HARMONIC ANALYSIS AND REPRESENTATION THEORY LECTURES GIVEN AT EL ESCORIAL, JUNE 2022

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1. SL$(2,\mathbb{R})$ and its representations

1.1. **Introduction.** I begin by thanking the organisers of this conference, who offered me the opportunity to speak here.

Harmonic analysts often analyse and synthesise functions on spaces with natural symmetries, such as translations or rotations; we often try to decompose in terms of functions which behave nicely relative to these symmetries. For instance, complex exponentials behave nicely under translations, and spherical harmonics behave nicely under rotations. Group representations has two parts. One is about understanding the functional analysis that underlies Fourier analysis and spherical harmonic expansions and similar expansions. The other is very specific, and is about the particular kind of special functions that arise in the study of particular symmetries. We will see a little of both. We will also see connections with fractional integrals, with van der Corput's lemma, and with partial differential equations.

These three lectures will focus on:

- unitary representations of $G := \mathrm{SL}(2,\mathbb{R})$;
- matrix coefficient decay and the Kunze–Stein phenomenon;
- a proof and some open problems.

The new content of these lectures is based on four papers: [ACD18, ACD19, BCNT, C].

Harmonic analysis on the group $\mathrm{SL}(2,\mathbb{R})$, often abbreviated to G here, has a long history. Bargmann [Ba47] described its representations and Plancherel formula. Ehrenpreis and Mautner [EM55a, EM55b] used it to clarify various conjectures in abstract harmonic analysis (though the *similarity problem* is still unresolved). Kunze and Stein [KS60] developed the work of Ehrenpreis and Mautner and proved the convolution inclusion

$$\mathsf{L}^p(G) * \mathsf{L}^2(G) \subseteq \mathsf{L}^2(G).$$

In recent times, this has been used in the study of dispersive equations on spaces of nonnegative curvature, such as hyperbolic spaces.

In the next two lectures we are going to use harmonic analysis on G to prove dispersive estimates for functions on $\mathbb{R}$ and on $\mathbb{R}^n$.

For those unfamiliar with noncommutative harmonic analysis, I point out that the definitions and many of the basic theorems are similar to those in euclidean space. For instance, we define convolution:

$$f_1 * f_2(x) = \int_G f_1(y)\, f_2(y^{-1}x)\,\mathrm{d}y.$$

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Actually, $\mathsf{L}^1(G)$ is a Banach star-algebra, when we define

$$f^*(x) = \bar{f}(x^{-1}).$$

1.2. **Structure of** $\mathrm{SL}(2,\mathbb{R})$**.** We now describe $\mathrm{SL}(2,\mathbb{R})$, abbreviated to G for convenience, and various decompositions and representations thereof. We adopt an approach learnt from R. A. Kunze in a course in Perugia many years ago, which may be found in Astengo, Cowling and Di Blasio [ACD18]. First, define subgroups K, M, A, N and $\bar{N}$ of G as follows:

$$K = \{k_\theta : \theta \in \mathbb{R}\} \qquad M = \{m_\pm\}$$
$$A = \left\{a_s : s \in \mathbb{R}^+\right\} \qquad N = \{n_b : b \in \mathbb{R}\} \qquad \bar{N} = \{\bar{n}_c : c \in \mathbb{R}\},$$

where

$$k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad m_\pm = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$$
$$a_s = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \qquad n_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \qquad \bar{n}_c = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}.$$

We write integrals in two ways, for instance,

$$\int_K u(k)\,\mathrm{d}k \qquad \text{or} \qquad \int_{-\pi}^{\pi} u(k_\theta)\,\mathrm{d}\theta.$$

There are a number of standard decompositions of G. The Iwasawa decomposition (which is based on the Gram–Schmidt process, and is called the QR decomposition by applied mathematicians and engineers) asserts that every element x of G may be expressed uniquely in the form $x = nak$, where $n \in N$, $a \in A$ and $k \in K$.

Another important decomposition, the Cartan decomposition, states that every element of G may be written in the form $k_\theta a_s k_\phi$, where $s \geq 1$; if $x \in K$ then $s = 1$ and there are many choices for k_θ and k_ϕ; otherwise, the decomposition is unique up to changes in θ and ϕ by adding (or subtracting) π to both or 2π to either. This result is derived using the polar decomposition of a matrix. The Haar measure of G may be written in polar coordinates:

$$\int_G u(x)\,\mathrm{d}x = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_0^\infty u(k_\theta a_s k_\phi)\sinh(r)\,\mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}s.$$

This formula gives rise to a similar formula for the measure in hyperbolic space, that implies that the measure of the ball of radius s grows exponentially in s as $s \to \infty$.

Bargmann [Ba47] identified four types of irreducible unitary representations of G, called the principal series, the discrete series, the complementary series, and the trivial representation. We describe these below. Amongst many other things, Bargmann found explicit formulae in terms of special functions for the "generalised spherical functions", that is the matrix coefficients $\langle \rho(\cdot)f, f'\rangle$ of these representations ρ when the vectors f and f' are particular normalised vectors that transform by scalars under the action of K, that is, $\rho(k_\phi)f = \mathrm{e}^{in\phi}f$ and $\rho(k_\theta)f' = \mathrm{e}^{im\theta}f'$. These led to asymptotic formulae of the form

$$\langle \rho(k_\theta a_r k_\phi)f, f'\rangle \asymp C(\rho, m, n)\exp(-\lambda r) \qquad \text{as } r \to +\infty,$$

or of a slightly more general form; the parameter λ depends on which representation is under consideration. These formulae were instrumental in Bargmann's proof of the Plancherel formula, which involves the representations of the principal and discrete series only.

1.3. **Action of** $\mathrm{SL}(2,\mathbb{R})$ **on homogeneous functions on** $\mathbb{R}^2$**.** Consider $\mathbb{R}^2$ as a space of row vectors, and let G act on $\mathbb{R}^2$ by right multiplication. Then G fixes the origin and acts transitively on $\mathbb{R}^2 \setminus \{(0,0)\}$. Further, if

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$$

then $(1,0)x = (1,0)$ if and only if $a = 1$ and $b = 0$; equivalently,

$$x = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \in \bar{N}.$$

Consequently we may identify $\bar{N}\backslash G$ with $\mathbb{R}^2 \setminus \{0\}$.

Fix $\zeta \in \mathbb{C}$ and $\epsilon \in \{0,1\}$. Consider the space $\mathsf{V}_{\zeta,\epsilon}$ of smooth functions on $\mathbb{R}^2 \setminus \{(0,0)\}$ that satisfy the homogeneity and parity conditions

$$f(\delta v) = \operatorname{sgn}^{\epsilon}(\delta)\, |\delta|^{2\zeta-1} f(v) \qquad \forall v \in \mathbb{R}^2 \setminus \{(0,0)\} \quad \forall \delta \in \mathbb{R} \setminus \{0\}. \tag{1}$$

We equip $\mathsf{V}_{\zeta,\epsilon}$ with the topology of locally uniform convergence of all partial derivatives. The partial derivatives may be taken in polar coordinates; for homogeneous functions, the radial derivatives are easy to deal with, and questions of convergence boil down to the behaviour of the angular derivatives on the unit circle.

Since G acts on $\mathbb{R}^2 \setminus \{(0,0)\}$ and commutes with scalar multiplication, G acts on $\mathsf{V}_{\zeta,\epsilon}$ by the formula

$$\rho_{\zeta,\epsilon}(x) f(v) = f(vx) \qquad \forall v \in \mathbb{R}^2 \setminus \{(0,0)\} \quad \forall x \in G.$$

This is a *group representation*, that is,

$$\rho_{\zeta,\epsilon}(x) = \rho_{\zeta,\epsilon}(x)\rho_{\zeta,\epsilon}(x) \qquad \text{and} \qquad \rho_{\zeta,\epsilon}(e) = I,$$

where e is the group identity and I the identity operator.

We obtain the "compact picture" of the representation by restricting v to lie on the unit circle $\{(s,t) \in \mathbb{R}^2 : s^2 + t^2 = 1\}$, and observing that

$$\rho_{\zeta,\epsilon}(x) f(v) = |vx|^{2\zeta-1} f(|vx|^{-1}\, vx).$$

Similarly, we obtain the "noncompact picture" by restricting v to lie on the vertical line $\{(1,t) : t \in \mathbb{R}\}$, and observing that

$$\rho_{\zeta,\epsilon}(x) f(1,t) = f((1,t)x) = f(a+tc, b+td) = \operatorname{sgn}^{\epsilon}(a+tc)\, |a+tc|^{2\zeta-1} f(1, x\cdot t),$$

where

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{and} \qquad x \cdot t = \frac{b+dt}{a+ct}.$$

Clearly a little care is required "at infinity" in this version of the representation.

The space $\mathsf{V}_{\zeta,\epsilon}$ is spanned (topologically) by functions whose restrictions to the unit circle are complex exponentials. Let $f_{\zeta,\mu}$ be the function in $\mathsf{V}_{\zeta,\epsilon}$ determined by the condition that

$$f_{\zeta,\mu}(\cos\theta, \sin\theta) = \frac{1}{\pi^{1/2}} \mathrm{e}^{i\mu\theta} \qquad \forall \theta \in \mathbb{R}, \tag{2}$$

with the restriction that $\mu - \epsilon$ must be even; every function in $\mathsf{V}_{\zeta,\epsilon}$ is the limit of finite linear combinations of the functions $f_{\zeta,\mu}$. It is easy to check that

$$f_{\zeta,\mu}(1,t) = \frac{1}{\pi^{1/2}} (1+it)^{(2\zeta-1+\mu)/2} (1-it)^{(2\zeta-1-\mu)/2}, \tag{3}$$

and so in particular, $f_{\zeta,-\mu}(1,t) = f_{\zeta,\mu}(1,-t)$ for all $t \in \mathbb{R}$.

