Regularity of solutions to elliptic PDE: the p-ellipticity condition

Jill Pipher
Brown University

El Escorial 2022

Background: elliptic PDE

Plan for the lectures:

- Brief background: *ellipticity* and the theory of second order elliptic PDE,

Background: elliptic PDE

Plan for the lectures:

- Brief background: *ellipticity* and the theory of second order elliptic PDE,
- Introduce *p-ellipticity*, a property of complex matrices and systems of real matrices

Background: elliptic PDE

Plan for the lectures:

- Brief background: *ellipticity* and the theory of second order elliptic PDE,
- Introduce *p-ellipticity*, a property of complex matrices and systems of real matrices
- Describe how this concept has been used in PDE: regularity of solutions, with an emphasis on solving boundary value problems

Plan for the lectures:

- Brief background: *ellipticity* and the theory of second order elliptic PDE,
- Introduce *p-ellipticity*, a property of complex matrices and systems of real matrices
- Describe how this concept has been used in PDE: regularity of solutions, with an emphasis on solving boundary value problems
- How p-ellipticity leads to extrapolation of solvability of the L^p-Dirichlet problem

Background: elliptic PDE

Plan for the lectures:

- Brief background: *ellipticity* and the theory of second order elliptic PDE,
- Introduce *p-ellipticity*, a property of complex matrices and systems of real matrices
- Describe how this concept has been used in PDE: regularity of solutions, with an emphasis on solving boundary value problems
- How p-ellipticity leads to extrapolation of solvability of the L^p-Dirichlet problem
- Using the notion of *p-ellipticity* for systems to solve classical elliptic systems (Lamé).

Context: ellipticity in PDE

The principle term of a linear second order PDE,
$\sum_{i,j} D_i A_{i,j} D_j + ...$(lower order terms), gives rise to a bilinear form:
$\sum_{i,j} A_{i,j}\xi_i\xi_j$.
Ellipticity: there exists a λ such that

$$\lambda|\xi|^2 < \sum_{i,j} A_{i,j}\xi_i\xi_j < (\lambda)^{-1}|\xi|^2$$

.

These lectures focus on second order equations (or systems) in divergence form $L =: \operatorname{div} A(X)\nabla$, one possible generalization of Laplace's equation. (Later we'll discuss adding certain lower order terms.)

If the coefficients are complex-valued, ellipticity means:

$$\lambda|\xi|^2 \leq \Re e \sum_{i,j=0}^{n-1} A_{ij}(x)\xi_i\overline{\xi_j} = \Re e \, \langle A(x)\xi, \xi\rangle \quad \text{and} \quad |\langle A\xi, \eta\rangle| \leq \Lambda|\xi||\eta|$$

Solutions to divergence form elliptic equations with non-smooth coefficients

For a matrix A which is bounded, elliptic, classical solvability via the Lax-Milgram lemma: in Ω, define the bilinear form $\mathcal{B} : \dot{W}^{1,2}(\Omega;\mathbb{C}) \times \dot{W}_0^{1,2}(\Omega;\mathbb{C}) \to \mathbb{C}$ by

$$\mathcal{B}[u,w] = \int_\Omega A_{ij}(x)\partial_j u(x)\partial_i w(x)\,dx.$$

Then $\mathcal{B}$ is bounded - A has coefficients in $L^\infty(\Omega)$ for all $w \in \dot{W}_0^{1,2}(\Omega;\mathbb{C})$.
Given $f \in \dot{B}^{2,2}_{1/2}(\partial\Omega;\mathbb{C})$, there exists $v \in \dot{W}^{1,2}(\Omega;\mathbb{C})$ such that $\mathrm{Tr}\, v = f$ on $\partial\Omega$.

The solution u will have the form $u = u_0 + v$, where $u_0 \in \dot{W}_0^{1,2}(\Omega; \mathbb{C})$ is such that

$$\mathcal{B}[u_0, w] = -B[v, w] \quad \text{for all} \quad w \in \dot{W}_0^{1,2}(\Omega; \mathbb{C}).$$

$-B[v, \cdot] \in \left(\dot{W}_0^{1,2}(\Omega; \mathbb{C})\right)^*$, so Lax-Milgram gives a unique solution $u_0 \in \dot{W}_0^{1,2}(\Omega; \mathbb{C})$, when $\mathcal{B}$ is coercive on the space $\dot{W}_0^{1,2}(\Omega; \mathbb{C})$. When $\mathcal{L}$ is uniformly elliptic:

$$Re \int_\Omega A_{ij} \partial_j u \overline{\partial_i u} \, dx \geq \lambda \int_\Omega |\nabla u|^2 \, dx,$$

and so for all $f \in \dot{B}^{2,2}_{1/2}(\partial\Omega; \mathbb{C})$, there us a unique (up to constants) solution $u \in \dot{W}^{1,2}$ such that $\mathrm{Tr}\, u = f$ on $\partial\Omega$ and $\mathcal{L}u = 0$ in Ω in the sense that

$$\int_\Omega \operatorname{div} A \nabla u \ \phi = 0$$

for all $\phi \in C_0^\infty(\Omega)$.

From weak solutions to continuous solutions

Background and context for our study of complex coefficient equations and systems: the real coefficient case.

The DeGiorgi - Nash - Moser theory of the 1950s-1960s is the remarkable collection of restults that show that solutions to **real** divergence form elliptic equations must in fact be Hölder continuous.

De Giorgi: Motivation for the study of regularity of solutions of elliptic equations was initially driven by Hilbert's 19th problem: are solutions to minimizers in the calculus of variations analytic, if the data is analytic?

From weak solutions to continuous solutions

Background and context for our study of complex coefficient equations and systems: the real coefficient case.
The DeGiorgi - Nash - Moser theory of the 1950s-1960s is the remarkable collection of restults that show that solutions to **real** divergence form elliptic equations must in fact be Hölder continuous.
De Giorgi: Motivation for the study of regularity of solutions of elliptic equations was initially driven by Hilbert's 19th problem: are solutions to minimizers in the calculus of variations analytic, if the data is analytic?

$$\mathcal{E}(w) =: \int_\Omega F(\nabla w)dx,$$

where $F : R^n \to \Omega$ is smooth.

From weak solutions to continuous solutions

Background and context for our study of complex coefficient equations and systems: the real coefficient case.
The DeGiorgi - Nash - Moser theory of the 1950s-1960s is the remarkable collection of restults that show that solutions to **real** divergence form elliptic equations must in fact be Hölder continuous.
De Giorgi: Motivation for the study of regularity of solutions of elliptic equations was initially driven by Hilbert's 19th problem: are solutions to minimizers in the calculus of variations analytic, if the data is analytic?

$$\mathcal{E}(w) =: \int_\Omega F(\nabla w)dx,$$

where $F : R^n \to \Omega$ is smooth.

The minimizer - in a class of w satisfying some boundary conditions - is a solution to

$$\operatorname{div} F'(\nabla w) = 0$$

Example: $F(\rho) = |\rho|^2$, and $\operatorname{div} F'(\nabla w) = \Delta u$.

Example: $F(\rho) = |\rho|^2$, and $\operatorname{div} F'(\nabla w) = \Delta u$.

De Giorgi reduced the problem to a step that required a bootstrapping the regularity of an equation, $Lu = \operatorname{div} A\nabla u = 0$, with A only elliptic, measurable: i.e., the weak solution in a Sobolev space is actually a Hölder continuous function $u \in C^\alpha$, where α only depends on the ellipticity parameter of A.

Example: $F(\rho) = |\rho|^2$, and $\operatorname{div} F'(\nabla w) = \Delta u$.

De Giorgi reduced the problem to a step that required a bootstrapping the regularity of an equation, $Lu = \operatorname{div} A\nabla u = 0$, with A only elliptic, measurable: i.e., the weak solution in a Sobolev space is actually a Hölder continuous function $u \in C^\alpha$, where α only depends on the ellipticity parameter of A.

One concrete, motivating example, is the minimal surface equation:

$$Lu = \operatorname{div}\left(\frac{1}{\sqrt{1 + |\nabla u|^2}} \nabla u\right).$$

Divergence form parabolic equations

Nash (independently, at the same time as De Giorgi, and with a different method of proof) found the same result for parabolic divergence form equations:

$$L =: D_t - \operatorname{div} A(X)\nabla,$$

which also gives the elliptic result in the steady state $t = 0$.

