The nonlinear Brascamp–Lieb inequality

Neal Bez
Saitama University

Based on joint papers with
Jon Bennett, Stefan Buschenhenke, Michael Cowling, Taryn Flock and Sanghyuk Lee

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$$\int_{\mathbb{R}^n} \prod_{j=1}^m f_j(L_j x)^{c_j}\,\mathrm{d}x \le \mathrm{B}(\mathbf{L},\mathbf{c}) \prod_{j=1}^m \left(\int_{\mathbb{R}^{n_j}} f_j \right)^{c_j}$$

$f_j \in L^1(\mathbb{R}^{n_j}),\ f_j \ge 0$

$L_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$ linear surjections

$c_j \in [0,1]$

$(\mathbf{L},\mathbf{c})$: Brascamp–Lieb data $\quad \mathbf{L} = (L_j)_{j=1}^m,\ \mathbf{c} = (c_j)_{j=1}^m$

$\mathrm{B}(\mathbf{L},\mathbf{c}) \in [0,\infty]$: Brascamp–Lieb constant $\quad$ best constant

$$\mathrm{B}(\mathbf{L},\mathbf{c}) = \sup_{\int f_j = 1} \int_{\mathbb{R}^n} \prod_{j=1}^m f_j(L_j x)^{c_j}\,\mathrm{d}x$$

$\mathrm{B}(\mathbf{L},\mathbf{c}) < \infty \ \Rightarrow\ n = \sum_{j=1}^m c_j n_j \quad f_j \rightsquigarrow f_j(R\,\cdot)$

Characterisation of finiteness

Theorem (Bennett–Carbery–Christ–Tao)

$\mathrm{B}(\mathbf{L}, \mathbf{c}) < \infty$ if and only if

(i) $n = \sum_{j=1}^{m} c_j n_j$ (scaling)

(ii) $\dim(V) \leq \sum_{j=1}^{m} c_j \dim(L_j V)$ for all $V \leq \mathbb{R}^n$ (dimension)

Special role of gaussians

Theorem (Lieb)

$$\mathrm{B}(\mathbf{L},\mathbf{c}) = \sup_{A_j>0} \frac{\prod_{j=1}^m \det(A_j)^{c_j/2}}{\det(\sum_{j=1}^m c_j L_j^* A_j L_j)^{1/2}}$$

Note

$$\int_{\mathbb{R}^n} \prod_{j=1}^m f_j(L_j x)^{c_j}\,\mathrm{d}x = \frac{\prod_{j=1}^m \det(A_j)^{c_j/2}}{\det(\sum_{j=1}^m c_j L_j^* A_j L_j)^{1/2}}$$

where

$$f_j(x) = (\det A_j)^{\frac{1}{2}} \exp(-\pi\langle A_j x, x\rangle)$$

The Loomis–Whitney inequality

$$\int_{\mathbb{R}^n} \prod_{j=1}^{n} f_j(\Pi_j x)^{\frac{1}{n-1}}\,\mathrm{d}x \le \prod_{j=1}^{n} \left(\int_{\mathbb{R}^{n-1}} f_j \right)^{\frac{1}{n-1}}$$

Here $\Pi_j x = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$

- $\ker \Pi_j = \mathrm{span}(e_j)$
- If $\ker \widetilde{\Pi}_j = \mathrm{span}(v_j)$ and $\mathrm{span}(v_1, \dots, v_n) = \mathbb{R}^n$ then

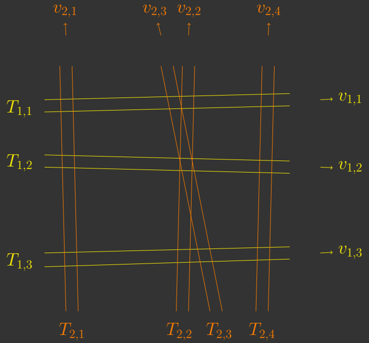
$$\int_{\mathbb{R}^n} \prod_{j=1}^{n} f_j(\widetilde{\Pi}_j x)^{\frac{1}{n-1}}\,\mathrm{d}x \le C \prod_{j=1}^{n} \left(\int_{\mathbb{R}^{n-1}} f_j \right)^{\frac{1}{n-1}}$$