Lemma 1.1. *Every function in $\mathsf{V}_{\zeta,\epsilon}$ is a sum $\sum_{\mu\in 2\mathbb{Z}+\epsilon} a_\mu\, f_{\zeta,\mu}$, where the coefficients a_μ tend to zero faster than any polynomial in μ. Conversely, every such sum is a function in $\mathsf{V}_{\zeta,\epsilon}$.*

Proof. Every function in $\mathsf{V}_{\zeta,\epsilon}$ is determined by its restriction to the unit circle, and the Fourier series of smooth functions on the unit circle are exactly of this form. □

Lemma 1.2. *Suppose that $f \in \mathsf{V}_{\zeta,\epsilon}$ and $g \in \mathsf{V}_{-\zeta,\epsilon}$. Then*

$$\begin{aligned}\int_{-\infty}^{\infty} f(1,t)\, g(1,t)\, \mathrm{d}t &= \int_{-\pi/2}^{\pi/2} f(\cos\theta,\sin\theta)\, g(\cos\theta,\sin\theta)\, \mathrm{d}\theta \\ &= \frac{1}{2}\int_{-\pi}^{\pi} f(\cos\theta,\sin\theta)\, g(\cos\theta,\sin\theta)\, \mathrm{d}\theta.R\end{aligned}$$

We define the pairing (f,g) to be any of the above integrals, then

$$(\rho_{\zeta,\epsilon}(x)f, \rho_{-\zeta,\epsilon}(x)g) = (f,g) \qquad \forall x \in G, \tag{4}$$

or equivalently,

$$\rho_{-\zeta,\epsilon}(x)^\top = \rho_{\zeta,\epsilon}(x^{-1}) \qquad \forall x \in G,$$

where $\rho_{-\zeta,\epsilon}(x)^\top$ denotes the transpose of $\rho_{-\zeta,\epsilon}(x)$.

Proof. The first integral is equal to the second by the change of variables $t = \tan(\theta)$, and the second is equal to the third because fg is even.

It is obvious that (4) holds when $x \in NA$ from a change of variable in the integral over $\mathbb{R}$ and when $x \in K$ from a change of variable in the integral over $[-\pi,\pi]$. By the Iwasawa decomposition, $G = NAK$, and so (4) holds for all $x \in G$. This implies that

$$(\rho_{-\zeta,\epsilon}(x)^\top \rho_{\zeta,\epsilon}(x)f, g) = (f,g) \qquad \forall x \in G.$$

The bilinear form $(\cdot,\cdot)$ gives us a duality between $\mathsf{V}_{\zeta,\epsilon}$ and $\mathsf{V}_{-\zeta,\epsilon}$. While $\mathsf{V}_{-\zeta,\epsilon}$ is smaller than the topological dual space of $\mathsf{V}_{\zeta,\epsilon}$, it is weak-star dense in the dual space, and the set $\{(f,g) : g \in \mathsf{V}_{-\zeta,\epsilon}\}$ determines f in $\mathsf{V}_{\zeta,\epsilon}$. Thus $\rho_{-\zeta,\epsilon}(x)^\top\rho_{\zeta,\epsilon}(x)$ is the identity operator on $\mathsf{V}_{\zeta,\epsilon}$, and hence $\rho_{-\zeta,\epsilon}(x)^\top = \rho_{\zeta,\epsilon}(x^{-1})$. □

Note that our normalisation of $f_{\zeta,\mu}$ means that $(f_{\zeta,\mu}, f_{-\zeta,\nu})$ is equal to 1 if $\mu + \nu = 0$ and to 0 otherwise.

For future use, we note that there is a family of functions ϕ_ζ on G, the *zonal spherical functions*, associated to the representations $\rho_{\zeta,0}$, which are left- and right-K-invariant, and on A are given by the formula

$$\begin{aligned}\phi_\zeta(a_r) &= (\rho_{\zeta,0}(a_r) f_{\zeta,0}, f_{-\zeta,0}) \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} f_{\zeta,0}((1,t)a_r)\, f_{-\zeta,0}(1,t)\,\mathrm{d}t \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} f_{\zeta,0}(e^r, e^{-r}t)\, f_{-\zeta,0}(1,t)\,\mathrm{d}t \\ &= \frac{e^{(2\zeta-1)}}{\pi}\int_{-\infty}^{\infty} f_{\zeta,0}(1, e^{-2r}t)\, f_{-\zeta,0}(1,t)\,\mathrm{d}t \\ &= \frac{e^{(2\zeta-1)r}}{\pi}\int_{-\infty}^{\infty} \frac{1}{(1+\mathrm{e}^{-4r}t^2)^{\zeta-1/2}}\,\frac{1}{(1+t^2)^{-\zeta-1/2}}\,\mathrm{d}t.\end{aligned} \tag{5}$$

These functions depend analytically on ζ and satisfy the identity $\phi_\zeta = \phi_{-\zeta}$. As shown by Bargmann [Ba47], it is possible to find series expansions for the functions $\phi_\zeta(a_r)$ when r is close to ∞. More precisely, $\phi_\zeta(a_r)$ is the sum of two terms, which are asymptotic to

$$\mathbf{c}(\zeta)e^{2\zeta-1} \qquad \text{and} \qquad \mathbf{c}(-\zeta)e^{-2\zeta-1}.$$

The remainder terms involve positive integer powers of e^{-r}, and $\mathbf{c}$ is a "known" function.

When $\mathrm{Re}(\zeta)$ is positive, only one term is significant, while if $\mathrm{Re}(\zeta) = 0$, both terms are important. At 0, $\mathbf{c}(\zeta)$ behaves like C/ζ, and by l'Hôpital's rule, $\phi_0(a_r)$ behaves like $4Cre^{-r}$.

1.4. **The principal series.**

Corollary 1.3. *Suppose that $\eta \in \mathbb{R}$. The operators $\rho_{i\eta,\epsilon}(x)$ are unitary for all $x \in G$ when $\mathsf{V}_{i\eta,\epsilon}$ is equipped with the inner product $\langle f, f'\rangle := (f, \bar{f}')$ and completed to a Hilbert space.*

Proof. Evidently, $\langle\cdot,\cdot\rangle$ is an inner product on $\mathsf{V}_{i\eta,\epsilon}$. Further, $(\rho_{i\eta,\epsilon}f')^- = \rho_{-i\eta,\epsilon}\bar{f}'$, and so

$$\langle \rho_{i\eta,\epsilon}(x)f, \rho_{i\eta,\epsilon}(x)f'\rangle = (\rho_{i\eta,\epsilon}(x)f, \rho_{-i\eta,\epsilon}(x)\bar{f}') = (f,\bar{f}') = \langle f, f'\rangle\,,$$

and each $\rho_{i\eta,\epsilon}(x)$ is unitary. □

This Hilbert space may be identified with the standard L^2 space on the unit circle or on the line $\{(1,t) : t \in \mathbb{R}\}$. We may show similarly that the representations $\rho_{\zeta,\epsilon}$ act isometrically on L^p spaces when $p(\mathrm{Re}(2\zeta) - 1) = 2$; see [ACD18].

1.5. **The complementary series.**

Lemma 1.4. *Suppose that f and f' are continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$ and satisfy the homogeneity condition* (1)*, where $-1/2 < \mathrm{Re}\,\zeta < 0$. Then*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(1,t)\, f'(1,u)\,\mathrm{sgn}^\epsilon(u-t)\,|u-t|^{-1-2\zeta}\;\mathrm{d}t\,\mathrm{d}u \tag{6}$$
$$= \int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} f(\cos\theta,\sin\theta)\, f'(\cos\phi,\sin\phi)\,\mathrm{sgn}^\epsilon(\sin(\phi-\theta))\,|\sin(\phi-\theta)|^{-1-2\zeta}\;\mathrm{d}\phi\,\mathrm{d}\theta.$$

If moreover f and f' satisfy the condition (1)*, then these double integrals are both equal to*

$$\frac{1}{4}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(\cos\theta,\sin\theta)\, f'(\cos\phi,\sin\phi)\,\mathrm{sgn}^\epsilon(\sin(\phi-\theta))\,|\sin(\phi-\theta)|^{-1-2\zeta}\;\mathrm{d}\phi\,\mathrm{d}\theta.$$

For $f, f' \in \mathsf{V}_{\zeta,\epsilon}$, write $(I_{\zeta,\epsilon} f, f')$ for any of the three double integrals above. Then

$$(I_{\zeta,\epsilon}\rho_{\zeta,\epsilon}(x)f, \rho_{\zeta,\epsilon}(x)f') = (I_{\zeta,\epsilon} f, f') \qquad \forall x \in G. \tag{7}$$

Proof. Since

$$(\tan\theta - \tan\phi)\cos\theta\cos\phi = \sin(\theta - \phi),$$

we see that $\operatorname{sgn}(\tan\phi - \tan\theta) = \operatorname{sgn}(\sin(\phi-\theta))$ when $\theta, \phi \in (-\pi/2, \pi/2)$, and hence

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(1,t)\, f'(1,u) \operatorname{sgn}^{\epsilon}(u-t)\, |u-t|^{-1-2\zeta}\, \mathrm{d}t\, \mathrm{d}u \\
&\quad = \int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} f(1,\tan\theta)\, f'(1,\tan\phi) \\
&\qquad\qquad \operatorname{sgn}^{\epsilon}(\tan\phi - \tan\theta)\, |\tan\phi - \tan\theta|^{-1-2\zeta} \sec^2\theta \sec^2\phi\, \mathrm{d}\theta\, \mathrm{d}\phi \\
&\quad = \int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} f(\cos\theta,\sin\theta)\, f'(\cos\phi,\sin\phi) \\
&\qquad\qquad \operatorname{sgn}^{\epsilon}(\tan\phi - \tan\theta)\, |\tan\phi - \tan\theta|^{-1-2\zeta} \sec^{1+2\zeta}\theta \sec^{1+2\zeta}\phi\, \mathrm{d}\theta\, \mathrm{d}\phi \\
&\quad = \int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} f(\cos\theta,\sin\theta)\, f'(\cos\phi,\sin\phi) \\
&\qquad\qquad \operatorname{sgn}^{\epsilon}(\sin(\phi-\theta))\, |\sin(\theta-\phi)|^{-1-2\zeta}\, \mathrm{d}\theta\, \mathrm{d}\phi;
\end{aligned}$$

all integrals converge absolutely. If moreover f and f' both satisfy the parity condition, the last integral is equal to

$$\begin{aligned}
&\frac{1}{4}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(\cos\theta,\sin\theta)\, f'(\cos\phi,\sin\phi) \\
&\qquad\qquad \operatorname{sgn}^{\epsilon}(\sin(\phi-\theta))\, |\sin(\theta-\phi)|^{-1-2\zeta}\, \mathrm{d}\theta\, \mathrm{d}\phi
\end{aligned}$$

for reasons of parity. We write any of the three double integrals above as $(I_{\zeta,\epsilon} f, f')$; at this stage, $I_{\zeta,\epsilon} f$ is given by various convolutions and may be a measurable function or a distribution.