Nash was motivated by understanding "flow for viscous, compressible, heat conducting fluids": non-linear parabolic system, and studied the equation case first. Guided by physical intuition inspired by diffusion, Brownian motion, flow of heat and electrical charges.

Moser (1964) gave yet another proof, using a bootstrapping technique known as *Moser iteration*, which increments the regularity of solutions, up to L^∞ and then to Hölder continuity.

Boundary value problems for divergence form elliptic equations

The De Giorgi-Nash-Moser regularity theory opened up the field of boundary value problems for these div form equations in non-smooth domains, and data in L^p.

(1963) Littman-Stampacchia-Weinberger. For continuous data, the classical Dirichlet problem seeks solutions continuous up to the boundary. [LSW] showed that the continuous Dirichlet problem was solvable for L in any domain in which it is solvable for Δ.

(1981) Caffarelli-Fabes-Mortola-Salsa. Boundary behavior of non-negative solutions of elliptic divergence form equations.

In summary, solutions to **real** elliptic divergence form equations share many of the properties that harmonic functions enjoy:

- *the maximum principle*: the maximum value of a solution in a bounded domain is taken on the boundary.

In summary, solutions to **real** elliptic divergence form equations share many of the properties that harmonic functions enjoy:

- *the maximum principle*: the maximum value of a solution in a bounded domain is taken on the boundary.
- *Harnack principle*: positive solutions have comparable values in compact subsets K - constants depending only on K.

Complex matrices and systems of equations: p-ellipticity

In summary, solutions to **real** elliptic divergence form equations share many of the properties that harmonic functions enjoy:

- *the maximum principle*: the maximum value of a solution in a bounded domain is taken on the boundary.
- *Harnack principle*: positive solutions have comparable values in compact subsets K - constants depending only on K.
- *boundary measure*: solutions can be recovered by integrating boundary values using an analog of harmonic measure

In summary, solutions to **real** elliptic divergence form equations share many of the properties that harmonic functions enjoy:

- *the maximum principle*: the maximum value of a solution in a bounded domain is taken on the boundary.
- *Harnack principle*: positive solutions have comparable values in compact subsets K - constants depending only on K.
- *boundary measure*: solutions can be recovered by integrating boundary values using an analog of harmonic measure
- None of these properties hold for complex elliptic equations, or for elliptic systems. No maximum principles, no boundary measures, etc.

2005-2007: Cialdea and Maz'ya define the notion L^p-dissipativity, motivated by understanding when the semigroups $\exp(-tL_A), \;\; t > 0$, generated by second order elliptic operators are contractive in L^p. (Always true for real second order elliptic operators.)

They showed that the following condition was sufficient for L^p dissipativity:

$$\frac{4}{pp'}\langle \mathcal{R}e\, A(x)\xi, \xi\rangle + \langle \mathcal{R}e\, A(x)\eta, \eta\rangle \; + \tag{1}$$

$$2\langle (p^{-1}\mathcal{I}m\, A(x) - p'^{-1}\mathcal{I}m\, A^t(x))\xi, \eta\rangle \geq 0, \tag{2}$$

for all ξ, $\eta \in \mathbb{R}^n$.

In Dindoš-P. (2018), we formulated an elliptic strengthening, and using a change of variables $\xi = \frac{\sqrt{pp'}}{2}\lambda$, it becomes: For some $\varepsilon > 0$ and all $\lambda, \eta \in \mathbb{R}^n$

$$\langle \mathcal{R}e\, A\,\lambda, \lambda\rangle + \langle \mathcal{R}e\, A\,\eta, \eta\rangle + \left\langle \left(\sqrt{\tfrac{p'}{p}}\mathcal{I}m\, A - \sqrt{\tfrac{p}{p'}}\mathcal{I}m\, A^t \right) \lambda, \eta \right\rangle \geq \varepsilon(|\lambda|^2 + |\eta|^2).$$

This condition was formulated independently and termed *p-ellipticity* in [Carbonaro, Dragičević] as follows:
For $p > 1$ define the $\mathbb{R}$-linear map $\mathcal{J}_p : \mathbb{C}^n \to \mathbb{C}^n$ by

$$\mathcal{J}_p(\alpha + i\beta) = \frac{\alpha}{p} + i\frac{\beta}{p'}$$

where $p' = p/(p-1)$ and $\alpha, \beta \in \mathbb{R}^n$. [CD] shows that the matrix A is p-elliptic iff for a.e. $x \in \Omega$

$$\mathcal{R}e\,\langle A(x)\xi, \mathcal{J}_p\xi\rangle \geq \lambda_p|\xi|^2, \qquad \forall \xi \in \mathbb{C}^n \tag{3}$$

for some $\lambda_p > 0$.

[CD] showed that this was equivalent to:

Definition

The matrix A is *p-elliptic* if

$$|1 - 2/p| < \mu(A) \tag{4}$$

where

$$\mu(A) = \operatorname*{ess\,inf}_{(x,\xi)\in\Omega\times\mathbb{C}^n\setminus\{0\}} Re\,\frac{\langle A(x)\xi, \xi\rangle}{|\langle A(x)\xi, \bar{\xi}\rangle|}. \tag{5}$$

From this, see that A is p-elliptic in the range $p \in (p_0, p_0')$, where $p_0 = 2/(1 + \mu(A))$ (because: $|1 - 2/p| = |1 - 2/p'|$.)

$p_0 = 1$ (i.e., $p_0' = \infty$) if and only if the matrix A is real.

And, 2-ellipticity is just the usual ellipticity condition for complex-valued matrices.

Complex matrices and systems of equations: p-ellipticity

[CD] gave several examples of the significance of p-ellipticity, including dimension-free bilinear embeddings, L^p contractivity of semigroups, and holomorphic functional calculus. For example, they prove a dimension-free bound for the bilinear form:

Theorem

Suppose that $p > 1$ and A, B are both p-elliptic. Then,

$$\int_0^\infty \int_{\mathbb{R}^n} |\nabla P_t^A f(x)||\nabla P_t^B g(x)| dxdt \leq C||f||_p||g||_{p'}. \qquad (6)$$

for all $f, g \in C^\infty(\mathbb{R}^n)$, where P_t^A denotes e^{-tL_A}.

And, recent work of Feneuil-Mayboroda-Zhao [FMZ] used p-ellipticity in the theory of degenerate elliptic equations (domains with low dimensional boundaries), and extended the theory.

Higher regularity from p-ellipticity

Lemma

[DP1] Let the matrix A be p-elliptic. Suppose that u is a $W^{1,2}_{loc}(\Omega;\mathbb{C})$ solution to $\mathcal{L}$ in Ω. Then, for any ball $B_r(x)$ with $r < \delta(x)/4$,

$$\int_{B_r(x)} |\nabla u(y)|^2 |u(y)|^{p-2} dy \lesssim r^{-2} \int_{B_{2r}(x))} |u(y)|^p dy \tag{7}$$

and, when $p > 2$,

$$\left(\fint_{B_r(x))} |u(y)|^q dy \right)^{1/q} \lesssim \left(\fint_{B_{2r}(x)} |u(y)|^2 dy \right)^{1/2} \tag{8}$$

for all $q \in (2, \frac{np}{n-2}]$ when $n > 2$, and where the implied constants depend only p-ellipticity and K. When $n = 2$, q can be any number in $(2, \infty)$. In particular, $|u|^{(p-2)/2} u$ belongs to $W^{1,2}_{loc}(\Omega;\mathbb{C})$.

When $p = 2$, (7) is Cacciopoli's inequality for solutions:

$$\int_{B_r(x)} |\nabla u(y)|^2 dy \lesssim r^{-2} \int_{B_{2r}(x))} |u(y)|^2 dy \qquad (9)$$

To prove (9), use ellipticity to bound $\int_{B_r(x)} |\nabla u(y)|^2 dy$ by $\mathcal{R}e \int_\Omega \langle A(x)\nabla u, \nabla u\rangle \eta^2 dy$.

Then use the equation and divergence theorem, with a bump $\eta = 1$ in B_r and $= 0$ outside B_{2r}:
$\langle A(x)\nabla u, \nabla u\rangle \eta^2 = \mathrm{div}(u\eta^2 A\nabla u) - \bar{u}A\nabla u \nabla\eta\,\eta$.

The second term is bounded using Cauchy-Schwarz: the $\nabla\eta$ gives a bound of r^{-2} on B_{2r}.

The key is using the equation. That is what p-ellipticity allows us to do to prove the Lemma.

Why *higher* regularity?