Theorem (Guth)

$$\int_{\mathbb{R}^n} \prod_{j=1}^{n} \Big(\sum_{k=1}^{N_j} 1_{T_{j,k}} \Big)^{\frac{1}{n-1}} \leq C \prod_{j=1}^{n} N_j^{\frac{1}{n-1}}$$

$T_{j,k}$: 1-tube, $|v_{j,k} - e_j| \leq c \ll 1$

Visibility lemma and factorisation (see also Carbery–Valdimarsson)



Theorem (Bennett–Carbery–Tao)

$\forall\, \varepsilon > 0$, $\exists\, C_\varepsilon < \infty$ s.t.

$$\int_{B(0,R)} \prod_{j=1}^{n} \Big(\sum_{k=1}^{N_j} 1_{T_{j,k}} \Big)^{\frac{1}{n-1}} \leq C_\varepsilon R^\varepsilon \prod_{j=1}^{n} N_j^{\frac{1}{n-1}}$$

$T_{j,k}$: 1-tube, $|v_{j,k} - e_j| \leq c \ll 1$

Heat-flow (Bennett–Carbery–Tao); Induction-on-scales (Guth)

k_j-plane version (Bennett–Carbery–Tao, Bennett–B–Flock–Lee, Zhang)

The nonlinear Brascamp–Lieb inequality

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$$\int_U \prod_{j=1}^m f_j(\varphi_j(x))^{c_j}\,\mathrm{d}x \le C \prod_{j=1}^m \left(\int_{\mathbb{R}^{n_j}} f_j \right)^{c_j} \qquad \text{(NBL)}$$

$f_j \in L^1(\mathbb{R}^{n_j}),\ f_j \ge 0$

$\varphi_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$ C^2 submersion near 0

$U \subseteq \mathbb{R}^n$

Conjecture (local version)

If $\mathrm{B}(\mathbf{L^0}) < \infty$ where $L_j^0 = \mathrm{d}\varphi_j(0)$, then there exists a neighbourhood $U \ni 0$ and $C < \infty$ such that (NBL) holds

Nonlinear Loomis–Whitney inequality (Bennett–Carbery–Wright)

Conjecture holds with $L_j^0 = \Pi_j$.

- B–C–W proof : Christ's method of refinements + tensorisation
- Further proofs : Bejenaru–Herr–Tataru (induction-on-scales), Koch–Steinerberger, Carbery–Hänninen–Valdimarsson (under C^1 regularity, holds with $C = 1 + \varepsilon$ on a neighbourhood U_ε)
- Nonlinear LW yields multilinear singular convolution estimates and these were applied to wellposedness of Zakharov system on $\mathbb{R}^2 \times \mathbb{R}$

$$\begin{aligned} i\partial_t u + \Delta u &= vu \\ \Box v &= \Delta |u|^2 \end{aligned}$$

by Bejenaru–Herr–Holmer–Tataru

- Further applications of nonlinear LW (type) : Bejenaru–Herr, Kinoshita, Hirayama–Kinoshita, Kinoshita–Schippa,...

Partition $U = \bigcup Q$, where cube Q has sidelength $\sqrt{\delta}$ and centre x_Q

Write

$$\int_U \prod_{j=1}^m (f_j \circ \varphi_j)^{c_j} = \sum_Q \int_Q \prod_{j=1}^m ((f_j 1_{\varphi_j(Q)}) \circ \varphi_j)^{c_j}$$

Since φ_j is C^2

$$\varphi_j(x) = \varphi_j(x_Q) + L_j^{x_Q}(x - x_Q) + O(\delta) \qquad (x \in Q)$$

where $L_j^x := \mathrm{d}\varphi_j(x)$

Want to replace $(f_j 1_{\varphi_j(Q)}) \circ \varphi_j$ by $(f_j 1_{\varphi_j(Q)}) \circ L_j^{x_Q}$ (modulo translations)

