{-1}(x)|_{\mathrm{op}} \, dy < \infty.$$

$$\|\mathbf{f}\|_{L^p(W)} = \left(\int_{\mathbb{R}^n} |W^{\frac{1}{p}}(x) \mathbf{f}(x)|^p \, dx \right)^{\frac{1}{p}} < \infty.$$

Matrix weights and SIOs

Theorem (NTV, CG, NPTV, DCU-JI-KM)

If $1 < p < \infty$, $W \in A_p$, *and* T *an SIO, then*

$$T : L^p(W) \to L^p(W)$$

$$\|T\|_{L^p(W)} \leqslant [W]_{A_p}^{1+\frac{1}{p-1}-\frac{1}{p}}.$$

Matrix weights and SIOs

Theorem (NTV, CG, NPTV, DCU-JI-KM)

If $1 < p < \infty$, $W \in A_p$, *and* T *an SIO, then*

$$T : L^p(W) \to L^p(W)$$

$$\|T\|_{L^p(W)} \lesssim [W]_{A_p}^{1+\frac{1}{p-1}-\frac{1}{p}}.$$

Approach I: maximal operator

For $1 \leqslant p < \infty$ define

$$M_W\mathbf{f}(x) = \sup_Q \fint_Q |W(x)^{\frac{1}{p}} W(y)^{-\frac{1}{p}}\mathbf{f}(y)|\,dy \cdot \chi_Q(x).$$

Theorem (CG (2003), JI-KM (2019))

If $1 < p < \infty$ and $W \in A_p$, then

$$M_W : L^p(\mathbb{R}^n, \mathbb{R}^d) \to L^p(\mathbb{R}^n)$$

$$\|M_W\|_{L^p(\mathbb{R}^n,\mathbb{R}^d)} \leqslant [W]_{A_p}^{\frac{1}{p-1}}.$$

Approach I: maximal operator

For $1 \leqslant p < \infty$ define

$$M_W \mathbf{f}(x) = \sup_Q \fint_Q |W(x)^{\frac{1}{p}} W(y)^{-\frac{1}{p}} \mathbf{f}(y)| \, dy \cdot \chi_Q(x).$$

Theorem (CG (2003), JI-KM (2019))

If $1 < p < \infty$ and $W \in A_p$, then

$$M_W : L^p(\mathbb{R}^n, \mathbb{R}^d) \to L^p(\mathbb{R}^n)$$

$$\|M_W\|_{L^p(\mathbb{R}^n,\mathbb{R}^d)} \leqslant [W]_{A_p}^{\frac{1}{p-1}}.$$

Approach I: maximal operator

For $1 \leqslant p < \infty$ define

$$M_W\mathbf{f}(x) = \sup_Q \fint_Q |W(x)^{\frac{1}{p}} W(y)^{-\frac{1}{p}}\mathbf{f}(y)|\,dy \cdot \chi_Q(x).$$

Theorem (CG (2003), JI-KM (2019))

If $1 < p < \infty$ and $W \in A_p$, then

$$M_W : L^p(\mathbb{R}^n, \mathbb{R}^d) \to L^p(\mathbb{R}^n)$$

$$\|M_W\|_{L^p(\mathbb{R}^n,\mathbb{R}^d)} \leqslant [W]_{A_p}^{\frac{1}{p-1}}.$$

Approach II: Convex body sparse domination

Theorem (NPTV (2017))

If T is an SIO, and $\mathbf{f} \in L^\infty_c(\mathbb{R}^n, \mathbb{R}^d)$, then there exists a sparse family $\mathcal{S}$ such that

$$T\mathbf{f}(x) \in \sum_{Q \in \mathcal{S}} \langle\langle \mathbf{f} \rangle\rangle_Q \cdot \chi_Q(x)$$

where

$$\langle\langle \mathbf{f} \rangle\rangle_Q = \left\{ \fint_Q \phi(x)\mathbf{f}(x)\,dx : \phi \in L^\infty(Q), \|\phi\|_\infty \leqslant 1 \right\}.$$

Three questions

- Is the A_2 conjecture true for matrix weights?
- Does Rubio de Francia extrapolation hold for matrix weights?
- Does the Jones factorization theorem hold for matrix weights?