First, (7) says that $|u(x)|^{p/2-1}u(x)) \in W^{1,2}_{\text{loc}}$, since

$$|\nabla(|u(x)|^{p/2-1}u(x))|^2 \approx |u(x)|^{p-2}|\nabla u(x)|^2.$$

Second, the higher integrability of u (which, as a weak solution to $\mathcal{L} = \text{div} A\nabla$ belongs to $L^{2n/(n-2)}$), via Sobolev, is extended up to $L^{np/(n-2)}$.

There is an analogous reverse Hölder result for $p < 2$ from [DP1], improved in [FMZ], will come back to later.

The p-ellipticity condition allows us to "insert the equation" as was needed in the Caccioppoli estimate:

Theorem

Assume that the matrix A is p-elliptic. Then there exists $\lambda'_p = \lambda'_p(\Lambda, \lambda_p) > 0$ such that for any nonnegative, bounded and measurable function χ and any u such that $|u|^{(p-2)/2}u \in W^{1,2}_{loc}(\Omega;\mathbb{C})$, we have

$$\Re e \int_\Omega \langle A(x)\nabla u, \nabla(|u|^{p-2}u)\rangle \chi(x)\, dx \geq \lambda'_p \int_\Omega |u|^{p-2}|\nabla u|^2 \chi(x)\, dx. \tag{10}$$

Proof of Theorem. We start with the p-ellipticity condition:

$$\langle \mathcal{R}e\, A\,\lambda, \lambda\rangle + \langle \mathcal{R}e\, A\,\eta, \eta\rangle + \left\langle \left(\sqrt{\tfrac{p'}{p}}\mathcal{I}m\, A - \sqrt{\tfrac{p}{p'}}\mathcal{I}m\, A^t \right) \lambda, \eta \right\rangle \geq \varepsilon(|\lambda|^2 + |\eta|^2).$$

For $v \in W^{1,2}_{loc}(\Omega;\mathbb{C})$, write $|v|^{-1}\overline{v}\nabla v$ as $X + iY$, that is

$$X = \mathcal{R}e(|v|^{-1}\overline{v}\nabla v) \quad \text{and} \quad Y = \mathcal{I}m(|v|^{-1}\overline{v}\nabla v).$$

Let $\chi(x)$ be a nonnegative bounded an measurable function on Ω. With X in place of ξ and Y in place of η, multiplying by $\chi(x)$ and integrating over Ω one obtains

$$\mathcal{R}e \int_\Omega \Big[\langle A\nabla v, \nabla v\rangle - (1-2/p)\langle (A - A^*)\nabla(|v|), |v|^{-1}\overline{v}\nabla v\rangle - (1-2/p)^2\langle A\nabla(|v|), \nabla(|v|)\rangle \Big] \chi(x)\, dx \geq \varepsilon' \int_\Omega |\nabla v|^2 \chi(x)\, dx$$

Following the calculations of [CMequations]: set $v = |u|^{p/2-1}u$ and

$$g_\varepsilon = (|v|^2 + \varepsilon^2)^{1/2}, \qquad u_\varepsilon = g_\varepsilon^{2/p-1} v,$$

for u, u_ε. Then, $|u|^{(p-2)/2}u$, $|u_\varepsilon|^{(p-2)/2}u_\varepsilon \in W^{1,2}_{loc}(\Omega, \mathbb{C}^n)$.
$\varepsilon \to 0+$. When $p > 2$,

$$\lim_{\varepsilon\to 0+} \mathcal{R}e \int_\Omega \langle A\nabla u_\varepsilon, \nabla(|u_\varepsilon|^{p-2}u_\varepsilon)\rangle \chi \, dx =$$

$$\mathcal{R}e \int_\Omega \Big[\langle A\nabla v, \nabla v\rangle - (1-2/p)\langle (A-A^*)\nabla(|v|), |v|^{-1}\overline{v}\nabla v\rangle -$$

$$(1-2/p)^2 \langle A\nabla(|v|), \nabla(|v|)\rangle\Big] \chi \, dx \geq \varepsilon' \int_\Omega |\nabla(|u|^{p/2-1}u)|^2 \chi \, dx.$$

and the first term is

$$\mathcal{R}e \int_\Omega \langle A(x)\nabla u, \nabla(|u|^{p-2}u)\rangle \chi(x) \, dx.$$

When $1 < p < 2$, can get the same conclusion using a duality argument.

As [CM] observed, if $w = |u|^{p-2}u$, so that $u = |w|^{p'-2}w$, then $|u|^{(p-2)/2}u \in W^{1,2}_{loc}(\Omega;\mathbb{C})$ implies that $|w|^{(p'-2)/2}w \in W^{1,2}_{loc}(\Omega;\mathbb{C})$.
Run the above argument for w noting that A^* is p'-elliptic.
Recall that we'd like to show (7):

$$\int_{B_r(x)} |\nabla u(y)|^2 |u(y)|^{p-2} dy \lesssim r^{-2} \int_{B_{2r}(x))} |u(y)|^p dy$$

We'll assume that the coefficients of A are smooth, and eliminate this assumption using a limiting argument (and that the smoothness doesn't appear in the constants).

Let $B_r(x)$ be ball in the interior of Ω with $r < \delta(x)/4$. Let φ be a smooth cutoff function, with $\varphi = 1$ on $B_r(x)$ and vanishing outside of $B_{2r}(x)$. Set $v = u\varphi$. Then,

$$\mathcal{L}v = u\mathcal{L}\varphi + A\nabla u \cdot \nabla\varphi + A^*\nabla u \cdot \nabla\varphi. \tag{11}$$

Multiply both sides of (11) by $|v|^{p-2}\overline{v}$ and integrate by parts to obtain

$$\begin{aligned}\int \nabla(|v|^{p-2}\overline{v}) \cdot A\nabla v = &\int \nabla(|v|^{p-2}\overline{v}u) \cdot A\nabla\varphi \\ &- \int |v|^{p-2}\overline{v}A\nabla u \cdot \nabla\varphi \\ &- \int |v|^{p-2}\overline{v}A^*\nabla u \cdot \nabla\varphi\end{aligned} \tag{12}$$

The real part of the left hand side of (12) is bounded from below by $\lambda_p \int |v|^{p-2}|\nabla v|^2\, dy$ by p-ellipticity and (10).

Since, $\overline{v}\nabla u = \overline{u}\nabla v - |u|^2\nabla\varphi$ the last two integrals are bounded by a constant times

$$\int |v|^{p-2}|u||\nabla\varphi||\nabla v| + |v|^{p-2}|u|^2|\nabla\varphi|^2\, dy \tag{13}$$

The second term in (13) is bounded by $r^{-2}\int_{B_{2r}(x)}|u|^p$, as is the first after using C-S.

The first term above requires computing the derivative to get an upper bound of

$$\int |v|^{p-2}||\nabla v||u||\nabla\varphi|dy \tag{14}$$

Using (7), and applying Sobolev embedding to $F = |v|^{p/2-1}v$ gives:

$$\left(\fint_{B_r(x)} |u|^{\tilde{p}}\, dy\right)^{1/\tilde{p}} \lesssim \left(\fint_{B_{2r}(x)} |v|^{\tilde{p}}\, dy\right)^{1/\tilde{p}}$$
$$\lesssim \left(r^2 \fint_{B_{2r}(x)} |\nabla(|v|^{p/2-1}v)|^2\, dy\right)^{1/p} \qquad (15)$$

where $\tilde{p} = \frac{pn}{n-2}$. For $n = 2$, this means $\tilde{p} \in (1, \infty)$. (In fact u is Hölder continuous in the interior.)
Use this to obtain a reverse Hölder inequality for u:

$$\left(\fint_{B_r(x)} |u|^{\tilde{p}}\, dy\right)^{1/\tilde{p}} \lesssim \left(\fint_{B_{2r}(x)} |u|^{p}\, dy\right)^{1/p} \qquad (16)$$

The assumptions of smoothness of the coefficients of A are removed by a limiting argument, approximating A by smooth A_j. There are analogs of (7) and (8) for $p < 2$: Here the reverse

Hölder inequality is from 2 to p, etc. But the arguments are more delicate. For example, for $v = u\varphi$ we used estimates like $|v|^{p-2} \leq |u|^{p-2}$ which do not hold when $p < 2$ near $\varphi = 0$.

The approach to the RH for $p < 2$ in [DP1] also gave an error term that was removed in the work of [FMZ], where they also showed a boundary estimate, which turns out to be a key tool in *extrapolating the solvability of the Dirichlet problem*.