Now, as in the definition of the pairing (Lemma 1.2), we may show that for $f, f' \in \mathsf{V}_{\zeta,\epsilon}$,

$$(I_{\zeta,\epsilon}\rho_{\zeta,\epsilon}(x)f, \rho_{\zeta,\epsilon}(x)f') = (I_{\zeta,\epsilon} f, f') \qquad \forall x \in G,$$

by considering the last integral when $x \in K$ and the first when $x \in NA$, and using the Iwasawa decomposition. ☐

Corollary 1.5. *Suppose that $\xi \in (-1/2, 0) \cup (0, 1/2)$ and that $\epsilon = 0$. Then $\langle\!\langle \cdot,\cdot \rangle\!\rangle_\xi$, given by*

$$\langle\!\langle f, f' \rangle\!\rangle_\xi := (I_{\xi,0} f, \bar{f}'),$$

is an inner product on $\mathsf{V}_{\xi,\epsilon}$, and the representation $\rho_{\xi,0}$ acts unitarily on the completion of $\mathsf{V}_{\xi,0}$ relative to this inner product.

Proof. It is evident that $\langle\!\langle f, f' \rangle\!\rangle_\xi$ is a sesquilinear pairing on $\mathsf{V}_{\xi,0}$; the question is whether it enjoys the necessary positivity property. We know the Fourier transform of the kernel of the (implicitly defined) operator $I_{\xi,\epsilon}$:

$$\mathcal{F}(\operatorname{sgn}^{\epsilon}(\cdot)\, |\cdot|^{-1-2\xi}) = c(\xi,\epsilon) \operatorname{sgn}^{\epsilon}(\cdot)\, |\cdot|^{2\xi},$$

where

$$c(\xi,\epsilon) = i^\epsilon \pi^{1/2} 2^{-2\xi} \frac{\Gamma((\epsilon - 2\xi)/2)}{\Gamma((1+\epsilon+2\xi)/2)} \tag{8}$$

(see, e.g., [GS64, p. 173] or [SW72, p. 160]); the condition on ξ shows that $c(\xi, 0)$ is positive. Hence

$$\begin{aligned} \langle\!\langle f, f \rangle\!\rangle_\xi &:= (I_{\xi,0} f, \bar{f}) \\ &= \int_{-\infty}^{\infty} c(\xi,0) \hat{f}(y) \, (\hat{f}(y))^- \, \mathrm{d}y \geq 0; \end{aligned}$$

equality holds only if $f = 0$. Hence $\langle\!\langle \cdot, \cdot \rangle\!\rangle_\xi$ is an inner product on $\mathsf{V}_{\xi,0}$. Further, ρ acts unitarily on the completion of this space from (7). □

This Hilbert space may be identified with a Sobolev-like space on the unit circle or a homogeneous Sobolev space $H^\xi(\mathbb{R})$ on the line $\{(1,t) : t \in \mathbb{R}\}$. In [ACD18, Lemma 3.6] there are explicit computations of the action of $I_{\zeta,\epsilon}$ on the basis elements $f_{\zeta,\mu}$.

Kunze and Stein [KS60] showed that ρ_ζ acts uniformly boundedly on $H^{\operatorname{Re}\zeta}(\mathbb{R})$; they were then able to use complex interpolation to prove the Kunze–Stein phenomenon. Recently, Astengo, Cowling and Di Blasio [ACD19] found optimal estimates for the norms of the uniformly bounded operators $\rho_\zeta(x)$.

1.6. **The discrete series of representations.** Bargmann [Ba47] showed that there are other unitary representations of $\mathrm{SL}(2,\mathbb{R})$ which are important for harmonic analysis.

It is well known that G acts on the upper half plane $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ by fractional linear transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{c+d}\,;$$

it can be checked that $x \cdot i = i$ if and only if $x \in K$, and then the mapping $x \mapsto x \cdot i$ identifies G/K with the upper half plane. The upper half plane is one of several models for the 2-dimensional real hyperbolic space $\mathbb{H}^2$, an important model space of negative curvature. If we take into account the fact that the elements of M act trivially on $\mathbb{H}^2$, then we may identify G with the connected component of the group of isometries of $\mathbb{H}^2$.

By conjugation with the Cayley transform, we get an action of G on the unit disc in $\mathbb{C}$:

$$\begin{pmatrix} a & b \\ d & d \end{pmatrix} \cdot z = \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}\,,$$

where $\alpha = (a - ib + ic + d)/2$ and $\beta = (-ia + b + c + id)/2$.

For $\ell \in \{1, 2, 3, \dots\}$, denote by H_ℓ the Hilbert space of holomorphic functions on the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ endowed with the inner product

$$\langle f, g \rangle_\ell = \frac{\ell - 1}{\pi} \int_D (1 - |z|^2)^{\ell-2} \bar{f}(z) g(x) \, \mathrm{d}\sigma(z) \qquad (\text{if } \ell > 1),$$

$$\langle f, g \rangle_1 = \lim_{\ell \to 1} \frac{\ell - 1}{\pi} \int_D (1 - |z|^2)^{\ell-2} \bar{f}(z) g(x) \, \mathrm{d}\sigma(z) \qquad (\text{if } \ell = 1).$$

Consider the representation $\rho_{\ell,+}$ on H_ℓ given by

$$\rho_{\ell,+}(x) f(z) = \mu^+(x, x^{-1} \cdot z)^\ell f(x^{-1} \cdot z) \qquad \text{where} \qquad \mu^+(x, x^{-1} \cdot z) = \alpha - \bar{\beta} z.$$

Then $\rho_{\ell,+}$ is irreducible and acts unitarily on H_ℓ for all $x \in G$, and we say it belongs to the holomorphic discrete series. The functions

$$g^+_{\ell,m}(z) = (-1)^{m-k} \binom{\ell - 1 + m - k}{m-k}^{1/2} z^{m-k} \qquad \forall m \in k + \mathbb{N},$$

are an orthonormal basis of H_ℓ.

The antiholomorphic discrete series $\rho_{\ell,-}$ is defined similarly, and has a similar basis. Bargmann showed [Ba47, eq. (10.29c)] that

$$\langle g^-_{\ell,m}, \rho^-_\ell(x) g^-_{\ell,n} \rangle_\ell = (-1)^{-m+n} \overline{\langle g^+_{\ell,-m}, \rho^+_\ell(x) g^+_{\ell,-n} \rangle_\ell}.$$

1.7. **The trivial representation.** The group $\mathrm{SL}(2,\mathbb{R})$ has a trivial representation ρ on $\mathbb{C}$; each $\rho(x)$ is just the identity mapping.

1.8. **The Plancherel formula.** Let ρ be a (strongly continuous) unitary representation of G on a Hilbert space H_ρ. A matrix coefficient of ρ is a function on G of the form $x \mapsto \langle \rho(x) f, f' \rangle$, where $f, f' \in \mathsf{H}_\rho$. These matrix coefficients encode the properties of ρ. For instance, when ρ is an irreducible unitary representation, the matrix coefficients of ρ are all eigenvalues of a differential operator on G, and the eigenvalue depends on ρ. A *matrix coefficient of a unitary representation ρ of G* is a function u of the form $\langle \rho(\cdot) f, f' \rangle$, that is,

$$u(x) = \langle \rho(x) f, f' \rangle \quad \forall x \in G,$$

where $\rho \in \bar{G}$ and $f, f' \in \mathsf{H}_\rho$. We abbreviate this formula to $u = \langle \rho(\cdot) f, f' \rangle$. The set of matrix coefficients of a fixed representation is not *a priori* a vector space, and to allow us to use the tools of functional analysis, we follow Arsac [Ar76] and define $A_\rho(G)$ to be the set of all infinite linear combinations

$$\sum_{j \in \mathbb{N}} \langle \rho(\cdot) f_j, f'_j \rangle \qquad \text{such that} \qquad \sum_{j \in \mathbb{N}} \|f_j\|_{\mathsf{H}_\rho} \left\|f'_j\right\|_{\mathsf{H}_\rho} < \infty;$$

the space $A_\rho(G)$ has a natural norm.

Bargmann [Ba47] found the Plancherel formula for $\mathrm{SL}(2,\mathbb{R})$. This formula has an L^2-version, or can be expressed as an inversion theorem. The inversion theorem version states that a "suitable" function u on G may be written in the following form:

$$u(x) = \int_0^\infty u_{i\eta,0}(x) \,\mathrm{d}\eta + \int_0^\infty u_{i\eta,1}(x) \,\mathrm{d}\eta + \sum_{\ell=1}^\infty u_{\ell,+}(x) + \sum_{\ell=1}^\infty u_{\ell,-}(x),$$

where the functions involved all belong to spaces A_ρ, where ρ is one of $\rho_{i\eta,0}$, $\rho_{i\eta,1}$, $\rho_{\ell,+}$ and $\rho_{\ell,-}$. Note that the complementary series and trivial representation do not appear.

1.9. **Spherical functions.** The space $\mathsf{L}^1(G)^\natural$ of integrable left- and right-K-invariant functions on G forms a commutative $*$-algebra. The Gel′fand space of multiplicative linear functionals on $\mathsf{L}^1(G)^\natural$ may be identified with the set of all bounded zonal spherical functions on G, that is,

$$\int_G f * f'(x)\, \phi_\zeta(x) \,\mathrm{d}x = \int_G f(x)\, \phi_\zeta(x) \,\mathrm{d}x \times \int_G f'(x')\, \phi_\zeta(x') \,\mathrm{d}x'$$

for all f, f' in $\mathsf{L}^1(G)^\natural$ and all bounded zonal spherical functions ϕ_ζ. These functions ϕ_ζ are all matrix coefficients of not necessarily unitary representations of G, the so-called class one representations, that is, that have nontrivial K-invariant vectors.

We are interested in positive star-linear functionals on $\mathsf{L}^1(G)^\natural$, that is, in positive-real-valued hermitean zonal spherical functions, that is, spherical functions such that $\phi_\zeta > 0$ and $\phi_\zeta^* = \phi_\zeta$. It may be checked that ϕ_ζ is hermitean if and only if ζ is either purely real or purely imaginary; ϕ_ζ is positive-real-valued if and only if $\zeta \in \mathbb{R}$. Finally, ϕ_ζ is bounded if and only if $\operatorname{Re}\zeta \in [-1, 1]$. The symmetries mentioned above imply that $\phi_\zeta^* = \phi_{\bar\zeta}$.