Three questions

- Is the A_2 conjecture true for matrix weights?
- Does Rubio de Francia extrapolation hold for matrix weights?
- Does the Jones factorization theorem hold for matrix weights?

Three questions

- Is the A_2 conjecture true for matrix weights?
- Does Rubio de Francia extrapolation hold for matrix weights?
- Does the Jones factorization theorem hold for matrix weights?

Three questions

- Is the A_2 conjecture true for matrix weights?
- Does Rubio de Francia extrapolation hold for matrix weights?
- Does the Jones factorization theorem hold for matrix weights?

How hard can this be?

Actually, the whole theory of scalar (A_p)-weights can be transferred to the matrix case except two results...the Peter Jones factorization and the Rubio-de-Francia extrapolation theory. But today (May 1, 1996) we do not know what the analogues of these two things are in high dimensions.

– F. Nazarov, S. Treil, The Hunt for the Bellman function.

Rubio de Francia extrapolation

Theorem (MB-DCU (2021))

Suppose that for some p_0, $1 \leqslant p_0 < \infty$, and every $W_0 \in A_{p_0}$,

$$\left(\int_{\mathbb{R}^n} |W_0^{\frac{1}{p_0}} T\mathbf{f}|^{p_0}\,dx\right)^{\frac{1}{p_0}} \leqslant N_{p_0}([W_0]_{p_0})\left(\int_{\mathbb{R}^n} |W_0^{\frac{1}{p_0}} \mathbf{f}|^{p_0}\,dx\right)^{\frac{1}{p_0}}.$$

Then for every $1 < p < \infty$ and $W \in A_p$,

$$\left(\int_{\mathbb{R}^n} |W^{\frac{1}{p}} T\mathbf{f}|^{p}\,dx\right)^{\frac{1}{p}} \leqslant N_{p}([W]_{p})\left(\int_{\mathbb{R}^n} |W^{\frac{1}{p}} f|^{p}\,dx\right)^{\frac{1}{p}}.$$

Rubio de Francia extrapolation

Theorem (MB-DCU (2021))

Suppose that for some p_0, $1 \leqslant p_0 < \infty$, and every $W_0 \in A_{p_0}$,

$$\left(\int_{\mathbb{R}^n} |W_0^{\frac{1}{p_0}} T\mathbf{f}|^{p_0}\,dx \right)^{\frac{1}{p_0}} \leqslant N_{p_0}([W_0]_{p_0}) \left(\int_{\mathbb{R}^n} |W_0^{\frac{1}{p_0}} \mathbf{f}|^{p_0}\,dx \right)^{\frac{1}{p_0}}.$$

Then for every $1 < p < \infty$ and $W \in A_p$,

$$\left(\int_{\mathbb{R}^n} |W^{\frac{1}{p}} T\mathbf{f}|^{p}\,dx \right)^{\frac{1}{p}} \leqslant N_p([W]_p) \left(\int_{\mathbb{R}^n} |W^{\frac{1}{p}} f|^{p}\,dx \right)^{\frac{1}{p}}.$$

Jones factorization

Theorem (MB-DCU (2021))

Given $1 < p < \infty$, then $W \in A_p$ if and only if

- *there exist $W_0, W_1 \in A_1$ commuting matrices*
- *$W = W_0 W_1^{1-p}$.*

Jones factorization

Theorem (MB-DCU (2021))

Given $1 < p < \infty$*, then* $W \in A_p$ *if and only if*

- *there exist* $W_0,\ W_1 \in A_1$ *commuting matrices*
- $W = W_0 W_1^{1-p}$.