Remark: In [DP1], we considered equations with lower order terms:

$$0 = \partial_i \left(A_{ij}(x)\partial_j u\right) + B_i(x)\partial_i u$$

defined in a domain Ω, with $\delta(x) = \mathrm{dist}(x, \partial\Omega)$ and with $|B(x)| \leq K\delta^{-1}(x)$. In [DP3], we observed that one could add lower terms and get a more general RH inequality.

The approach of [FMZ] is direct - in the sense that it does not use smooth approximations - and close in spirit to the Moser iteration scheme. Consider the case $q < 2$, and the interior estimate on balls B for the moment, where [FMZ] obtain an improved RH estimate. They first establish, for balls B with $3B \subset \Omega$:
If Ψ is smooth s.t. $0 \leq \Psi \leq 1$, $|\nabla \Psi(X)| \lesssim (\delta(X))^{-1}$, and A is q-elliptic, then if $k > 2$,

$$\int_B |u|^{q-2} |\nabla u|^2 \Psi^k \, dX \leq \frac{C}{r^2} \int_{2B} |u|^q \Psi^{k-2} \, dX. \qquad (17)$$

Using this fact, and applying Sobolev with $p = qn/(n-2)$

$$\begin{aligned} \left(\fint_B |u|^p \Psi^{k'} \right)^{\frac{1}{p}} &= \left(\fint_B \left(|u|^{q/2} \Psi^{k' q/2p} \right)^{2p/q} \right)^{\frac{1}{p}} \\ &\leq C \left(r^2 \fint_B \left| \nabla \left(|u|^{q/2} \Psi^{k' q/2p} \right) \right|^2 \right)^{\frac{1}{q}} \end{aligned}$$

This is bounded by:

$$\left(r^2 \fint_B |u|^{q-2} |\nabla u|^2 \Psi^{k'q/p} + r^2 \fint_B |u|^q \Psi^{k'q/p-2} |\nabla \Psi|^2 \right)^{\frac{1}{q}}.$$

and now choose k' big enough to use the gradient bounds.
This reverse Hölder can be iterated a finite number of times, replacing the ball $2B$ by sB, where $1 < s < 2$ depending on the q, and with a suitably larger choice of k'.

[FMZ] proves (17) by choosing a cut-off function $\Phi = \Psi\eta_B$, where $\eta_B \in C_0^\infty(2B)$ is s.t. $0 \leq \eta_B \leq 1$, $\eta_B \equiv 1$ on B, $|\nabla \eta_B| \leq C/r$. They take care of a priori finiteness by defining $u_N = \min\{|u|, N\}$ if $q \geq 2$, and $u_N = \max\{|u|, 1/N\}$ if $q < 2$, which guarantees that $u_N^{\frac{q}{2}-1}|u| \in W^{1,2}_{loc}(\Omega)$.

For the case $q \geq 2$, let $E_1 = \{X \in \Omega : |u| \leq N\}$ and $E_2 = \{X \in \Omega : |u| > N\}$. Then

$$\int_\Omega u_N^{q-2} |\nabla u|^2 \Phi^k = \int_{E_1} |u|^{q-2} |\nabla u|^2 \Phi^k + \int_{E_2} N^{q-2} |\nabla u|^2 \Phi^k.$$

By q-ellipticity, with $\chi = \Phi^k \mathbb{1}_{E_1}$, get

$$\mathcal{R}e \int_{E_1} A\nabla u \cdot \nabla[|u|^{q-2}\overline{u}]\, \Phi^k \geq \lambda_q' \int_{E_1} |u|^{q-2} |\nabla u|^2 \Phi^k$$

Using the equation, they arrive at:

$$\int_\Omega u_N^{q-2} |\nabla u|^2 \Phi^k \lesssim \int_\Omega u_N^{q-2} |u|^2 \Phi^{k-2} |\nabla \Phi|^2, \tag{18}$$

By Fatou's lemma, and $u_N^{q-2} \leq |u|^{q-2}$ for both $q \leq 2$ and $q \geq 2$, this gives

$$\begin{aligned}
\int_\Omega |u|^{q-2}|\nabla u|^2 \Phi^k \, dm &\leq \liminf_{N\to\infty} \int_\Omega u_N^{q-2} |\nabla u|^2 \Phi^k \, dm \\
&\lesssim \liminf_{N\to\infty} \int_\Omega u_N^{q-2} |u|^2 \Phi^{k-2} |\nabla \Phi|^2 \, dm \\
&\leq \int_\Omega |u|^q \Phi^{k-2} |\nabla \Phi|^2 \, dm. \qquad (19)
\end{aligned}$$

In summary, [FMZ] plus some small modifications from [DP4] lead to the following boundary result.

Lemma

Let Ω be a chord-arc domain. Let $\mathcal{L} = \partial_i (A_{ij}(x)\partial_j u) + B_i(x)\partial_i$ be a q-elliptic operator. There exists $K = K(n, q, \lambda_q, \|A\|_\infty) > 0$ of the following significance. Suppose that $|B_i(x)| \leq K\delta(x)^{-1}$ for all $x \in \Omega$. Let $u \in W$ be a weak solution to $\mathcal{L}u = 0$ in Ω and B be a ball of radius r centered on $\partial\Omega$ such that $Tr(u) = 0$ on $2B \cap \partial\Omega$. Then,

$$\int_{B\cap\Omega} |u|^{q-2}|\nabla u|^2\, dx \leq \frac{C}{r^2} \int_{(2B\setminus B)\cap\Omega} |u|^q\, dx.$$

And,

$$\left(\frac{1}{|B\cap\Omega|}\int_{B\cap\Omega} |u|^q\, dx\right)^{\frac{1}{q}} \leq C\left(\frac{1}{|2B\cap\Omega|}\int_{2B\cap\Omega} |u|^p\, dx\right)^{\frac{1}{p}} \quad (20)$$

for any $p > 0$. The constant $C > 0$ depends only on n, q, the constant λ_q and $\|A\|_\infty$.

Applications of p-ellipticity

Our applications so far include:

Solvability of the Dirichlet problem for p-elliptic matrices satisfying a Carleson condition. [DP1]

Perturbation theory for p-elliptic operators [DP2]

Regularity problem for p-elliptic matrices satisfying a Carleson condition. [DP3]

Extrapolation of solvability the Dirichlet problem [DP4]

Definition of p-ellipticity for systems (three types of ellipticity) and applications to solving the Dirichlet problem for classical systems of Lamé type, and also Kenig-Shen homogenization type. [DLP]

The Dirichlet Problem

Problem: find solutions to $\mathcal{L}u = 0$ in Ω, with $u(Q) = f(Q)$ for σ-a.e. $Q \in \partial\Omega$, where $f \in L^p(\partial\Omega)$.
Challenges: lack of smoothness of coefficients, how to interpret $u = f$ on the boundary, plus other issues if the coefficients/solution are complex valued.

Problem: find solutions to $\mathcal{L}u = 0$ in Ω, with $u(Q) = f(Q)$ for σ-a.e. $Q \in \partial\Omega$, where $f \in L^p(\partial\Omega)$.
Challenges: lack of smoothness of coefficients, how to interpret $u = f$ on the boundary, plus other issues if the coefficients/solution are complex valued.
Even in the case of Laplace's equation in, say $\mathbb{R}^n_+$, the convergence of u to its measurable boundary values exists in terms of *nontangential* approach regions to the boundary.
When the coefficients are merely bounded, measurable but real, and the matrix A is elliptic, solutions are Hölder continuous and the idea is to get nontangential convergence by controlling $N(u)(Q) := \sup\{u(x) : x \in \gamma_a(Q)\}$ in $L^p(\partial\Omega)$ in terms of $||f||_p$ where $\gamma_a(Q) = \{x \in \Omega : |x - Q| < (1 + a)\delta(x)\}$.

There is an additional consideration if the coefficients are complex - the solutions are not necessarily defined *pointwise*.

We can say the Dirichlet problem is solvable for a given $p \in (1, \infty)$ if there exists a $C = C(p, \Omega) > 0$ such that for all boundary data $f \in L^p(\partial\Omega; \mathbb{C}) \cap \dot{B}^{2,2}_{1/2}(\partial\Omega; \mathbb{C})$ the unique energy solution satisfies the estimate

$$\|\tilde{N}_{2,a}(u)\|_{L^p(\partial\Omega; d\sigma)} \leq C\|f\|_{L^p(\partial\Omega; d\sigma)}, \tag{21}$$

where $d\sigma$ denotes surface measure on the boundary, i.e., the restriction of H^{n-1} to $\partial\Omega$.