Theorem 1.6. *Suppose that $\xi, \xi' \in [0, \infty)$. Then $\phi_\xi \leq \phi_{\xi'}$ if and only if $\xi \leq \xi'$.*

Proof. Suppose that $\xi \leq \xi'$. From the integral formula (5), $|\phi_\zeta| \leq \phi_{\operatorname{Re}(\zeta)}$ for all $\zeta \in \mathbb{C}$. Hence from the maximum principle, applied pointwise, and the symmetry $\phi_{-\zeta} = \phi_\zeta$,

$$\phi_\xi \leq \max\{|\phi_{\xi'+i\eta}|, |\phi_{-\xi'+i\eta}| : \eta \in \mathbb{R}\} = \max\{|\phi_{\xi'+i\eta}| : \eta \in \mathbb{R}\} = \phi_{\xi'}.$$

first if $\operatorname{Re}(z)$ is either 0 or 1 and then if $0 \leq \operatorname{Re}(z) \leq 1$ by the maximum principle, applied pointwise. Iterating this interpolation result yields one direction of the theorem.

The converse follows from Bargmann's formulae for the asymptotic behaviour of spherical functions. □

Finally we introduce the generalised spherical functions. Given integers μ, ν in $2\mathbb{Z} + \epsilon$, we define

$$\phi_{\zeta,\epsilon}^{\mu,\nu}(x) = (\rho_{\zeta,\epsilon}(x) f_{\zeta,\mu}, f_{-\zeta,-\nu}) \qquad \forall x \in G, \tag{9}$$

where $f_{\zeta,\mu}$ are the functions defined in equation (2).

Then $\phi_{\zeta,0}^{0,0}$ is a standard left- and right-K-invariant zonal spherical function; the generalised spherical functions $\phi_{\zeta,\epsilon}^{\mu,\nu}$ were considered by Ehrenpreis and Mautner [EM55a, EM55b].

2. Estimates for matrix coefficients

In this lecture, we describe various decay estimates for matrix coefficients of unitary representations. We recall that, given a unitary representation ρ of G on a Hilbert space H_ρ, a matrix coefficient of ρ is a function on G of the form

$$x \mapsto \langle \rho(x) f, f' \rangle,$$

where $f, f' \in \mathsf{H}_\rho$.

Bargmann observed that all the generalised spherical functions associated to the discrete series belong to $\mathsf{L}^2(G)$; those associated to the principal series belong to L^{2+}, by which we mean that they belong to $\mathsf{L}^{2+\delta}(G)$ for all $\delta \in \mathbb{R}^+$; those associated to the complementary series belong to $\mathsf{L}^{q+}(G)$ for some q depending only on the representation. The matrix coefficients of the trivial representation are all constant functions and do not decay at infinity at all. The estimates that follow from his analysis do not seem to be uniform in r when we consider different "K-types" μ and ν and different representations.

Kunze and Stein [KS60] took a great leap forwards. They showed that every matrix coefficient of every representation involved in the Plancherel formula, and hence every coefficient of the regular representation, lies in $\mathsf{L}^{2+}(G)$. This is a dual formulation of the Kunze–Stein convolution theorem. They also established L^{p+} estimates for the complementary series, of the form

$$\|\langle\rho_{\xi,0}(\cdot)f,f'\rangle\|_q \le C(\xi,q)\,\|f\|_{\mathsf{H}_\rho}\,\|f'\|_{\mathsf{H}_\rho} \qquad \forall f,f'\in\mathsf{H}^\xi \quad \forall q\in(p,+\infty),$$

for all $q\in(2/(1-2\xi),+\infty)$, when $0<\xi<1/2$.

The asymptotic formulae found by Bargmann are very precise "at infinity", but they only hold for some matrix coefficients and it is hard to see the sort of uniform behaviour that the Kunze–Stein phenomenon tells us must occur. On the other hand, L^{p+} estimates hold for all matrix coefficients of a given irreducible unitary representation, but the fact that a function lies in some L^q-space gives no pointwise information.

One aim of these lectures is to present a new form of estimate for matrix coefficients, which we believe has many of the best features of both of the forms of estimate above. These estimates were inspired by similar estimates due to Herz [Hz70] and Haagerup [Ha79].

We define the averaging operators $\mathcal{A}$ on functions on G by

$$\mathcal{A}(u)(g) = \left(\int_K\int_K |u(kgk')|^2\,\mathrm{d}k\,\mathrm{d}k'\right)^{1/2} \qquad \forall x\in G.$$

It is evident that $\mathcal{A}(u)$ is left- and right-K-invariant.

We aim to show estimates of the form

$$\mathcal{A}\left(\langle\rho(k_\theta a_r k_\phi)f,f'\rangle\right) \le \phi_\lambda(r)\,\|f\|_{\mathsf{H}_\rho}\,\|f'\|_{\mathsf{H}_\rho} \tag{10}$$

for all $f,f'\in\mathsf{H}_\rho$ and all $r\in[0,+\infty)$; the zonal spherical function ϕ_λ depends on ρ. These are equivalent to uniform estimates for generalised spherical functions, as the following shows.

Proposition 2.1. *Let ρ be a unitary representation of G on a Hilbert space H_ρ which has an orthonormal basis $\{f_\mu:\mu\in\mathbb{B}\}$, where $\mathbb{B}$ is a subset of $\mathbb{Z}$, such that $\rho(k_\theta)f_\mu = \exp \mathrm{e}^{i\mu\theta}f_\mu$ for all $\theta\in\mathbb{R}$. Then the following are equivalent, for all $r\in\mathbb{R}^+$,*

1. *for all $\mu,\nu\in\mathbb{B}$,*
$$|\langle\rho(a_r)f_\mu,f_\nu\rangle| \le \phi_\lambda(r);$$
2 *for all $f,f'\in\mathsf{H}_\rho$,*
$$\mathcal{A}\left(\langle\pi(k_{\theta_1}a_r k_{\theta_2})f,f'\rangle\right) \le \phi_\lambda(r)\|f\|_{\mathsf{H}_\rho}\|f'\|_{\mathsf{H}_\rho}.$$

Proof. It is trivial that (2) implies (1). For the converse, let $f,f'\in\mathsf{H}_\rho$, and write

$$f=\sum_{\mu\in\mathbb{B}} b_\mu f_\mu, \qquad f'=\sum_{\nu\in\mathbb{B}} b'_\nu f_\nu,$$

Then

$$\begin{aligned}
&|\langle\rho(k_{\theta_1}a_r k_{\theta_2})f,f'\rangle|^2\\
&\quad= \langle\rho(a_r)\rho(k_{\theta_2})f,\rho(k_{\theta_1}^{-1})f'\rangle\overline{\langle\rho(a_r)\rho(k_{\theta_2})f,\rho(k_{\theta_1}^{-1})f'\rangle}\\
&\quad= \sum_{\mu,\mu',\nu,\nu'\in\mathbb{B}} b_\mu\bar{b}_{\mu'}b'_\nu\bar{d}_{\nu'}\mathrm{e}^{i\theta_2(\mu-\mu')}\mathrm{e}^{i\theta_1(\nu-\nu')}\langle\rho(a_r)f_\mu,f_\nu\rangle\overline{\langle\rho(a_r)f_{\mu'},f_{\nu'}\rangle}.
\end{aligned}$$

Then, integrating twice,

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|\langle\rho(k_{\theta_1}a_r k_{\theta_2})f,f'\rangle|^2\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2 = \sum_{\mu,\nu\in\mathbb{B}}|b_\mu|^2|b'_\nu|^2|\langle\rho(a_r)f_\mu,f_\nu\rangle|^2,$$

as required. □

2.1. **Herz' principe de majoration.** In [Hz70], Herz proved two important lemmas. We define the *right regular representation* ρ of G on $\mathsf{L}^2(G)$:

$$\rho(x)f(y) = f(yx) \qquad \forall x, y \in G.$$

The matrix coefficients u of ρ are given by

$$u(x) = \langle \rho(x)f, f' \rangle = \int_G f(yx)\, \bar{f}'(y)\, \mathrm{d}y = \int_G f(y^{-1}x)\, \bar{f}'(y^{-1})\, \mathrm{d}y = f^* * f'(x).$$

Lemma 2.2. *The matrix coefficients of the right regular representation of G acting on $\mathsf{L}^2(G)$ may be dominated, pointwise, by matrix coefficients of the representation $\rho_{0,0}$ acting on the completion of $\mathsf{V}_{0,0}$.*

Proof. We write the measure on G as an integral over $\bar{N}$ followed by an integral over $\bar{N}\backslash G$, and apply the Cauchy–Schwarz inequality.

$$\begin{aligned} |u(x)| &= \left| \int_{\bar{N}\backslash G} \int_{\bar{N}} f(\bar{n}\dot{y}x)\, T f'(\bar{n}\dot{y})\, \mathrm{d}\bar{n}\, \mathrm{d}\dot{y} \right| \\ &\leq \int_{\bar{N}\backslash G} \left(\int_{\bar{N}} |f(\bar{n}\dot{y}x)|^2\, \mathrm{d}\bar{n} \right)^{1/2} \left(\int_{\bar{N}} |\, T f'(\bar{n}\dot{y})\, \mathrm{d}\bar{n} \right)^{1/2} \mathrm{d}\dot{y} \\ &= \int_{\bar{N}\backslash G} F(\dot{y}x)\, F'(\dot{y})\, \mathrm{d}\dot{y}, \end{aligned}$$

where F and F' are nonnegative-real-valued functions on $\bar{N}\backslash G$, with the property that $\|F\|_2 \leq \|f\|_2$ and $\|F'\|_2 \leq \|f'\|_2$.

Since we may identify $\bar{N}\backslash G$ with $\mathbb{R}^2 \setminus \{0\}$, we may rewrite this as an integral over $\mathbb{R}^2 \setminus \{0\}$. This is a matrix coefficient of a representation of G on $\mathsf{L}^2(\mathbb{R}^2 \setminus \{0\})$:

$$|u(x)| = \int_{\mathbb{R}^2\setminus\{0\}} F(vx)\, F'(v)\, \mathrm{d}v,$$

where the product vx is just the vector v multiplied by the matrix x.

The next step is a similar application of the Cauchy–Schwarz inequality, where we now integrate over rays coming out of the origin. □

Lemma 2.3. *If $f, f' \in \mathsf{V}_{0,0}$, and $u = \langle \rho_{0,0}(\cdot)f, f' \rangle$, then*

$$\mathcal{A}(u) \leq \|f\|_2\, \|f'\|_2\, \phi_0,$$

where ϕ_0 is the zonal spherical function associated to $\rho_{0.0}$.