Jones factorization

Theorem (MB-DCU (2021))

Given $1 < p < \infty$, then $W \in A_p$ if and only if

- *there exist $W_0,\ W_1 \in A_1$ commuting matrices*
- $W = W_0 W_1^{1-p}$.

(Very short) Outline of the proofs

- Follow the scalar proofs
- Replace scalar tools with matrix equivalents

Biggest conceptual obstacle: how to replace the maximal operator?

(Very short) Outline of the proofs

- Follow the scalar proofs
- Replace scalar tools with matrix equivalents

Biggest conceptual obstacle: how to replace the maximal operator?

(Very short) Outline of the proofs

- Follow the scalar proofs
- Replace scalar tools with matrix equivalents

Biggest conceptual obstacle: how to replace the maximal operator?

(Very short) Outline of the proofs

- Follow the scalar proofs
- Replace scalar tools with matrix equivalents

Biggest conceptual obstacle: how to replace the maximal operator?

Convex sets

Let $\mathcal{K}$ bet the collection of all convex sets K in $\mathbb{R}^d$ that are:

- closed & bounded
- symmetric: $x \in K \Rightarrow -x \in K$
- absorbing: $0 \in \mathrm{Int}(K)$

Every such K is the unit ball of a norm on $\mathbb{R}^n$.

John Ellipsoid: given K there exists a unique ellipsoid E such that $E \subset K \subset \sqrt{d}E$.

Convex sets

Let $\mathcal{K}$ bet the collection of all convex sets K in $\mathbb{R}^d$ that are:

- closed & bounded
- symmetric: $x \in K \Rightarrow -x \in K$
- absorbing: $0 \in \mathrm{Int}(K)$

Every such K is the unit ball of a norm on $\mathbb{R}^n$.

John Ellipsoid: given K there exists a unique ellipsoid E such that $E \subset K \subset \sqrt{d}E$.

Convex sets

Let $\mathcal{K}$ bet the collection of all convex sets K in $\mathbb{R}^d$ that are:

- closed & bounded
- symmetric: $x \in K \Rightarrow -x \in K$
- absorbing: $0 \in \text{Int}(K)$

Every such K is the unit ball of a norm on $\mathbb{R}^n$.

John Ellipsoid: given K there exists a unique ellipsoid E such that $E \subset K \subset \sqrt{d}E$.

Convex sets

Let $\mathcal{K}$ bet the collection of all convex sets K in $\mathbb{R}^d$ that are:

- closed & bounded
- symmetric: $x \in K \Rightarrow -x \in K$
- absorbing: $0 \in \text{Int}(K)$

Every such K is the unit ball of a norm on $\mathbb{R}^n$.

John Ellipsoid: given K there exists a unique ellipsoid E such that $E \subset K \subset \sqrt{d}E$.

Convex sets

Let $\mathcal{K}$ bet the collection of all convex sets K in $\mathbb{R}^d$ that are:

- closed & bounded
- symmetric: $x \in K \ \Rightarrow -x \in K$
- absorbing: $0 \in \mathrm{Int}(K)$

Every such K is the unit ball of a norm on $\mathbb{R}^n$.

John Ellipsoid: given K there exists a unique ellipsoid E such that $E \subset K \subset \sqrt{d}E$.

Convex sets

Let $\mathcal{K}$ bet the collection of all convex sets K in $\mathbb{R}^d$ that are:

- closed & bounded
- symmetric: $x \in K \ \Rightarrow -x \in K$
- absorbing: $0 \in \mathrm{Int}(K)$

Every such K is the unit ball of a norm on $\mathbb{R}^n$.

John Ellipsoid: given K there exists a unique ellipsoid E such that $E \subset K \subset \sqrt{d}E$.

Convex-set valued functions

We consider measurable functions $F : \mathbb{R}^n \to \mathcal{K}$.