Let $0 < \delta \leq 1$ and

$$L^1(\delta) = \{f \in L^1_+ : \tfrac{1}{2} f(y) \leq f(x) \leq 2f(y) \text{ whenever } |x - y| \leq \delta\}$$

E.g. $P_{c\delta} * f \in L^1(\delta)$ for general $f \in L^1_+$

Then (modulo translations) for $f_j \in L^1(\delta)$

$$\int_U \prod_{j=1}^m (f_j \circ \varphi_j)^{c_j} = \sum_Q \int_Q \prod_{j=1}^m ((f_j 1_{\varphi_j(Q)}) \circ \varphi_j)^{c_j}$$
$$\lesssim \sum_Q \int_Q \prod_{j=1}^m ((f_j 1_{\varphi_j(Q)}) \circ L_j^{x_Q})^{c_j}$$

where $L_j^x := \mathrm{d}\varphi_j(x)$

We are assuming $\mathrm{B}(\mathbf{L}^0) < \infty$, to proceed we need

$$\mathrm{B}(\mathbf{L}^x) \lesssim 1 \qquad (x \in U)$$

Since φ_j is C^2, $\mathbf{L}^x$ will be close to $\mathbf{L}^0$

Local boundedness of BL constant (Bennett–B–Flock–Lee)

Suppose $\mathrm{B}(\mathbf{L}^0) < \infty$. Then $\exists \delta > 0$ and $\exists C < \infty$ s.t.

$$\mathrm{B}(\mathbf{L}) \leq C \qquad \text{whenever } \|\mathbf{L} - \mathbf{L}^0\| \leq \delta$$

Partition $U = \bigcup Q$, where cube Q has sidelength $\sqrt{\delta}$ and centre x_Q

Then (modulo translations) for $f_j \in L^1(\delta)$

$$\begin{aligned}
\int_U \prod_{j=1}^m (f_j \circ \varphi_j)^{c_j} &= \sum_Q \int_Q \prod_{j=1}^m ((f_j 1_{\varphi_j(Q)}) \circ \varphi_j)^{c_j} \\
&\lesssim \sum_Q \int_Q \prod_{j=1}^m ((f_j 1_{\varphi_j(Q)}) \circ \mathrm{d}\varphi_j(x_Q))^{c_j} \\
&\lesssim \sum_Q \prod_{j=1}^m \left(\int_{\varphi_j(Q)} f_j \right)^{c_j}
\end{aligned}$$

We have $\int_{\varphi_j(Q)} f_j \lesssim \delta^{n_j/2} P_{\sqrt{\delta}} * f_j(\varphi_j(x_Q))$ uniformly in $x_Q \in Q$

Induction-on-scales

We obtain

$$\mathcal{C}(\delta) \leq C_0 \mathcal{C}(\sqrt{\delta})$$

for some constant $C_0 > 1$

Here

$$\mathcal{C}(\delta) = \sup_{\substack{\int f_j = 1 \\ f_j \in L^1(\delta)}} \int_U \prod_{j=1}^{m} (f_j \circ \varphi_j)^{c_j}$$

Iterating ($O(\log\log(\frac{1}{\delta}))$ times) gives

$$\mathcal{C}(\delta) \lesssim (\log(1/\delta))^{\kappa}$$

See Bennett–B–Flock–Lee

A tighter induction

For a given data $(\mathbf{L}, \mathbf{c})$, define the functional

$$\Lambda(\mathbf{f}) = \int_{\mathbb{R}^n} \prod_{j=1}^{m} (f_j \circ L_j)^{c_j} \qquad \left(\int f_j = 1\right)$$

Setting $h_j^x(z) = f_j(z) g_j(L_j x - z)$, we have

$$\begin{aligned}
\Lambda(\mathbf{f})\Lambda(\mathbf{g}) &= \int \prod_j f_j(L_j y)^{c_j} \int \prod_j g_j(L_j(x-y))^{c_j} \, \mathrm{d}x \, \mathrm{d}y \\
&= \int \left(\int \prod_j (h_j^x(L_j y))^{c_j} \, \mathrm{d}y \right) \mathrm{d}x \\
&= \int \Lambda(\mathbf{h}^x) \prod_j (f_j * g_j(L_j x))^{c_j} \, \mathrm{d}x \\
&\leq \sup_x \Lambda(\mathbf{h}^x) \int \prod_j (f_j * g_j(L_j x))^{c_j} \, \mathrm{d}x \\
&= \sup_x \Lambda(\mathbf{h}^x) \Lambda(\mathbf{f} * \mathbf{g})
\end{aligned}$$