Proof. This is an application of Minkowski's integral inequality. □

The zonal spherical function ϕ_0 has an asymptotic expansion:

$$\phi_0(a_s) \asymp C s \exp \mathrm{e}^{-s} \qquad \text{as } s \to \infty$$

that is anomalous in the sense that it contains a factor s as well as an exponential. Herz's lemmas give us estimates for root-mean-square averages of matrix coefficients of the regular representation, and by functional analysis, of all the representations $\rho_{is,0}$ and $\rho_{is,1}$. However, these involve terms $s \exp \mathrm{e}^{-s}$ while Bargmann's asymptotic expansions involve only exponential terms when $s \neq 0$. So it is natural to ask whether there are decay estimates for matrix coefficients of the representations $\rho_{is,0}$ and $\rho_{is,1}$, when $s \neq 0$, that do not involve the additional polynomial term.

The solution of this exercise involves a rather delicate application of van der Corput's lemma, which is particularly suited to dealing with functions built of exponential functions. Since oscillatory integrals of the type considered here arise in analysis on spaces of negative curvature, it is likely that the technique here can be applied elsewhere.

We recall van der Corput's lemma.

Lemma 2.4. *Consider the oscillatory integral*

$$\mathrm{I}(\lambda) = \int_a^b \mathrm{e}^{i\lambda\phi(t)}\,\mathrm{d}t,$$

where the phase function ϕ is a real-valued function on $[a,b]$. If $\phi^{(1)}$ is monotone and $|\phi^{(1)}| \geq \lambda$, then

$$\mathrm{I}(\lambda) \lesssim \frac{1}{\lambda}\,.$$

If $j \geq 2$ and $|\phi^{(j)}| \geq \lambda$, then

$$\mathrm{I}(\lambda) \lesssim \frac{1}{\lambda^{1/j}}\,.$$

2.2. **The principal series.** Recall that, if $\zeta = i\eta$ where $\eta \in \mathbb{R}$, the completion of the space $\mathsf{V}_{is,\varepsilon}$ with respect to the inner product

$$\langle f, g\rangle := (f, \bar{g})$$

is an L^2 space on which $\rho_{i\eta,\varepsilon}$ acts unitarily, and the functions $f_{i\eta,\mu}$ form an orthonormal basis.

Theorem 2.5. *For all $\eta \in \mathbb{R} \setminus \{0\}$, all $\varepsilon \in \{0,1\}$, all $r \in \mathbb{R}^+$ and all $\mu, \nu \in 2\mathbb{Z} + \varepsilon$,*

$$|\langle \rho_{i\eta,\varepsilon}(a_r) f_{i\eta,\mu}, f_{i\eta,\nu}\rangle| \lesssim \frac{|\eta|+1}{|\eta|}\mathrm{e}^{-r}.$$

Proof. In this proof, we abbreviate $f_{i\eta,\mu}$ to f_μ. By definition,

$$\begin{aligned}
\langle \rho_{i\eta,\varepsilon}(x) f_\mu, f_\nu\rangle &= (\rho_{i\eta,\varepsilon}(x) f_\mu, \bar{f}_\nu) \\
&= \int_{-\infty}^{\infty} f_\mu(\mathrm{e}^{-r}, \mathrm{e}^r t) f_{-\nu}(1,t)\,\mathrm{d}t \\
&= \mathrm{e}^{-r}\int_{-\infty}^{\infty} f_\mu(\mathrm{e}^{-r}, t) f_{-\nu}(1, \mathrm{e}^{-r}t)\,\mathrm{d}t \qquad (11)\\
&= \frac{1}{\pi}\mathrm{e}^{-r}\int_{-\infty}^{\infty} (\mathrm{e}^{-2r}+t^2)^{i\eta-1/2}\exp(i\mu\arctan(\mathrm{e}^r t)) \\
&\qquad\qquad (1+\mathrm{e}^{-2r}t^2)^{-i\eta-1/2}\exp(-i\nu\arctan(\mathrm{e}^{-r}t))\,\mathrm{d}t,
\end{aligned}$$

where we used the properties of f_μ. We now split the last integral in (11) into three parts:

$$\int_{|t|<\mathrm{e}^{-r+1}} \dots\,\mathrm{d}t + \int_{\mathrm{e}^{-r+1}<|t|<\mathrm{e}^{r-1}} \dots\,\mathrm{d}t + \int_{|t|>\mathrm{e}^{r-1}} \dots\,\mathrm{d}t,$$

and we consider the three terms separately. First,

$$\left|\int_{|t|<\mathrm{e}^{-r+1}} (\mathrm{e}^{-2r}+t^2)^{i\eta-1/2}\exp(i\mu\arctan(\mathrm{e}^r t))(1+\mathrm{e}^{-2r}t^2)^{-i\eta-1/2}\exp(-i\nu\arctan(\mathrm{e}^{-r}t))\,\mathrm{d}t\right|$$
$$\le \int_{|t|<\mathrm{e}^{-r+1}} \mathrm{e}^r\,\mathrm{d}t \lesssim 1.$$

Similarly,

$$\left|\int_{|t|>\mathrm{e}^{r-1}} (\mathrm{e}^{-2r}+t^2)^{i\eta-1/2}\exp(i\mu\arctan(\mathrm{e}^r t))(1+\mathrm{e}^{-2r}t^2)^{-i\eta-1/2}\exp(-i\nu\arctan(\mathrm{e}^{-r}t))\,\mathrm{d}t\right|$$
$$\begin{aligned}&\le \int_{|t|>\mathrm{e}^{r-1}} \frac{1}{t}\frac{1}{(1+\mathrm{e}^{-2r}t^2)^{1/2}}\,\mathrm{d}t\\ &\lesssim 1.\end{aligned}$$

The second term requires a more detailed analysis. We split it further, into parts where $t>0$ or $t<0$. We treat only the case $t>0$, for the other is similar. First, observe that

$$\begin{aligned}&\int_{\mathrm{e}^{-r+1}}^{\mathrm{e}^{r-1}} (\mathrm{e}^{-2r}+t^2)^{i\eta-1/2}\exp(i\mu\arctan(\mathrm{e}^r t))(1+\mathrm{e}^{-2r}t^2)^{-i\eta-1/2}\exp(-i\nu\arctan(\mathrm{e}^{-r}t))\,\mathrm{d}t\\
&\quad= \int_{\mathrm{e}^{-r+1}}^{\mathrm{e}^{r-1}} t^{2i\eta-1}\left(1+\frac{\mathrm{e}^{-2r}}{t^2}\right)^{i\eta-1/2}\left(1+\frac{t^2}{\mathrm{e}^{2r}}\right)^{-i\eta-1/2}\\
&\qquad\qquad \exp(i\mu\arctan(\mathrm{e}^r t)-i\nu\arctan(\mathrm{e}^{-r}t))\,\mathrm{d}t\\
&\quad= \mathrm{I}_1+\mathrm{I}_2,\end{aligned}$$

say, where

$$\mathrm{I}_1 = \int_{\mathrm{e}^{-r+1}}^{\mathrm{e}^{r-1}} t^{2i\eta-1}\left(1+\frac{\mathrm{e}^{-2r}}{t^2}\right)^{i\eta}\left(1+\frac{t^2}{\mathrm{e}^{2r}}\right)^{-i\eta}\exp(i\mu\arctan(\mathrm{e}^r t)-i\nu\arctan(\mathrm{e}^{-r}t))\,\mathrm{d}t$$

and

$$\begin{aligned}\mathrm{I}_2 = \int_{\mathrm{e}^{-r+1}}^{\mathrm{e}^{r-1}} t^{2i\eta-1}\left(1+\frac{\mathrm{e}^{-2r}}{t^2}\right)^{i\eta}\left(1+\frac{t^2}{\mathrm{e}^{2r}}\right)^{-i\eta}&\exp(i\mu\arctan(\mathrm{e}^r t)-i\nu\arctan(\mathrm{e}^{-r}t))\\
\times&\left[\left(1+\frac{\mathrm{e}^{-2r}}{t^2}\right)^{-\frac12}\left(1+\frac{t^2}{\mathrm{e}^{2r}}\right)^{-\frac12}-1\right]\mathrm{d}t.\end{aligned}$$

We first estimate I_2. Since

$$|(1+x)^{-1/2}(1+y)^{-1/2}-1| \lesssim x+y$$

for all $x,y\in(0,\mathrm{e}^{-2})$, we see that

$$|\mathrm{I}_2| \lesssim \int_{\mathrm{e}^{-r+1}}^{\mathrm{e}^{r-1}} t^{-3}\mathrm{e}^{-2r}\,\mathrm{d}t + \int_{\mathrm{e}^{-r+1}}^{\mathrm{e}^{r-1}} t\mathrm{e}^{-2r}\,\mathrm{d}t \lesssim 1.$$

We now consider I_1. Observe that I_1 is equal to

$$\begin{aligned}
&\int_{\mathrm{e}^{-r+1}}^{\mathrm{e}^{r-1}} t^{2i\eta-1}\exp(i\eta[\log(1+\mathrm{e}^{-2r}t^{-2})-\log(1+t^2\mathrm{e}^{-2r})]+i\mu\arctan(\mathrm{e}^r t)-i\nu\arctan(\mathrm{e}^{-r}t))\,\mathrm{d}t\\
&\qquad=\mathrm{e}^{i\mu\frac{\pi}{2}}\int_{\mathrm{e}^{-r+1}}^{\mathrm{e}^{r-1}} t^{2i\eta-1}\exp(i\eta[\log(1+\mathrm{e}^{-2r}t^{-2})-\log(1+t^2\mathrm{e}^{-2r})])\\
&\qquad\qquad\qquad\exp(-i\mu\arctan(\mathrm{e}^{-r}/t)-i\nu\arctan(\mathrm{e}^{-r}t))\,\mathrm{d}t\\
&\qquad=\mathrm{e}^{i\mu\frac{\pi}{2}}\int_{1-r}^{r-1}\exp(ig_{\mu,\nu,r,\eta}(y))\,\mathrm{d}y,
\end{aligned}$$

where $g_{\mu,\nu,r,\eta}(y)$ is equal to

$$2y\eta+\eta[\log(1+\mathrm{e}^{-2(r+y)})-\log(1+\mathrm{e}^{2(y-2)})]-\mu\arctan(\mathrm{e}^{-r-y})-\nu\arctan(\mathrm{e}^{y-r});$$

here we set $t=\mathrm{e}^y$ and used the identity $\arctan(z)=\pi/2-\arctan(1/z)$. If N is the number of times that the second derivative of the function $g_{\mu,\nu,r,\eta}$ changes sign, which may depend on μ, ν, r and s, then

$$\begin{aligned}
|\mathrm{I}_1| &= \left|\int_{1-r}^{r-1}\exp(ig_{\mu,\nu,r,\eta}(y))\,\mathrm{d}y\right|\\
&\le \left|\int_{|g'_{\mu,\nu,r,\eta}|\ge|\eta|}\exp(ig_{\mu,\nu,r,\eta}(y))\,\mathrm{d}y\right|+\left|\int_{|g'_{\mu,\nu,r,\eta}|<|\eta|}\exp(ig_{\mu,\nu,r,\eta}(y))\,\mathrm{d}y\right|\\
&\lesssim \frac{N+1}{|\eta|}+|\{y\in[1-r,r-1]:|g'_{\mu,\nu,r,\eta}(y)|<|\eta|\}|
\end{aligned}$$

by van der Corput's lemma. An easy computation shows that

$$g'_{\mu,\nu,r,\eta}(y)=2\eta-2\eta\left[\frac{\mathrm{e}^{-2(r+y)}}{1+\mathrm{e}^{-2(r+y)}}+\frac{\mathrm{e}^{2(y-r)}}{1+\mathrm{e}^{2(y-r)}}\right]+\frac{\mu}{2\cosh(r+y)}-\frac{\nu}{2\cosh(r-y)}.$$