Given $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^d$, define

$$F_{\mathbf{f}}(x) = \overline{\operatorname{conv}}\{\mathbf{f}(x), -\mathbf{f}(x)\}.$$

Define $L^p_{\mathcal{K}}(\mathbb{R}^n, W)$

$$\|F\|_{L^p_{\mathcal{K}}(\mathbb{R}^n,W)} = \left(\int_{\mathbb{R}^n} |W(x)^{\frac{1}{p}} F(x)|^p \, dx\right)^{\frac{1}{p}}.$$

Convex-set valued functions

We consider measurable functions $F : \mathbb{R}^n \to \mathcal{K}$.

Given $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^d$, define

$$F_{\mathbf{f}}(x) = \overline{\mathrm{conv}}\{\mathbf{f}(x), -\mathbf{f}(x)\}.$$

Define $L^p_{\mathcal{K}}(\mathbb{R}^n, W)$

$$\|F\|_{L^p_{\mathcal{K}}(\mathbb{R}^n,W)} = \left(\int_{\mathbb{R}^n} |W(x)^{\frac{1}{p}} F(x)|^p \, dx\right)^{\frac{1}{p}}.$$

Convex-set valued functions

We consider measurable functions $F : \mathbb{R}^n \to \mathcal{K}$.

Given $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^d$, define

$$F_{\mathbf{f}}(x) = \overline{\text{conv}}\{\mathbf{f}(x), -\mathbf{f}(x)\}.$$

Define $L^p_{\mathcal{K}}(\mathbb{R}^n, W)$

$$\|F\|_{L^p_{\mathcal{K}}(\mathbb{R}^n,W)} = \left(\int_{\mathbb{R}^n} |W(x)^{\frac{1}{p}} F(x)|^p \, dx \right)^{\frac{1}{p}}.$$

Aumann integral

Given $F : \mathbb{R}^n \to \mathcal{K}$ and $\Omega \subset \mathbb{R}^n$

$S^1(\Omega, F) = \{\mathbf{f} \in L^1(\Omega) : \mathbf{f}(x) \in F(x)\}$

$$\int_\Omega F(x)\,dx = \left\{ \int_\Omega \mathbf{f}(x)\,dx : \mathbf{f} \in S^1(\Omega, F) \right\}.$$

$$\langle\langle \mathbf{f} \rangle\rangle_Q = \fint_Q F_{\mathbf{f}}(x)\,dx.$$

Aumann integral

Given $F : \mathbb{R}^n \to \mathcal{K}$ and $\Omega \subset \mathbb{R}^n$

$S^1(\Omega, F) = \{\mathbf{f} \in L^1(\Omega) : \mathbf{f}(x) \in F(x)\}$

$$\int_\Omega F(x)\,dx = \left\{ \int_\Omega \mathbf{f}(x)\,dx : \mathbf{f} \in S^1(\Omega, F) \right\}.$$

$$\langle\langle \mathbf{f} \rangle\rangle_Q = \fint_Q F_{\mathbf{f}}(x)\,dx.$$

Aumann integral

Given $F : \mathbb{R}^n \to \mathcal{K}$ and $\Omega \subset \mathbb{R}^n$

$S^1(\Omega, F) = \{\mathbf{f} \in L^1(\Omega) : \mathbf{f}(x) \in F(x)\}$

$$\int_\Omega F(x)\,dx = \left\{ \int_\Omega \mathbf{f}(x)\,dx : \mathbf{f} \in S^1(\Omega, F) \right\}.$$

$$\langle\langle \mathbf{f} \rangle\rangle_Q = \fint_Q F_{\mathbf{f}}(x)\,dx.$$

Aumann integral

Given $F : \mathbb{R}^n \to \mathcal{K}$ and $\Omega \subset \mathbb{R}^n$

$S^1(\Omega, F) = \{\mathbf{f} \in L^1(\Omega) : \mathbf{f}(x) \in F(x)\}$

$$\int_\Omega F(x)\,dx = \left\{ \int_\Omega \mathbf{f}(x)\,dx : \mathbf{f} \in S^1(\Omega, F) \right\}.$$

$$\langle\langle \mathbf{f} \rangle\rangle_Q = \fint_Q F_{\mathbf{f}}(x)\,dx.$$

Convex-set valued maximal operator

Given $F : \mathbb{R}^n \to \mathcal{K}$, define

$$MF(x) = \overline{\mathrm{conv}} \left(\bigcup_Q \fint_Q F(y)\,dy \cdot \chi_Q(x) \right).$$