Ball's inequality

If $h_j^x(z) = f_j(z)g_j(L_j x - z)$ then

$$\Lambda(\mathbf{f}) \le \frac{\sup_x \Lambda(\mathbf{h}^x)\Lambda(\mathbf{f} * \mathbf{g})}{\Lambda(\mathbf{g})}$$

- **If** we additionally assume that g is a **maximiser** then

$$\Lambda(\mathbf{f}) \le \sup_x \Lambda(\mathbf{h}^x)$$

  - $\mathbf{h}^x$ is a certain "localised" version $\mathbf{f}$ (w.r.t. maximiser g)
  - If g_j has compact (tiny) support near 0, then $h_j^x \approx f_j$ near $L_j x$
  - Strong indication we should try to induct on size of $\operatorname{supp} \mathbf{f}$

- Or, $\Lambda(\mathbf{f}) \le \Lambda(\mathbf{f} * \mathbf{g})$ $\rightsquigarrow$ induct on scale of constancy of $\mathbf{f}$

Let $F = \prod_{j=1}^{m} (f_j \circ \varphi_j)^{c_j}$ and define

$$\mathcal{C}(u,\delta) = \sup_{\int f_j = 1} \int_{B(u,\delta)} F$$

Roughly, hope for $\mathcal{C}(u,\delta) \leq (1+\delta^{\beta})\mathcal{C}(u,\delta^{\alpha})$, where $\alpha \in (1,2)$, $\beta > 0$

Now

$$\mathrm{B}(\mathbf{L}^u) \int_{B(u,\delta)} F(y)\,\mathrm{d}y = \int_{B(u,\delta)} F(y)\,\mathrm{d}y \int_{\mathbb{R}^n} \prod_j g^u_{\delta^\alpha,j}(L^u_j x)^{c_j}\,\mathrm{d}x$$

if g^u is a gaussian maximiser for $\mathbf{L}^u$, and

$$g^u_{\delta^\alpha,j}(w) = \delta^{-\alpha'} g^u_j(\delta^{-\alpha'} w) \qquad (\alpha' > \alpha)$$

Recall $L^u_j = \mathrm{d}\varphi_j(u)$

$$\mathrm{B}(\mathbf{L}^u)\int_{B(u,\delta)}\prod_{j=1}^m f_j(\varphi_j(y))^{c_j}\,\mathrm{d}y$$

$$=\int_{B(u,\delta)} F(y)\,\mathrm{d}y\int_{\mathbb{R}^n}\prod_j g^u_{\delta^\alpha,j}(L^u_j x)^{c_j}\,\mathrm{d}x$$

$$\leq (1+\delta^\beta)\int_{B(u,\delta)} F(y)\,\mathrm{d}y\int_{B(0,\delta^\alpha)}\prod_j g^u_{\delta^\alpha,j}(L^u_j x)^{c_j}\,\mathrm{d}x$$

$$=(1+\delta^\beta)\int_{B(u,\delta)} F(y)\int_{B(y,\delta^\alpha)}\prod_j g^u_{\delta^\alpha,j}(L^u_j(x-y))^{c_j}\,\mathrm{d}x\mathrm{d}y$$