Observe that $g'_{\mu,\nu,r,\eta}$ is a rational function of e^y whose dependence on μ, ν, r and η is only in the coefficients, so the same holds for $g''_{\mu,\nu,r,\eta}$; therefore, the number of the zeroes of $g''_{\mu,\nu,r,\eta}$ is uniformly bounded with respect to these parameters. Thus N is bounded independent of these parameters.

It now remains to show that

$$|\{y\in[1-r,r-1]:|g'_{\mu,\nu,r,\eta}(y)|<|\eta|\}|\lesssim 1. \tag{12}$$

We suppose that $\eta>0$, as the case where $\eta<0$ is analogous.

If $y\in[-r+1,r-1]$, then $r+y\ge 1$ and $y-r\le -1$. Thus,

$$0\le\frac{\mathrm{e}^{-2(r+y)}}{1+\mathrm{e}^{-2(r+y)}}+\frac{\mathrm{e}^{2(y-r)}}{1+\mathrm{e}^{2(y-r)}}\le 2\mathrm{e}^{-2},$$

so that, if

$$\phi_{\mu,\nu,r}(y)=\frac{\nu}{2\cosh(r-y)}-\frac{\mu}{2\cosh(r+y)},$$

then

$$\{y\in[1-r,r-1]:|g'_{\mu,\nu,r,\eta}(y)|<\eta\}\subseteq\{y\in[1-r,r-1]:\phi_{\mu,\nu,r}(y)\in[2\eta/5,3\eta]\}$$

since $1-4\mathrm{e}^{-2}>2/5$. We consider two cases.

If $\mu\nu \geq 0$, then $\phi_{\mu,\nu,r}$ is monotone (increasing if either $\mu, \nu > 0$ or $\mu = 0$, $\nu > 0$; decreasing if either $\mu, \nu < 0$ or $\mu < 0$, $\nu = 0$). Moreover

$$\begin{aligned}
&|\{y \in [1-r, r-1] : \phi_{\mu,\nu,r}(y) \in [2\eta/5, 3\eta]\}| \\
&\qquad = |\{y \in [1-r, r-1] : \phi_{\mu,\nu,r}(y) > 0,\ \log(\phi_{\mu,\nu,r}(y)) \in [\log(2\eta/5), \log(3\eta)]\}|.
\end{aligned}$$

To prove that this quantity is bounded by a constant independent of μ, ν, r and η, it is enough to prove that the inverse function of $\log \circ \phi_{\mu,\nu,r}$ is Lipschitz, uniformly in μ, ν and r, or more precisely, that

$$|\phi'_{\mu,\nu,r}(y)| \geq c_0 \phi_{\mu,\nu,r}(y)$$

where c_0 is a constant. If, for instance, $\mu, \nu > 0$, then

$$\begin{aligned}
|\phi'_{\mu,\nu,r}(y)| = \phi'_{\mu,\nu,r}(y) &= \frac{\nu \sinh(r-y)}{2\cosh^2(r-y)} + \frac{\mu \sinh(r+y)}{2\cosh^2(r+y)} \\
&= \frac{\nu}{2\cosh(r-y)} \tanh(r-y) + \frac{\mu}{2\cosh(r+y)} \tanh(r+y) \\
&\geq \tanh(1) \frac{\nu}{2\cosh(r-y)} \\
&\geq \tanh(1) \phi_{\mu,\nu,r}(y).
\end{aligned}$$

The other cases for which $\mu\nu \geq 0$ can be treated analogously.

If $\mu\nu < 0$, then the set $\{y \in [1-r, r-1] : \phi_{\mu,\nu,r}(y) \in [2\eta/5, 3\eta]\}$ is empty unless $\mu < 0$ and $\nu > 0$. In this case, $\phi_{\mu,\nu,r} > 0$. Moreover, there are uniform estimates

$$\cosh(r+y) \approx \mathrm{e}^{r+y} \qquad \text{and} \qquad \cosh(r-y) \approx \mathrm{e}^{r-y}$$

since $r + y > 0$ and $r - y > 0$. Therefore

$$\begin{aligned}
&|\{y \in [1-r, r-1] : \phi_{\mu,\nu,r}(y) \in [2\eta/5, 3\eta]\}| \\
&\qquad \lesssim |\{y \in [1-r, r-1] : \nu \mathrm{e}^{y} - \mu \mathrm{e}^{-y} \in [c_1 \eta \mathrm{e}^{r}, c_2 \eta \mathrm{e}^{r}]\}| \\
&\qquad = |\{y \in [1-r, r-1] : \log(\mathrm{e}^{y} + a\mathrm{e}^{-y}) \in [\log(c_1 \eta \mathrm{e}^{r}/\nu), \log(c_2 \eta \mathrm{e}^{r}/\nu)]\}|
\end{aligned}$$

where $a := -\mu/\nu > 0$ and c_1 and c_2 are constants. This is what we are going to estimate.

Write $\psi_{\mu,\nu}(y) = \mathrm{e}^{y} + a\mathrm{e}^{-y}$ and let

$$y_+ := \frac{1}{2}(\log a + \log 3), \qquad y_- = \frac{1}{2}(\log a - \log 3).$$

Then $\log(\psi_{\mu,\nu})$ is invertible in $(-\infty, y_-]$ and in $[y_+, +\infty)$. Moreover, since

$$\begin{aligned}
\psi'_{\mu,\nu}(y) &\geq \frac{1}{2} \psi_{\mu,\nu}(y) \qquad \forall y \in [y_+, +\infty) \\
\psi'_{\mu,\nu}(y) &\leq -\frac{1}{2} \psi_{\mu,\nu}(y) \qquad \forall y \in (-\infty, y_-],
\end{aligned}$$

the restrictions of $\log(\psi_{\mu,\nu})$ to $(-\infty, y_-]$ and to $[y_+, +\infty)$ have uniformly Lipschitz inverse. Since $[y_-, y_+]$ has constant length, the proof of (12) and thus of the theorem is complete. $\square$

2.3. **The discrete series.** In [BCNT], the following theorem is proved.

Theorem 2.6. *Let $\ell \in \mathbb{Z}^+$ and let $k = \frac{1}{2}\ell$. Then for all $r \in \mathbb{R}$ and all $m, n \in k + \mathbb{N}$*

$$|\langle g^+_{\ell,m}, \rho^+_\ell(a_r) g^+_{\ell,n}\rangle_\ell| \le (\cosh \zeta(r))^{-1},$$

while for all $r \in \mathbb{R}$ and $m, n \in -k - \mathbb{N}$,

$$|\langle g^-_{\ell,m}, \rho^-_\ell(a_r) g^-_{\ell,n}\rangle_\ell| \le (\cosh(r))^{-1}.$$

Proof. We omit the proof, which involves a delicate study of special functions. □

2.4. **The complementary series.** It is not hard to see that many of the matrix coefficients of the complementary series representations have exponential decay, and indeed to prove a corresponding result for the complementary series. But we are going to prove a general result later that deals with the complementary series as well as with another representation that we have not yet defined.

2.5. **The representation with many names.** The group $\mathrm{SL}(2,\mathbb{R})$ is not simply connected: topologically it is a circle. In Lie theory, one constructs a double cover $\tilde{G}$ of G and a covering map from $\tilde{G}$ to G that is a *local isomorphism*. The group $\tilde{G}$ cannot be represented as a group of matrices. However, it contains subgroups A, N and $\bar{N}$ that are isomorphic to the subgroups A, N and $\bar{N}$ of G and a compact subgroup $\tilde{K}$ that is a double cover of the rotation group K. While it is hard to write an element of G it is easy to write elements of these subgroups.

Our next representation, written ω, is a unitary representation of $\tilde{G}$ but is *odd*, and not defined on G. It is known as the *Weil representation*, the *Segal–Shale–Weil* representation, the *harmonic representation*, the *metaplectic representation*, and the *oscillator representation*. It appears in number theory, mathematical physics, and in harmonic analysis, particularly in connection with the "phase space" viewpoint; see, e.g., [Fo89]. It acts on $\mathsf{L}^2(\mathbb{R})$, and we describe it by saying how it behaves on N, $\bar{N}$, A and $\tilde{K}$.

First, we describe briefly the Hermite functions. Recall that the differential operator $\mathcal{H}$, given by

$$\mathcal{H}f(y) = y^2 f(y) - \frac{d^2}{dy^2} f(y)$$

has eigenfunctions h_j, where $j \in \mathbb{N}$, which form an orthonormal basis for $\mathsf{L}^2(\mathbb{R})$. Further, $\mathcal{H}h_j = (2j+1)h_j$ for each $j \in \mathbb{N}$. It is well known that

$$h_0(y) = \mathrm{e}^{-y^2/2}$$

and more generally h_j is a polynomial of degree j multiplied by h_0. Then $\operatorname{span}\{h_j : 0 \le j \le J\}$ coincides with the collection of all polynomials of degree at most J multiplied by h_0. It may be checked that $f \in \mathsf{L}^2(\mathbb{R})$ lies in the Schwartz space if and only if the coefficients c_j in its Hermite expansion $\sum_{j\in\mathbb{N}} c_j h_j$ vanish faster than any power of j at infinity.