- $F(x) \subset MF(x)$
- $M(F+G)(x) \subset MF(x) + MG(x)$
- $M(\alpha F)(x) = \alpha MF(x).$
- $M : L^p_{\mathcal{K}}(\mathbb{R}^n, |\cdot|) \to L^p_{\mathcal{K}}(\mathbb{R}^n, |\cdot|)$

Convex-set valued maximal operator

Given $F : \mathbb{R}^n \to \mathcal{K}$, define

$$MF(x) = \overline{\mathrm{conv}}\left(\bigcup_{Q} \fint_{Q} F(y)\,dy \cdot \chi_Q(x) \right).$$

- $F(x) \subset MF(x)$
- $M(F+G)(x) \subset MF(x) + MG(x)$
- $M(\alpha F)(x) = \alpha MF(x)$.
- $M : L^p_{\mathcal{K}}(\mathbb{R}^n, |\cdot|) \to L^p_{\mathcal{K}}(\mathbb{R}^n, |\cdot|)$

Convex-set valued maximal operator

Given $F : \mathbb{R}^n \to \mathcal{K}$, define

$$MF(x) = \overline{\text{conv}} \left(\bigcup_Q \fint_Q F(y)\, dy \cdot \chi_Q(x) \right).$$

- $F(x) \subset MF(x)$
- $M(F+G)(x) \subset MF(x) + MG(x)$
- $M(\alpha F)(x) = \alpha MF(x)$.
- $M : L^p_{\mathcal{K}}(\mathbb{R}^n, |\cdot|) \to L^p_{\mathcal{K}}(\mathbb{R}^n, |\cdot|)$

Convex-set valued maximal operator

Given $F : \mathbb{R}^n \to \mathcal{K}$, define

$$MF(x) = \overline{\text{conv}} \left(\bigcup_Q \fint_Q F(y)\, dy \cdot \chi_Q(x) \right).$$

- $F(x) \subset MF(x)$
- $M(F+G)(x) \subset MF(x) + MG(x)$
- $M(\alpha F)(x) = \alpha MF(x)$.
- $M : L^p_{\mathcal{K}}(\mathbb{R}^n, |\cdot|) \to L^p_{\mathcal{K}}(\mathbb{R}^n, |\cdot|)$

Convex-set valued maximal operator

Given $F : \mathbb{R}^n \to \mathcal{K}$, define

$$MF(x) = \overline{\mathrm{conv}}\left(\bigcup_Q \fint_Q F(y)\,dy \cdot \chi_Q(x) \right).$$

- $F(x) \subset MF(x)$
- $M(F+G)(x) \subset MF(x) + MG(x)$
- $M(\alpha F)(x) = \alpha MF(x)$.
- $M : L^p_{\mathcal{K}}(\mathbb{R}^n, |\cdot|) \to L^p_{\mathcal{K}}(\mathbb{R}^n, |\cdot|)$

Weighted norm inequalities

Theorem

Given $1 < p < \infty$, $W \in A_p$,

$$M : L^p_{\mathcal{K}}(\mathbb{R}^n, W) \to L^p_{\mathcal{K}}(\mathbb{R}^n, W)$$

$$\|M\|_{L^p_{\mathcal{K}}(\mathbb{R}^n,W)} \leqslant C(n,p)[W]_{A_p}^{\frac{1}{p-1}}.$$

Weighted norm inequalities

Theorem

Given $1 < p < \infty$, $W \in A_p$,

$$M : L^p_{\mathcal{K}}(\mathbb{R}^n, W) \to L^p_{\mathcal{K}}(\mathbb{R}^n, W)$$

$$\|M\|_{L^p_{\mathcal{K}}(\mathbb{R}^n,W)} \leqslant C(n,p)[W]_{A_p}^{\frac{1}{p-1}}.$$