Specifically, given $w \in L^2_{loc}(\Omega; \mathbb{C})$ we define

$$\tilde{N}_{2,a}(w)(Q) := \sup_{x \in \Gamma_a(Q)} w_2(x) \tag{22}$$

where, at each $x \in \Omega$,

$$w_2(x) := \left(\fint_{B_{\delta(x)/2}(x)} |w(z)|^2 \, dz \right)^{1/2}. \tag{23}$$

The nontangential approach regions require some modifications when the domain is not as regular as Lipschitz, such as a CAD.

We consider the problem of extrapolation of the Dirichlet problem for complex-valued operators.

Real-valued second order elliptic operators in divergence form satisfy a maximum principle; in the language of nontangential boundary value problems, this implies (trivially) solvability of the Dirichlet problem with data in L^∞ on the boundary of the domain.

This means that solvability of boundary value problems with data in a Lebesgue space $L^q(\partial\Omega)$ *extrapolates*, via interpolation with the endpoint L^∞, to solvability in all L^p, $p \geq q$.

We consider the problem of extrapolation of the Dirichlet problem for complex-valued operators.

Real-valued second order elliptic operators in divergence form satisfy a maximum principle; in the language of nontangential boundary value problems, this implies (trivially) solvability of the Dirichlet problem with data in L^∞ on the boundary of the domain.

This means that solvability of boundary value problems with data in a Lebesgue space $L^q(\partial\Omega)$ *extrapolates*, via interpolation with the endpoint L^∞, to solvability in all L^p, $p \geq q$.

Various techniques can establish solvability for a single value of p: the Kato-type techniques for a special class of complex operators, or the Rellich-type inequalities for real symmetric operators, or methods that apply when coefficients satisfy a Carleson measure condition. But in the complex valued case, there is no maximum principle in general.

Theorem

[DP3] Let Ω be a chord-arc domain in $\mathbb{R}^n$ and $\mathcal{L} = \partial_i (A_{ij}(x)\partial_j) + B_i(x)\partial_i$ be a second order operator with bounded and measurable coefficients A and $|B| \lesssim \delta(x)^{-1}$. Define

$$p_0 = \sup\{p > 1 : A \text{ is } p\text{-elliptic}\}.$$

Assume that the L^q Dirichlet problem is solvable for $\mathcal{L}$ for some $q \in (1, \frac{p_0(n-1)}{(n-2)})$ (if $p_0 = \infty$ or $n = 2$ we require $q \in (1, \infty)$). Then the L^p Dirichlet problem is solvable for $\mathcal{L}$ for p in the range $[q, \frac{p_0(n-1)}{(n-2)})$, if one of the following constraints holds on the size of the vector B.

- *Ω is bounded and $B(x) = o(\delta(x)^{-1})$ as $x \to \partial\Omega$.*
- *Ω is bounded and $\limsup_{x\to\partial\Omega} |B(x)\delta(x)| \leq K$. Here $K = K(A, p, n) > 0$ is sufficiently small.*
- *Ω is unbounded and $|B(x)\delta(x)| \leq K$ for all $x \in \Omega$. Here $K = K(A, p, n) > 0$ is sufficiently small.*

To show how to extrapolate using p-ellipticity, [DM3] makes use of an abstract result [Sh] (and also [Theorem 3.1, WZ] for a version on an arbitrary bounded domains).

Theorem

Let Ω be an open set in $\mathbb{R}^n$ and let T be a bounded sublinear operator on $L^q(\partial\Omega;\mathbb{C}^m)$, $q > 1$. Suppose that for some $p > q$, T satisfies the following L^p localization property. For any ball $\Delta = \Delta_d \subset \partial\Omega$ and C^∞ function f supported in $\partial\Omega \setminus 3\Delta$ the following estimate holds:

$$\left(\fint_{\Delta} |Tf|^p \, dx' \right)^{1/p} \leq \tag{24}$$
$$\lesssim \left\{ \left(\fint_{2\Delta} |Tf|^q \, dx' \right)^{1/q} + \sup_{\Delta' \supset \Delta} \left(\fint_{\Delta'} |f|^q \, dx' \right)^{1/q} \right\},$$

for some $C > 0$ independent of f. Then T is bounded on $L^r(\partial\Omega;\mathbb{C}^m)$ for any $q \leq r < p$.

We use this with T the sublinear operator $f \mapsto \tilde{N}_{2,a}(u)$, and u the solution of the Dirichlet problem $\mathcal{L}u = 0$ with boundary data f. In the statement of the theorem above, the specific enlargement factors $(2\Delta, 3\Delta)$ do not play a significant role. Thus, for $p_0 = \sup\{p > 1 : A \text{ is p-elliptic}\}$, associated to an operator $\mathcal{L}$ with lower order terms B, we prove:

Theorem

For any $p \in (1, \frac{p_0(n-1)}{n-2})$ there exist $K(p) > 0$ such that if $|B_i(x)| \leq K(p)\delta(x)^{-1}$ holds for the first order coefficients of the operator $\mathcal{L}$ then

$$\left(\frac{1}{|\Delta|}\int_\Delta |\tilde{N}_{2,a}(u)|^p\, dx'\right)^{1/p} \leq C\left(\frac{1}{|8m\Delta|}\int_{8m\Delta} |\tilde{N}_{2,a}(u)|^q\, dx'\right)^{1/q}, \tag{25}$$

holds for all $1 \leq q \leq p$. Here u solves $\mathcal{L}u = 0$ in Ω and $f = u\big|_{\partial\Omega}$ vanishes on $16m\Delta$.

The proof begins by breaking up the nontangential maximal function into two parts:

$$\mathcal{M}_1(u)(Q) = \sup_{y \in \Gamma_a(Q)} \{u_2(y) : \delta(y) \leq d\}, \tag{26}$$

$$\mathcal{M}_2(u)(Q) = \sup_{y \in \Gamma_a(Q)} \{u_2(y) : \delta(y) > d\}.$$

Here $d = \mathrm{diam}(\Delta)$ and u_2 is the L^2 average of u

$$u_2(y) = \left(\fint_{B_{\delta(y)/2}(y)} |u(z)|^2 \, dz \right)^{1/2}.$$

$\mathcal{M}_2(u)$ can be shown to have the right estimate (a real variables argument), so we turn to the estimate of $\mathcal{M}_1(u)$.

Let $s \in (p_0', p_0)$ and set $v = |u|^{s/2-1}u$. The expression v_2 denotes the L^2 average over the appropriate interior ball.
Claim: Because u vanishes on $3\Delta \subset m\Delta$, for any $Q \in \Delta$ and for $x \in \Gamma_a(Q)$ with $\delta(x) = h$, and for $P \in C(Q,h) := \{P : x \in \Gamma_a(P),\ h/2 < |P-Q| < h\}$,

$$v_2(x) \lesssim (hd)^{1/2} A_{\tilde{a}}(\nabla v)(P) \tag{27}$$

where

$$A^2_{\tilde{a}}(\nabla v)(P) = d^{-1} \int_{\Gamma^{2d}_{\tilde{a}}(P)} |\nabla v|^2(z)\delta(z)^{1-n} dz.$$

Here, the parameter $\tilde{a}$ is to be determined, and the truncated nontangential corkscrew region is defined to be $\Gamma^{2d}_{\tilde{a}}(P) := \Gamma_{\tilde{a}}(P) \cap B(P, 2d)$.
This also follows from a real variables argument, using Poincaré type inequalities (somewhat more delicate on CADs).

Averaging over $P \in C(Q,h)$, and xince $\sigma(C(Q,h)) \approx h^{n-1}$, for $x \in \Gamma(Q)$:

$$\begin{aligned} v_2(x) &\lesssim h^{3/2-n} d^{1/2} \int_{C(Q,h)} A(\nabla v)(P) d\sigma(P) \\ &\lesssim d^{1/2} \int_{C(Q,h)} \frac{A(\nabla v)(P)}{|P-Q|^{n-3/2}} d\sigma(P) \end{aligned}$$

Because $C(Q,h) \subset 2\Delta$, it follows that

$$\mathcal{M}_1(v_2)(Q) \lesssim d^{1/2} \int_{2\Delta} \frac{A(\nabla v)(P)}{|P-Q|^{n-3/2}} d\sigma(P) \tag{28}$$

By the fractional integral estimate, this implies that

$$\begin{aligned} \left(\fint_{\Delta} [\mathcal{M}_1(v_2)(P)]^r \, d\sigma(P) \right)^{1/r} &\leq Cd \left(\fint_{2\Delta} [A(\nabla v)(P)]^2 \, d\sigma(P) \right)^{1/2} \\ &\leq Cd \left(\fint_{T(m\Delta)} |\nabla v(x)|^2 \, dx \right)^{1/2} \quad (29) \end{aligned}$$

for $\frac{1}{r} = \frac{1}{2} - \frac{1}{2(n-1)}$, and $\{x \in \Gamma_{\tilde{a}}^{2d}(P) : P \in 2\Delta\} \subset T(m\Delta)$.