We define the *fractional Fourier transform* using this basis:

$$\mathcal{F}_\theta \sum_{j\in\mathbb{N}} c_j h_j := \sum_{j\in\mathbb{N}} c_j \exp((2j+1)i\theta/2) h_j.$$

The point is that $\mathcal{F}_\pi$ is the Fourier transform, and we have included the Fourier transform in a group of unitary operators.

We now describe the oscillator representation ω. The elements of N act by unitary multiplication operators:

$$\omega(n_b)f(y) = \mathrm{e}^{iby^2/2}f(y).$$

The elements of $\bar{N}$ act by unitary convolutions, easily expressed as Fourier multiplier operators:

$$[\omega(\bar{n}_c)f]\,\hat{}\,(y) = \mathrm{e}^{icy^2/2}\hat{f}(y).$$

The elements of A act by suitably normalised dilations:

$$[\omega(a_r)f](y) = \mathrm{e}^{-r/2}f(e^{-r}y).$$

Finally, $\tilde{K}$ acts by the fractional Fourier transform:

$$\omega(\tilde{k_\theta})f(y) = \mathcal{F}_\theta f(y)$$

for all $y \in \mathbb{R}$.

Observe that if f and f' have finite Hermite expansions $\sum_{j\leq J} c_j h_j$ and $\sum_{j\leq J} c'_j h_j$, then for all θ and ϕ in $\mathbb{R}$,

$$\begin{aligned}
\left|\left\langle \omega(\tilde{k}_\theta a_r \tilde{k}_\phi f, f'\right\rangle\right| &= \left|\left\langle \omega(\tilde{k}_\theta a_r \tilde{k}_\phi f, f'\right\rangle\right| \\
&\leq \sum_{j,j'\leq J} |c_j||c'_{j'}|\,|\langle \omega(a_r)h_j, h_{j'}\rangle| \\
&\lesssim_{f,f'} \max_{j,j'\leq J} \int_{-\infty}^{\infty} \mathrm{e}^{-r/2}|p_j(e^{-r}y)|\mathrm{e}^{-\mathrm{e}^{-2r}y^2/2}|p_{j'}(y)|\mathrm{e}^{-y^2/2}\,\mathrm{d}y \\
&\lesssim_{j,j'} \mathrm{e}^{-r/2}
\end{aligned}$$

for all $r \in \mathbb{R}^+$. In light of what we know about the Haar measure on G, this implies that these matrix coefficients belong to $\mathsf{L}^{4+}(G)$.

In the next lecture, we will obtain estimates for all matrix coefficients of ω. These will follow from the following theorem

Theorem. *Suppose that ρ is a unitary representation of G on a Hilbert space H_ρ, and $\xi \in (0, 1/2)$. The following are equivalent:*
(a) there is a dense subspace H^0_ρ of H_ρ such that, for all $f_1, f_2 \in \mathsf{H}^0_\rho$,

$$\mathcal{A}(\langle \rho(\cdot)f, f_2\rangle) \lesssim_{f_1,f_2} \phi_\xi$$

(b) for all $f_1, f_2 \in \mathsf{H}_\rho$,

$$\mathcal{A}(\langle \rho(\cdot)f_1, f_2\rangle) \leq \|f_1\|\,\|f_2\|\,\phi_\xi$$

When (a) and (b) hold, all the matrix coefficients of ρ belong to the same L^q spaces as ϕ_ξ.

This has several corollaries; for the complementary series and for the metaplectic representation.

Corollary 2.7. *Fix $\xi \in (0, 1/2)$, and take $f, f_2 \in H_\xi$. Then*

$$\mathcal{A}\left(\langle\!\langle \rho_\xi(a_r)f, f_2\rangle\!\rangle_\xi\right) \lesssim \frac{1}{|\lambda|}\mathrm{e}^{-r(1-2|\lambda|)} \qquad \forall r \in \mathbb{R}^+.$$

In light of Proposition 2.1, this gives a family of uniform estimates for the hypergeometric functions that appear in Bargmann's interpretation of the generalised spherical functions.

Recall that ω denotes the metaplectic representation.

Corollary 2.8. *The following estimate holds for all $f, f_2 \in \mathsf{L}^2(\mathbb{R})$:*

$$\mathcal{A}\left(\langle \omega(a_r) f, f_2 \rangle\right) \lesssim \mathrm{e}^{-r/2} \qquad \forall r \in \mathbb{R}^+.$$

3. Some functional analysis and applications

Recently, Samei and Wiersma [SW18] found a functional analytic proof that if one matrix coefficient of an irreducible representation satisfies an L^{q+} estimate (where $q \in [2, +\infty)$), then all matrix coefficients do. A related result appears in de Laat and Siebenand [LS21].

Their elegant proof does not apply to all locally compact groups, but it does to many, including all semisimple Lie groups. The key to their argument is understanding various C^*-algebras associated to group representations. We present a variation on their result. This implies uniform estimates for complementary series representations and for the metaplectic representation.

Many of the following results on abstract harmonic analysis and C^*-algebras may be found in standard texts such as [HR63] or [Di60].

3.1. **Construction of an exotic algebra.** Recall the averaging operators $\mathcal{A}$ on functions on G:

$$\mathcal{A}(f)(x) = \left(\int_K \int_K |f(kxk')|^2 \,\mathrm{d}k \,\mathrm{d}k' \right)^{1/2} \qquad \forall x \in G.$$

Lemma 3.1. *Suppose that $f_1, f_2 \in C_c(G)$. Then*

$$\mathcal{A}(f_1 * f_2) \leq \mathcal{A}(f_1) * \mathcal{A}(f_2).$$

Hence, if ϕ_λ is a positive-real-valued spherical function, then

$$\int_G \mathcal{A}(f_1 * f_2)(x)\, \phi_\lambda(x) \,\mathrm{d}x \leq \left(\int_G \mathcal{A}(f_1)(y)\, \phi_\lambda(y) \,\mathrm{d}y \right) \left(\int_G \mathcal{A}(f_2)(x)\, \phi_\lambda(x) \,\mathrm{d}x \right).$$

Proof. This is an exercise in integration. Indeed, for $k, k', k'' \in K$,

$$\begin{aligned} |f_1 * f_2(k'xk'')| &= \left| \int_G f_1(y)\, f_2(y^{-1}k'xk'') \,\mathrm{d}y \right| \\ &\leq \int_G |f_1(y)|\, |f_2(y^{-1}k'xk'')| \,\mathrm{d}y \\ &= \int_G |f_1(k'yk)|\, |f_2(k^{-1}y^{-1}xk'')| \,\mathrm{d}y, \end{aligned}$$

whence

$$\begin{aligned} |f_1 * f_2(k'xk'')| &\leq \int_K \int_G |f_1(k'yk)|\, |f_2(k^{-1}y^{-1}xk'')| \,\mathrm{d}y \,\mathrm{d}k \\ &= \int_G \int_K |f_1(k'yk)|\, |f_2(k^{-1}y^{-1}xk'')| \,\mathrm{d}k \,\mathrm{d}y \\ &\leq \int_G \left(\int_K |f_1(k'yk)|^2 \,\mathrm{d}k \right)^{1/2} \left(\int_K |f_2(k^{-1}y^{-1}xk'')|^2 \,\mathrm{d}k \right)^{1/2} \mathrm{d}y. \end{aligned}$$

We deduce that

$$
\begin{aligned}
&\mathcal{A}(f_1 * f_2)(x) \\
&\quad = \left(\int_K \int_K |f_1 * f_2(k'xk'')|^2 \,\mathrm{d}k' \,\mathrm{d}k'' \right)^{1/2} \\
&\quad \le \left(\int_K \int_K \left| \int_G \left(\int_K |f_1(k'yk)|^2 \,\mathrm{d}k \right)^{1/2} \left(\int_K |f_2(k^{-1}y^{-1}xk'')|^2 \,\mathrm{d}k \right)^{1/2} \mathrm{d}y \right|^2 \mathrm{d}k' \,\mathrm{d}k'' \right)^{1/2} \\
&\quad \le \int_G \left(\int_K \int_K \left| \left(\int_K |f_1(k'yk)|^2 \,\mathrm{d}k \right) \left(\int_K |f_2(k^{-1}y^{-1}xk'')|^2 \,\mathrm{d}k \right) \right| \mathrm{d}k' \,\mathrm{d}k'' \right)^{1/2} \mathrm{d}y \\
&\quad = \int_G \left(\int_K \int_K |f_1(k'yk)|^2 \,\mathrm{d}k \,\mathrm{d}k' \right)^{1/2} \left(\int_K \int_K |f_2(k^{-1}y^{-1}xk'')|^2 \,\mathrm{d}k \,\mathrm{d}k'' \right)^{1/2} \mathrm{d}y \\
&\quad = \mathcal{A}(f_1) * \mathcal{A}(f_2)(x).
\end{aligned}
$$

The second part of the theorem follows immediately. □

For the next results, we fix a spherical function ϕ_λ that is bounded, positive-real-valued, and Hermitean. For $f \in C_c(G)$, we define

$$
\|f\|_\lambda = \int_G \mathcal{A}(f)(x)\, \phi_\lambda(x) \,\mathrm{d}x. \tag{13}
$$

Lemma 3.2. *The norm $\|\cdot\|_\lambda$ is a Banach star-algebra norm on $C_c(G)$, that is,*
(a) $\|f\|_\lambda \ge 0$ and equality holds if and only if $f = 0$
(b) $\|f_1 + f_2\|_\lambda \le \|f_1\|_\lambda + \|f_2\|_\lambda$
(c) $\|cf\|_\lambda = |c|\, \|f\|_\lambda$
(d) $\|f_1 * f_2\|_\lambda \le \|f_1\|_\lambda \|f_2\|_\lambda$
(e) $\|f\| = \|f^*\|$
for all $f, f_1, f_2 \in C_c(G)$ and all $c \in \mathbb{C}$.