Convex body A_1

Given $F : \mathbb{R}^n \to \mathcal{K}$, define $F \in A_1^{\mathcal{K}}$ if

$$MF(x) \subset CF(x)$$

Theorem

Given a matrix weight W, $W \in A_1$ if and only if

$$F = W\mathbf{B} \in \mathbf{A}_1^{\mathcal{K}}$$

Convex body A_1

Given $F : \mathbb{R}^n \to \mathcal{K}$, define $F \in A_1^{\mathcal{K}}$ if

$$MF(x) \subset CF(x)$$

Theorem

Given a matrix weight W, $W \in A_1$ if and only if

$$F = W\mathbf{B} \in \mathbf{A}_1^{\mathcal{K}}$$

Rubio de Francia Iteration operator

Given $F : \mathbb{R}^n \to \mathcal{K}$ and $W \in A_p$ define

$$\mathcal{R}F(x) = \sum_{k=0}^{\infty} \frac{M^k F(x)}{2^k \|M\|^k_{L^p_{\mathcal{K}}(\mathbb{R}^n, W)}}.$$

- $F(x) \subset \mathcal{R}F(x)$
- $\|\mathcal{R}F\|_{L^p_{\mathcal{K}}(\mathbb{R}^n,W)} \leqslant 2\|F\|_{L^p_{\mathcal{K}}(\mathbb{R}^n,W)}$
- $\mathcal{R}F \in A_1^{\mathcal{K}}$, $[\mathcal{R}F]_{A_1^{\mathcal{K}}} \leqslant 2\|M\|_{L^p_{\mathcal{K}}(\mathbb{R}^n,W)}$.

Rubio de Francia Iteration operator

Given $F : \mathbb{R}^n \to \mathcal{K}$ and $W \in A_p$ define

$$\mathcal{R}F(x) = \sum_{k=0}^{\infty} \frac{M^k F(x)}{2^k \|M\|^k_{L^p_{\mathcal{K}}(\mathbb{R}^n, W)}}.$$

- $F(x) \subset \mathcal{R}F(x)$
- $\|\mathcal{R}F\|_{L^p_{\mathcal{K}}(\mathbb{R}^n, W)} \leqslant 2\|F\|_{L^p_{\mathcal{K}}(\mathbb{R}^n, W)}$
- $\mathcal{R}F \in A_1^{\mathcal{K}}$, $[\mathcal{R}F]_{A_1^{\mathcal{K}}} \leqslant 2\|M\|_{L^p_{\mathcal{K}}(\mathbb{R}^n, W)}$.

Rubio de Francia Iteration operator

Given $F : \mathbb{R}^n \to \mathcal{K}$ and $W \in A_p$ define

$$\mathcal{R}F(x) = \sum_{k=0}^{\infty} \frac{M^k F(x)}{2^k \|M\|^k_{L^p_{\mathcal{K}}(\mathbb{R}^n,W)}}.$$

- $F(x) \subset \mathcal{R}F(x)$
- $\|\mathcal{R}F\|_{L^p_{\mathcal{K}}(\mathbb{R}^n,W)} \leqslant 2\|F\|_{L^p_{\mathcal{K}}(\mathbb{R}^n,W)}$
- $\mathcal{R}F \in A_1^{\mathcal{K}}$, $[\mathcal{R}F]_{A_1^{\mathcal{K}}} \leqslant 2\|M\|_{L^p_{\mathcal{K}}(\mathbb{R}^n,W)}$.