Now, p-ellipticity will be used.
Since $|\nabla v|^2 = |u|^{s-2}|\nabla u|^2$:

$$\left(\fint_{T(m\Delta)} |\nabla v(x)|^2 \, dx\right)^{1/2} \lesssim d^{-1} \left(\fint_{T(2m\Delta)} |u(x)|^s \, dx\right)^{1/2},$$

whenever the solution $\mathcal{L}u = 0$ vanishes on at least $3m\Delta$.
By Lemma 5, we therefore have that

$$\begin{aligned} \left(\fint_{\Delta} [\mathcal{M}_1(v_2)(P)]^r \, d\sigma(P)\right)^{1/r} &\leq C\left(\fint_{T(2m\Delta)} |u(x)|^s \, dx\right)^{1/2} \\ &\leq C\left(\fint_{T(4m\Delta)} |u(x)|^q \, dx\right)^{s/2q} \end{aligned}$$

This gives the desired estimate in terms of $N_q(u)$, which again is controlled by $N_2(u)$ by the *reverse Hölder on boundary balls*.

In [DP1], we solve a Dirichlet problem for complex elliptic equations. Let $1 < p < \infty$, let $\Omega := \mathbb{R}^n_+ = \{(x_0, x') : x_0 > 0 \text{ and } x' \in \mathbb{R}^{n-1}\}$. Consider the operator

$$\mathcal{L}u = \partial_i \left(A_{ij}(x)\partial_j u\right) + B_i(x)\partial_i u$$

and assume that the matrix A is p-elliptic with constants λ_p, Λ and $\mathcal{I}m\, A_{0j} = 0$ for all $1 \leq j \leq n-1$ and $A_{00} = 1$. Assume that

$$d\mu(x) = \sup_{B_{\delta(x)/2}(x)} \left[|\nabla A(x)|^2 + |B(x)|^2\right] \delta(x)\, dx \qquad (30)$$

is a Carleson measure in Ω. Let us also denote

$$d\mu'(x) = \sup_{B_{\delta(x)/2}(x)} \left[\sum_j |\partial_0 A_{0j}|^2 + \left|\sum_j \partial_j A_{0j}\right|^2 + |B(x)|^2\right] \delta(x)\, dx. \qquad (31)$$

Then there exist $C, K = C, K(\lambda_p, \Lambda, \|\mu\|_{\mathcal{C}}, n, p) > 0$ such that if $\|\mu'\|_{\mathcal{C}} < K$, then the L^p-Dirichlet problem is solvable for $\mathcal{L}$.

An improvement in range of p is given in [DP3]. Further, there is a corollary to this result for matrices A in "block form". It uses the fact that Carleson measure norm condition is needed only on the last row (A_{0j}) of the matrix A, an observation pointed out to us by S. Mayboroda.

Corollary

Suppose the operator $\mathcal{L}$ on $\mathbb{R}^n_+$ has the form

$$\mathcal{L}u = \partial_0^2 u + \sum_{i,j=1}^{n-1} \partial_i (A_{ij} \partial_j u)$$

where the matrix A has coefficients satisfying the Carleson condition (30)*.*
Then for all $1 < p < \infty$ for which A is p-elliptic, the L^p-Dirichlet problem is solvable for $\mathcal{L}$ and the estimate

$$\|\tilde{N}_{p,a} u\|_{L^p(\partial\Omega)} \leq C \|f\|_{L^p(\partial\Omega;\mathbb{C})} \tag{32}$$

Solving boundary value problems

The complex coefficient equation $\mathcal{L}u = \partial_i \left(A_{ij}(x)\partial_j u\right) + B_i(x)\partial_i u$ can be formulated as a skew-symmetric system of equations for the real and imaginary parts of the solution u.

For this reason, the $p = 2$ case of the solvability of the Dirichlet problem would be a consequence of the the L^2 solvability of the analogous Dirichlet problem for (non-symmetric) elliptic systems. Under the (small) Carleson measure assumption, this non-symmetric L^2 Dirichlet problem was solved in Dindos-Hwang-M.Mitrea ([DHM]).

Extrapolation extends the range of L^p-solvability. Importantly, Shen's extrapolation method has a further refinement for those settings where the regularity problem is also solvable. The improvement comes from estimating $\mathcal{M}_1(u)$ in terms of a fractional integral involving the *gradient of* u. This latter estimate requires a boundary RH for gradients, as well as for nontangential maximal functions of gradients.

This was exploited in [DP4] to arrive at the improved estimate:

$$\left(\fint_{\Delta}[\mathcal{M}_1(u)(x')]^p\,dx'\right)^{1/p} \leq C\left(\fint_{3\beta m\Delta}\left[\tilde{N}_{2,a}(u)(z')\right]^2\,dz'\right)^{1/2}$$

where $\frac{1}{p}=\frac{1}{q}-\frac{1}{n-1}$ and $1<q<n-1$ such that A is q-elliptic and Carleson norm of μ is small. When A is q-elliptic for $q\in(p_0,p_0')$ and $p_0'>2$ this implies in dimensions 2 and 3 that we can consider any $2<p<\infty$, while in dimensions $n\geq 4$ we can have $2<p<p_{\max}=p_0'(n-1)/(n-1-p_0')$ when $p_0'<n-1$, $p_{\max}=\infty$ otherwise. Also note that is always the case that $p_{max}>2(n-1)/(n-3)$.

p-elliptic systems

In [Dindoš-J. Li-P.] we investigated p-ellipticity for systems of equations: note that even for *real* elliptic systems, we lack lack the tools of elliptic equations: (weak) maximum principles, Hölder continuity of solutions,etc. The systems have the form:

$$(\mathcal{L}u)_\alpha = \partial_h(A^{hk}_{\alpha\beta}(x)\partial_k u^\beta) + B^h_{\alpha\beta}(x)\partial_h u^\beta, \quad \alpha = 1, 2, \ldots, m; \tag{33}$$

$for\, u : \Omega \to \mathbb{C}^m.$

Several notions of ellipticity for systems: Legendre (strong), integral (to apply Lax-Milgram), and Legendre-Hadamard (weak). We formulate the analogs of these, namely three distinct notions of p-ellipticity, and prove the corresponding relationships which are used to study specific elliptic systems of interest.

In particular, we consider the consequences of p-ellipticity for the variable coefficient Lamé system:

$$\mathcal{L}u = \nabla \cdot (\lambda(x)(\nabla \cdot u)I + \mu(x)(\nabla u + (\nabla u)^T)). \tag{34}$$

in order to extrapolate solvability of the Dirichlet problem from certain instances when the L^2-Dirichlet problem has been established.

For systems of equations, there are at least three different notions of classical ellipticity of the tensor $A = (A^{hk}_{\alpha\beta})$.
The coefficient tensor A is *strongly elliptic* if there exists a $C > 0$ such that

$$\Re e \langle A(x)\xi, \xi \rangle_{\mathbb{C}^{n\times m}} := \Re e \left(A^{hk}_{\alpha\beta}(x) \xi^{\alpha}_{h} \overline{\xi^{\beta}_{k}} \right) \geq C|\xi|^2 \qquad (35)$$

for all $\xi = (\xi^h_\alpha) \in \mathbb{C}^{n\times m}$ and a.e. $x \in \Omega$. Strong ellipticity is traditionally referred to as the **Legendre** condition.