Now $L^u_j(x-y)=\varphi_j(u)+L^u_j(x-u)-\varphi_j(y)+O(\delta^2)$ so

$$\cdots\leq(1+\delta^\beta)^2\int_{B(u,2\delta)}\int_{B(x,\delta^\alpha)}\prod_j h^x_j(\varphi_j(y))^{c_j}\,\mathrm{d}y\mathrm{d}x$$

where $h^x_j(z)=f_j(z)g^u_{\delta^\alpha,j}(\varphi_j(u)+L^u_j(x-u)-z)$

So, with $h_j^x(z) = f_j(z) g^u_{\delta^\alpha,j}(\varphi_j(u) + L_j^u(x-u) - z)$,

$$\begin{aligned}
&\mathrm{B}(\mathbf{L}^u) \int_{B(u,\delta)} \prod_{j=1}^m f_j(\varphi_j(y))^{c_j}\,\mathrm{d}y \\
&\leq (1+\delta^\beta) \int_{B(u,2\delta)} \int_{B(x,\delta^\alpha)} \prod_j h_j^x(\varphi_j(y))^{c_j}\,\mathrm{d}y\mathrm{d}x \\
&\leq (1+\delta^\beta) \int_{B(u,2\delta)} \mathcal{C}(x,\delta^\alpha) \prod_j \left(\int h_j^x\right)^{c_j} \mathrm{d}x
\end{aligned}$$

so (modulo translations)

$$\begin{aligned}
&\mathrm{B}(\mathbf{L}^u) \int_{B(u,\delta)} \prod_{j=1}^m f_j(\varphi_j(y))^{c_j}\,\mathrm{d}y \\
&\leq (1+\delta^\beta) \sup_{x\in B(u,2\delta)} \mathcal{C}(x,\delta^\alpha) \int \prod_j (f_j * g^u_{\delta^\alpha,j}(L_j^u x))^{c_j}\,\mathrm{d}x
\end{aligned}$$

An argument like the above gives

$$\int_{B(u,\delta)} \prod_{j=1}^{m} (f_j \circ \varphi_j)^{c_j} \leq (1+\delta^\beta) \sup_{x \in B(u,2\delta)} \mathcal{C}(x,\delta^\alpha)$$

and thus

$$\mathcal{C}(u,\delta) \leq (1+\delta^\beta) \sup_{x \in B(u,2\delta)} \mathcal{C}(x,\delta^\alpha)$$

- At the very start we assumed gaussian maximisers exist – not always the case!
- Lieb's theorem guarantees gaussian near-maximisers but to keep the argument tight, we need a quantitative version of Lieb's theorem

- Using
$$\mathcal{C}(u,\delta) \le (1+\delta^{\beta}) \max_{x\in B(u,2\delta)} \mathcal{C}(x,\delta^{\alpha})$$
want to zoom in enough to do something like
$$f_j(\varphi_j(x)) \le \kappa f_j(\mathrm{d}\varphi_j(u)x) \qquad (x \in B(u,\delta))$$
Recall $\mathcal{C}(u,\delta) = \sup_{\int f_j=1} \int_{B(u,\delta)} \prod_{j=1}^{m} (f_j \circ \varphi_j)^{c_j}$
- For this, need f_j "locally constant" so ... need more parameters (functions κ-constant at scale μ)

Also, to keep things tight, we use

Theorem (Bennett–B–Cowling–Flock)

$\mathbf{L} \mapsto \mathrm{B}(\mathbf{L})$ is continuous

Theorem (Bennett–B–Buschenhenke–Cowling–Flock)

Suppose $\varphi_j : \mathbb{R} \to \mathbb{R}^{n_j}$ is a C^2 submersion near 0 s.t. $\mathrm{B}(\mathbf{L^0}) < \infty$, where $L_j^0 = \mathrm{d}\varphi_j(0)$.

Then $\forall \varepsilon > 0$, $\exists U \ni 0$ s.t.

$$\int_U \prod_{j=1}^m (f_j \circ \varphi_j)^{c_j} \leq (1+\varepsilon)\mathrm{B}(\mathbf{L^0}) \prod_{j=1}^m \Big(\int_{\mathbb{R}^{n_j}} f_j \Big)^{c_j}$$

- Yields rather general multilinear singular convolution estimates (when combined with Fourier duality of the BL inequality)
- Local version of Young convolution on Lie groups (addressing a question of Cowling–Martini–Müller–Parcet)
- Exciting recent developments: Duncan (an alternative viewpoint on nonlinear Ball's inequality, global nonlinear BL),...