Rubio de Francia Iteration operator

Given $F : \mathbb{R}^n \to \mathcal{K}$ and $W \in A_p$ define

$$\mathcal{R}F(x) = \sum_{k=0}^{\infty} \frac{M^k F(x)}{2^k \|M\|^k_{L^p_{\mathcal{K}}(\mathbb{R}^n, W)}}.$$

- $F(x) \subset \mathcal{R}F(x)$
- $\|\mathcal{R}F\|_{L^p_{\mathcal{K}}(\mathbb{R}^n, W)} \leqslant 2\|F\|_{L^p_{\mathcal{K}}(\mathbb{R}^n, W)}$
- $\mathcal{R}F \in A_1^{\mathcal{K}}$, $[\mathcal{R}F]_{A_1^{\mathcal{K}}} \leqslant 2\|M\|_{L^p_{\mathcal{K}}(\mathbb{R}^n, W)}$.

Proof of Jones factorization

- Factorization follows scalar proof using a modified Rubio de Francia iteration operator
- Reverse factorization is **very** delicate and uses an interpolation argument and the equivalent definition of matrix A_p in terms of norms.

Proof of Jones factorization

- Factorization follows scalar proof using a modified Rubio de Francia iteration operator
- Reverse factorization is **very** delicate and uses an interpolation argument and the equivalent definition of matrix A_p in terms of norms.

Proof of Jones factorization

- Factorization follows scalar proof using a modified Rubio de Francia iteration operator
- Reverse factorization is **very** delicate and uses an interpolation argument and the equivalent definition of matrix A_p in terms of norms.

Proof of Extrapolation

- Again follows the scalar proof.
- Uses both reverse factorization and the iteration algorithm.
- Must pass between convex-set and ellipsoid valued functions.

Proof of Extrapolation

- Again follows the scalar proof.
- Uses both reverse factorization and the iteration algorithm.
- Must pass between convex-set and ellipsoid valued functions.

Proof of Extrapolation

- Again follows the scalar proof.
- Uses both reverse factorization and the iteration algorithm.
- Must pass between convex-set and ellipsoid valued functions.

Proof of Extrapolation

- Again follows the scalar proof.
- Uses both reverse factorization and the iteration algorithm.
- Must pass between convex-set and ellipsoid valued functions.

The A_2 conjecture

- Using our techniques, we can prove bounds for singular integrals with constant $[W]_{A_2}^2$, and think we can reduce it to $[W]_{A_2}^{3/2}$.
- Key missing tool: “universal” maximal operator not bounded in matrix case. (Counter-example Nazarov, Petermichl, Škreb, Treil, 01/22.)

The A_2 conjecture

- Using our techniques, we can prove bounds for singular integrals with constant $[W]_{A_2}^2$, and think we can reduce it to $[W]_{A_2}^{3/2}$.
- Key missing tool: “universal” maximal operator not bounded in matrix case. (Counter-example Nazarov, Petermichl, Škreb, Treil, 01/22.)

The A_2 conjecture

- Using our techniques, we can prove bounds for singular integrals with constant $[W]_{A_2}^2$, and think we can reduce it to $[W]_{A_2}^{3/2}$.
- Key missing tool: "universal" maximal operator not bounded in matrix case. (Counter-example Nazarov, Petermichl, Škreb, Treil, 01/22.)

Gracias!

Roll Tide!

New Directions

- Off diagonal and limited range extrapolation
- Matrix weighted Hardy spaces
- Extend to matrix valued A_p^+ weights
- Extend to multilinear operators

New Directions

- Off diagonal and limited range extrapolation
- Matrix weighted Hardy spaces
- Extend to matrix valued A_p^+ weights
- Extend to multilinear operators

New Directions

- Off diagonal and limited range extrapolation
- Matrix weighted Hardy spaces
- Extend to matrix valued A_p^+ weights
- Extend to multilinear operators

New Directions

- Off diagonal and limited range extrapolation
- Matrix weighted Hardy spaces
- Extend to matrix valued A_p^+ weights
- Extend to multilinear operators

New Directions

- Off diagonal and limited range extrapolation
- Matrix weighted Hardy spaces
- Extend to matrix valued A_p^+ weights
- Extend to multilinear operators