It follows from strong ellipticity via integration:

$$\Re e \int_\Omega \langle A\nabla v, \nabla v\rangle \, dx = \Re e \int_\Omega A^{hk}_{\alpha\beta}(x)\partial_k v^\beta \overline{\partial_h v^\alpha} \, dx \geq C \int_\Omega |\nabla v|^2 dx. \tag{36}$$

This *integral condition* of ellipticity is the starting point for the *Lax-Milgram* lemma which allows one to find weak solutions. The **weakest** form of ellipticity is the **Legendre-Hadamard** condition:

$$\Re e \left\langle (A^{hk}(x) q_h q_k)\xi, \xi \right\rangle = \Re e \left(A^{hk}_{\alpha\beta}(x) \xi^\alpha \overline{\xi^\beta} q_h q_k \right) \geq \lambda |\xi|^2 |q|^2 \tag{37}$$

for all $\xi = (\xi^\alpha) \in \mathbb{C}^m$, $q = (q_h) \in \mathbb{R}^n$, and a.e. $x \in \Omega$.

It is always the case that (36) implies (37), and all three conditions are equivalent when the operator is scalar ($m = 1$) and real-valued. Furthermore, if the coefficients of A are uniformly continuous (or, if Ω is a bounded domain, only need $A^{hk}_{\alpha\beta} \in C(\overline{\Omega})$), then ([Yan, B.]) condition (37) implies something similar to (36), namely a Gårding-type integral inequality

$$\Re e \int_\Omega \langle A\nabla v, \nabla v\rangle \, dx + M \int_\Omega |v|^2 dx \geq C \int_\Omega |\nabla v|^2 dx, \qquad (38)$$

for all $v \in W^{1,2}_0(\Omega, \mathbb{C}^m)$ and sufficiently large $M = M(A, \Omega) > 0$.

We formulate these three notions of p-ellipticity, and relate them to one another. In the following theorem, $\Omega \subset \mathbb{R}^n$ is open and $A = (A^{hk}_{\alpha\beta}(x)) : \Omega \to \mathbb{C}^{n\times m}$ is a bounded tensor-valued function.

Theorem

(i) *For any $p \in (1, \infty)$ the strong p-ellipticity condition implies the integral p-ellipticity condition.*

(ii) *For any $p \in (1, \infty)$ the integral ellipticity condition implies the weak p-ellipticity condition.*

(iii) *When $p = 2$ all of these coincide with the classical Legendre, integral and Legendre-Hadamard conditions.*

(iv) *Given any $p \in (1, \infty)$ if A satisfies one of these p-ellipticity conditions, then A^* satisfies the same p'-ellipticity condition. If A satisfies the Legendre-Hadamard p-ellipticity condition for some $p \in (1, \infty)$ then it also satisfies it for $p = 2$.*

(v) *If A satisfies any one of these p-ellipticity conditions for a given p and q in $(1, \infty)$, then A satisfies the same condition for any r from an open interval containing the points p and q.*

The integral p-ellipticity condition is important in two ways. First, it leads to higher integrability of weak solutions as in the scalar case. Second, it is used to prove extrapolation of the Dirichlet problem, just as in the scalar case.

These general results are applied to two well studied cases: the Lamé equations, and real elliptic systems of equations with rapidly oscillating periodic coefficients (homogenization).

Under certain natural conditions on the Lamé coefficients, the L^2 Dirichlet problem was shown to be solvable in [DHM]. In light of the extrapolation results, it is of interest to consider what further conditions on the coefficients give rise to p-ellipticity, for $p > 2$. For this investigation, we see the significance of using the pointwise p-ellipticity conditions, one of which produces "necessary conditions" on the Lamé coefficients, while the other gives "sufficient conditions".

Definition

The tensor-valued function $A^{hk}_{\alpha\beta}(x)$ satisfies the **strong** p-**ellipticity** condition if the following pointwise condition holds.

$$\Re e \left\langle A(x)\left(\xi - \left(1-\frac{2}{p}\right)\xi(\omega)\right), \xi + \left(1-\frac{2}{p}\right)\xi(\omega)\right\rangle_{\mathbb{C}^{n\times m}} \geq C|\xi|^2,$$

for all $\xi = (\xi^\alpha_h) \in \mathbb{C}^{n\times m}$, $\omega \in \mathbb{C}^m$ with $|\omega| = 1$ and a.e. $x \in \Omega$. For a tensor $\xi \in \mathbb{C}^{n\times m}$ and $\omega \in \mathbb{C}^m$, the expression $\xi(\omega)$ denotes an element of $\mathbb{C}^{n\times m}$ defined by

$$\xi(\omega)^\alpha_h = \omega^\alpha \Re e(\omega^\beta \overline{\xi^\beta_h}) = \omega^\alpha(\Re e\langle\omega, \xi_h\rangle). \tag{39}$$

When $p = 2$ this coincides with the usual strong (Legendre) ellipticity.

Consider a function $v \in W_0^{1,2}(\Omega, \mathbb{C}^m)$. In the definition of strong p-ellipticity, take $\xi_h^\alpha = \partial_h v^\alpha$ and choose $\omega = \frac{v}{|v|}$. It follows that

$$\xi(\omega)_h^\alpha = \frac{v^\alpha}{|v|^2} \Re e \langle v, \partial_h v \rangle \quad \text{i.e.,} \quad \xi(\omega) = \frac{v}{|v|} \nabla |v|.$$

Hence if (39) holds, integrating over Ω gives that for some $C > 0$,

$$\Re e \int_\Omega \left\langle A \left(\nabla v - \left(1 - \frac{2}{p}\right) \frac{v}{|v|} \nabla |v| \right), \nabla v + \left(1 - \frac{2}{p}\right) \frac{v}{|v|} \nabla |v| \right\rangle dx \tag{40}$$

$$\geq C \int_\Omega |\nabla v|^2 \, dx,$$

for all $v \in W_0^{1,2}(\Omega, \mathbb{C}^m)$.

Definition

$A_{\alpha\beta}^{hk}(x)$ satisfies the **integral p-ellipticity** condition if (40) holds.

We note that (40) is closely related to the notion of L^p-dissipativity of second order operators as defined by Cialdea and Maz'ya ([CMSystems]). L^p-dissipativity corresponds to having $C = 0$ on the righthand side.
The weakest form of p-ellipticity, analogous to the Legendre-Hadamard condition, also comes from generalizing the two dimensional condition formulated in [CM] which they showed to be necessary for L^p-dissipativity.
The trickiest relationship to establish is that the (40) condition implies the Legendre-Hadamard. Although we borrow heavily from ideas of [CM], we have a new twist, also improving on some of their results regarding L^p-dissipativity.

With $\beta(p,\eta,\omega) := \left(1-\frac{2}{p}\right)\omega(\Re e\langle\omega,\eta\rangle)$

Definition

$A^{hk}_{\alpha\beta}(x)$ satisfies the **Legendre-Hadamard** ***p*-ellipticity** condition if

$$\Re e\left\langle \left(A^{hk}(x)q_h q_k\right)\left(\eta-\beta(p,\eta,\omega)\right), \eta+\beta(p,\eta,\omega)\right\rangle \geq C|\eta|^2|q|^2, \tag{41}$$

for all $\eta,\omega\in\mathbb{C}^m$, $q\in\mathbb{R}^n$ with $|\omega|=1$, and a.e. $x\in\Omega$.

In [DHM], under certain assumptions on the Lamé coefficients $\mu(x)$ and $\lambda(x)$, they solved an L^p-Dirichlet problem for $p \in (2-\epsilon, \frac{2(n-1)}{n-2}+\epsilon)$.Their proof of solvability of the Dirichlet problem relies on the fact that the Lamé equation can be written as (33) in multiple ways. For an auxiliary function $r(x) \in \mathrm{Lip}_{loc}(\Omega)$, it can also be written:

$$\mathcal{L}(u) = \partial_h(A^{hk}_{\alpha\beta}(r)\partial_k u^\beta) + B^h_{\alpha\beta}(r)\partial_h u^\beta, \tag{42}$$

where

$$A^{hk}_{\alpha\beta}(r)(x) = \mu(x)\delta^{hk}\delta_{\alpha\beta} + (\lambda(x)+r(x))\delta^h_\alpha\delta^k_\beta + (\mu(x)-r(x))\delta^h_\beta\delta^k_\alpha, \tag{43}$$

$$B^h_{\alpha\beta}(r)(x) = \partial_k r(x)(\delta^h_\alpha\delta^k_\beta - \delta^h_\beta\delta^k_\alpha).$$

We use arguments similar to that of the scalar case to prove an extrapolation theorem for systems to study this equation and the system of homogenization studied by Kenig-Shen.
For operators in the form,

$$(\mathcal{L}u)_\alpha = \partial_h(A^{hk}_{\alpha\beta}(x)\partial_k u^\beta) + B^h_{\alpha\beta}(x)\partial_h u^\beta, \quad \alpha = 1, 2, \dots, m, \tag{44}$$

where the vector B satisfies one of the following ($\delta(x)$ is the distance of x to the boundary of Ω).

- Ω is bounded and $B(x) = o(\delta(x)^{-1})$ as $x \to \partial\Omega$.
- Ω is bounded and $\limsup_{x\to\partial\Omega} |B(x)\delta(x)| \leq K$. Here $K = K(A, p, n) > 0$ is sufficiently small.
- Ω is unbounded and $|B(x)\delta(x)| \leq K$ for all $x \in \Omega$. Here $K = K(A, p, n) > 0$ is sufficiently small.

we have the following theorem.

Theorem (**Extrapolation**)

[DLP] Let $\Omega \subset \mathbb{R}^n$ be a bounded or unbounded Lipschitz domain. Let

$$(\mathcal{L}u)_\alpha = \partial_h(A^{hk}_{\alpha\beta}(x)\partial_k u^\beta) + B^h_{\alpha\beta}(x)\partial_h u^\beta, \quad \alpha = 1, 2, \ldots, m; \tag{45}$$

be a second order operator with bounded and measurable coefficients A and $|B| \lesssim \delta(x)^{-1}$, where A satisfies (36). Define

$$p_0 = \sup\{p : A \text{ satisfies condition } (40)\}.$$

Assume that the L^q Dirichlet problem is solvable for $\mathcal{L}$ for some $q \in (1, \frac{p_0(n-1)}{(n-2)})$ (if $p_0 = \infty$ or $n = 2$ we require $q \in (1, \infty)$). Then the L^p Dirichlet problem is solvable for $\mathcal{L}$ for p in the range $[q, \frac{p_0(n-1)}{(n-2)})$, if one of the above constraints holds on the size of the vector B.

Remarks.

The symmetry, $A^{hk}_{\alpha\beta}(r) = A^{kh}_{\beta\alpha}(r)$, will simplify the p-ellipticity condition.

With respect to the classical ellipticity conditions ($p = 2$), it is the case that $A(r)$ satisfies the Legendre-Hadamard property when $\mu > 0$ and $\lambda > -2\mu$ [BM]. And, $A(r)$ is strongly elliptic if, in addition, $0 < r < 2\mu$ and $\lambda + 2\mu - r \geq 0$. Strong ellipticity may also hold even when $\lambda + 2\mu - r \leq 0$ depending on the relationship between r and μ. ([BM09]).

Aim: understand when the system (42) satisfies the integral p-ellipticity condition. It's hard to check the integral condition directly. Instead, aim to find the largest p-interval in which strong p-ellipticity (39) holds.

The computations start by noting that $A(r)$ is real and $A^{hk}_{\alpha\beta}(r) = A^{kh}_{\beta\alpha}(r)$ the cross-terms (containing both ξ and $\xi(\omega)$) cancel out and we get:

$$\inf_{|\omega|=1} \left[\langle A(r)(x)\xi, \xi \rangle_{\mathbb{R}^{n\times n}} - \left(1 - \frac{2}{p}\right)^2 \langle A(r)(x)\xi(\omega), \xi(\omega) \rangle_{\mathbb{R}^{n\times n}} \right] \geq C|\xi|^2. \tag{46}$$

After *some computations**, we derive *sufficient* conditions on the parameters λ, μ so that with a choice of r the equation (43) has coefficients $A(r)$ that are p-elliptic in an interval that depends on dimension and those parameters.

Turning to *necessary* conditions, we make use of the weak (Legendre-Hadamard) p-ellipticity and find that

$$\mu|\xi|^2|q|^2 + (\lambda+\mu)\langle q, \xi\rangle^2 - \left(1 - \frac{2}{p}\right)^2 \langle \omega, \xi\rangle^2 \left[\mu|q|^2 + (\lambda+\mu)\langle q, \omega\rangle^2\right] \\ \geq C|\xi|^2|q|^2,$$

which does not depend on the choice of r in (43).

* *...actually, a lot of computations*

The coefficients $A(r)$ of the operator $\mathcal{L}'$ satisfy the integral p-ellipticity condition (40) if and only if

$$\left(1-\frac{2}{p}\right)^2 < C(n,\lambda,\mu), \tag{47}$$

where

$$C(2,\lambda,\mu) = 1 - \operatorname{ess}\sup_{x\in\Omega}\left(\frac{\lambda+\mu}{\lambda+3\mu}\right)^2, \quad \text{for } n=2, \tag{48}$$

and for $n \geq 3$ we have estimates for $C(n,\lambda,\mu)$ from above and below by

$$C(n,\lambda,\mu) \leq 1 - \operatorname{ess}\sup_{x\in\Omega}\left(\frac{\lambda+\mu}{\lambda+3\mu}\right)^2, \tag{49}$$

and

$$C(n,\lambda,\mu) \geq 1 - \operatorname{ess}\sup_{x\in\Omega}\left(\frac{\lambda+\mu}{\max\{\mu,\lambda+2\mu\}}\right)^2. \tag{50}$$

From [D], the L^2 Dirichlet and regularity problems for the Lamé system are solvable under the assumption that λ, μ satisfy a certain Carleson measure with small constants, under the ellipticity assumption

$$\operatorname{ess}\inf_{x\in\Omega}\{(\sqrt{8}-1)\mu(x)+\lambda(x),(\sqrt{8}+1)\mu(x)-\lambda(x)\}\geq\mu_0. \quad (51)$$

the condition (51) implies that

$$1-\sqrt{8}<\lambda/\mu<1+\sqrt{8},$$

which implies a particular lower bound on the value of $C(n,\lambda,\mu)$. In particular, it implies that the solvability range $p\in(2-\varepsilon,p(n))$ for the L^p Dirichlet problem is at least:

$$p(2)=\infty, \qquad p(3)>11.50, \quad p(4)>8.055$$

and in general

$$p(n)>\frac{2(n-1)}{(n-2)(1-\sqrt{8\sqrt{2}-11})}\approx\frac{4.546(n-1)}{n-2}.$$

Many thanks to the organizers for a wonderful and inspiring conference in El Escorial, and for the opportunity to share these ideas with the nearly 200 participants here.

Some selected references follow.

A Selection of References

Please see the papers below for a comprehensive list of references.

[CD] Carbonaro - Dragičević, Convexity of power functions and bilinear embedding for divergence-form operators with complex coefficients, J. of EMS, Volume 22, Issue 10, 2020, 3175–3221.

[CMequations] Cialdea-Maz'ya, Criterion for the L^p-dissipativity of second order differential operators with complex coefficients, J. Math. Pures Appl., 84(9):1067–1100, 2005.

[CMsystems] Cialdea-Maz'ya, Criteria for the L^p-dissipativity of systems of second order differential equations, Ric. Mat., 55, 2006, 233–265.

[D] Dindoš, The L^p Dirichlet and Regularity problems for second order elliptic systems with application to the Lamé system, Comm. in PDE, 2021,

[DHM] Dindoš-Hwang-Mitrea, The L^p Dirichlet boundary problem for second order Elliptic Systems with rough coefficients,Trans. AMS, 374 2021, no. 5, 3659–3701.

[DP1] Dindoš-Pipher, Regularity of solutions to divergence form elliptic equations with complex coefficients, Adv. in Math. Vol. 341, 2018, 255-298,

[DP2] Dindoš-Pipher, Perturbation of p-elliptic complex coefficient operators, with M. Dindos, Acta Math. Sinica, English series, 2019.

[DP3] Dindoš-Pipher, Boundary value problems for second order elliptic operators with complex coefficients, APDE, Vol. 13, No. 6, 2020, 1897–1938

[DP4] Dindoš-Pipher, Extrapolation of the Dirichlet problem for elliptic equations with complex coefficients, J.Functional Analysis, Vol. 279, Issue 7, 2020

[DLP] Dindoš-J. Li-Pipher, The *p*-ellipticity condition for second order elliptic systems and applications to the Lamé systems and homogenization, J. Differential Equations, Vol. 302, (25), 2021, p. 367-405

[FMZ] Feneuil, S. Mayboroda, Z. Zhao, Dirichlet problem in domains with lower dimensional boundaries, Rev. Mat. Iberoam. 37, 2021, no. 3, 821–910.

[Shen] Z. Shen. Bounds of Riesz transforms on L^p spaces for second order elliptic operators, Ann. Inst. Fourier (Grenoble), 55(1):173–197, 2005.