Proof. Items (a) to (c) are obviously true. Part (d) follows from Lemma 4.1. Finally, if ϕ_λ is hermitean, then

$$
\begin{aligned}
\int_G \mathcal{A}(f^*)(x)\, \phi_\lambda(x) \,\mathrm{d}x &= \int_G \mathcal{A}(f)(x^{-1})\, \phi_\lambda(x) \,\mathrm{d}x \\
&= \int_G \mathcal{A}(f)(x)\, \phi_\lambda(x^{-1}) \,\mathrm{d}x \\
&= \int_G \mathcal{A}(f)(x)\, \phi_\lambda(x) \,\mathrm{d}x
\end{aligned}
$$

for all $f \in C_c(G)$, as claimed. □

We define $\mathfrak{A}_\lambda$ to be the convolution algebra $C_c(G)$, equipped with the norm $\|\cdot\|_\lambda$. Note that there is no reason to suppose that $\|f * f^*\|_\lambda = \|f\|_\lambda^2$, nor does the star-algebra $\mathfrak{A}_\lambda$ have a bounded approximate identify. We are going to construct the C^* enveloping algebra of $\mathfrak{A}_\lambda$, and we show that $\mathfrak{A}_\lambda$ has a nondegenerate representation on a Hilbert space.

Lemma 3.3. *If $f \in \mathfrak{A}_\lambda$, then $f' * f \in \mathsf{L}^2(G)$ and $\|f' * f\|_{\mathsf{L}^2(G)} \le \|f\|_\lambda \|f'\|_{\mathsf{L}^2(G)}$ for all $f' \in \mathsf{L}^2(G)$. Further, if $f' * f = 0$ in $\mathsf{L}^2(G)$ for all $f' \in \mathsf{L}^2(G)$, then $f = 0$ in $\mathfrak{A}_\lambda$.*

Proof. First, $\phi_0 \leq \phi_\lambda$ by Theorem 1.6. From the definition and Herz's *principe de majoration*, if $f_1, f_2 \in \mathsf{L}^2(G)$, then $f_1^* * f_2$ is a matrix coefficient of the right regular representation of G on $\mathsf{L}^2(G)$, and

$$\mathcal{A}(f_1^* * f_2) \leq \|f_1\|_2 \, \|f_1\|_2 \, \phi_0.$$

It follows that

$$\begin{aligned}
\left| \int_G f_2(x) \, f_1 * f_3(x) \, \mathrm{d}x \right| &= \left| \int_G f_1^* * f_2(x) \, f_3(x) \, \mathrm{d}x \right| \\
&\leq \int_G \mathcal{A}(f_1^* * f_2)(x) \, \mathcal{A}(f_3)(x) \, \mathrm{d}x \\
&\leq \|f_1\|_2 \, \|f_2\|_2 \int_G \phi_0(x) \, \mathcal{A}(f_3)(x) \, \mathrm{d}x \\
&= \|f_1\|_2 \, \|f_2\|_2 \, \|f_3\|_0 \\
&\leq \|f_1\|_2 \, \|f_2\|_2 \, \|f_3\|_\lambda
\end{aligned}$$

for all $f_3 \in \mathfrak{A}_\lambda$. By duality,

$$\|f_1 * f_3\|_2 \leq \|f_1\|_2 \, \|f_3\|_\lambda$$

and $\mathfrak{A}_\lambda$ may be embedded in the algebra of right convolution operators on $\mathsf{L}^2(G)$. □

This embedding is a star-representation. Now we examine the dual space of $\mathfrak{A}_\lambda$.

Lemma 3.4. *Suppose that u is a continuous function on G and ϕ_λ is bounded, positive-real-valued, and Hermitean. Then the following are equivalent:*
(a) $\left| \int_G u(x) \, f(x) \, \mathrm{d}x \right| \leq C \, \|f\|_\lambda$ *for all* $f \in C_c(G)$;
(b) $\mathcal{A}u \leq C\phi_\lambda$.

Proof. This is an exercise in standard inequalities in integration theory. □

Lemma 3.5. *Let ρ be a unitary representation of G. Then the following are equivalent:*
(a) for all f_1 and f_2 in a dense subspace H_ρ^0 of H_ρ, there is a constant $C(f_1, f_2)$ such that

$$\left| \int_G f(x) \, \langle \rho(x) f_1, f_2 \rangle \, \mathrm{d}x \right| \leq C(f_1, f_2) \, \|f\|_\lambda \qquad \forall f \in C_c(G);$$

(b) for all f_1 and f_2 in a dense subspace H_ρ^0 of H_ρ, there is a constant $C(f_1, f_2)$ such that

$$\mathcal{A}(\langle \rho(\cdot) f_1, f_2 \rangle) \leq C(f_1, f_2) \, \phi_\lambda;$$

(c) for all f_1 and f_2 in H_ρ,

$$\left| \int_G f(x) \, \langle \rho(x) f_1, f_2 \rangle \, \mathrm{d}x \right| \leq \|f_1\|_{\mathsf{H}_\rho} \, \|f_2\|_{\mathsf{H}_\rho} \, \|f\|_\lambda \quad \forall f \in C_c(G);$$

(d) for all f_1 and f_2 in H_ρ,

$$\mathcal{A}(\langle \rho(\cdot) f_1, f_2 \rangle) \leq \|f_1\|_{\mathsf{H}_\rho} \, \|f_2\|_{\mathsf{H}_\rho} \, \phi_\lambda.$$

Proof. Duality shows that (a) and (b) are equivalent and that (c) and (d) are equivalent. It is obvious that (d) implies (b). The spectral radius formula and more functional analysis imply that for any continuous star-representation ρ of $\mathfrak{A}_\lambda$ as operators on a Hilbert space,

$$\|\rho(f)\| \leq \|f\|_\lambda \qquad \forall f \in C_c(G).$$

Suppose that (a) holds, and take $f_1 \in \mathsf{H}^0_\rho$. Then

$$\left| \int_G f(x) \, \langle \rho(x) f_1, f_1 \rangle \right| \mathrm{d}x \leq C(f_1) \, \|f\|_\lambda \qquad \forall f \in C_c(G);$$

then the Gel′fand–Naĭmark–Segal (GNS) representation of $\mathfrak{A}_\lambda$ associated to $\langle \rho(\cdot) f_1, f_1 \rangle$ is bounded on $\mathfrak{A}_\lambda$, so we may assume that $C(f_1) = \|f_1\|^2_{\mathsf{H}_\rho}$. From this, (c) follows. □

Effectively we are dealing with the enveloping C^*-algebra of the convolution algebra $C_c(G)$ with the norm $\|\cdot\|_\lambda$. The algebra $C^*_{(\lambda)}(G)$ is an "exotic C^*-algebra", in the sense that it lies "between" the reduced C^*-algebra of G and the full C^*-algebra of G. It is also exotic in the sense that it arise as the completion of a star-algebra with no approximate identity.

3.2. **Application to the oscillator representation.** In the previous lecture, we defined the oscillator representation, and showed that there is a dense subspace H^0_ω of $\mathsf{H}_\omega = \mathsf{L}^2(\mathbb{R})$ such that

$$\left| \left\langle \omega(\tilde{k}_\theta a_r \tilde{k}_\phi) f_1, f_2 \right\rangle \right| \lesssim_{f_1, f_2} \mathrm{e}^{-r/2}$$

for all $r \in \mathbb{R}^+$. This implies immediately that

$$\mathcal{A}(\langle \omega(a_r) f_1, f_2 \rangle) \lesssim_{f_1, f_2} \phi_{1/4}(a_r)$$

The theorem now implies that

$$\mathcal{A}(\langle \omega(a_r) f_1, f_2 \rangle) \leq \|f_1\|_2 \, \|f_2\|_2 \, \phi_{1/4}(a_r),$$

or, since both sides of this inequality are left- and right-K-invariant functions,

$$\mathcal{A}(\left\langle \omega(\tilde{k}_\theta a_r \tilde{k}_\phi) f_1, f_2 \right\rangle) \leq \|f_1\|_2 \, \|f_2\|_2 \, \phi_{1/4}(\tilde{k}_\theta a_r \tilde{k}_\phi)$$

for any real θ and ϕ, and hence

$$\mathcal{A}(\langle \omega(x) f_1, f_2 \rangle) \leq \|f_1\|_2 \, \|f_2\|_2 \, \phi_{1/4}(x)$$

for all $x \in G$.

Consider $\bar{n}_b \in \bar{N}$ (where $b \neq 0$). By the Cartan decomposition, for the group $\tilde{G}$, it is possible to write

$$\bar{n}_b = \tilde{k}_\theta a_r \tilde{k}_\phi,$$

for some $r \in \mathbb{R}^+$ and $\theta, \phi \in [-2\pi, 2\pi]$. We compute r in terms of b, by first projecting from $\tilde{G}$ to G: this implies that

$$\bar{n}_b = k_{\theta'} a_r k_{\phi'}.$$

Take the transpose of both sides and then products:

$$n_b \bar{n}_b = (k_{\theta'} a_r k_{\phi'})^\top k_{\theta'} a_r k_{\phi'} = k_{-\phi'} a_r a_r k_{\phi'} = k_{-\phi'} a_{2r} k_{\phi'}.$$

Taking traces of both sides shows that

$$2 + b^2 = e^{2r} + e^{-2r},$$

and it follows that

$$\phi_{1/4}(\bar{n}_b) \simeq e^{-r/2} \simeq |b|^{-1/2}.$$

We conclude that

$$\left(\int_{-2\pi}^{2\pi} \int_{-2\pi}^{2\pi} |\langle \omega(\bar{n}_b) \mathcal{F}_\phi f_1, \mathcal{F}_\theta f_2 \rangle|^2 \,\mathrm{d}\theta \,\mathrm{d}\phi \right)^{1/2} \lesssim |b|^{-1/2} \left\| f_1 \right\|_2 \left\| f_2 \right\|_2 \tag{14}$$

for all $f_1, f_2 \in \mathsf{L}^2(\mathbb{R})$.

If there is a little smoothness in the integrand, as a function of θ and ϕ, then we can replace the L^2 norms by L^∞ norms. It then follows that, if $f_1, f_2 \in \mathrm{Dom}(\mathcal{H}^\alpha)$, where $\alpha > 1/2$, then

$$|\langle \omega(\bar{n}_b) \mathcal{F}_\phi f_1, \mathcal{F}_\theta f_2 \rangle| \lesssim |b|^{-1/2} \left\| \mathcal{H}^\alpha f_1 \right\|_2 \left\| \mathcal{H}^\alpha f_2 \right\|_2, \tag{15}$$

for all θ, ϕ, and r, so in particular,

$$|\langle \omega(\bar{n}_b) f_1, f_2 \rangle| \lesssim |b|^{-1/2} \left\| \mathcal{H}^\alpha f_1 \right\|_2 \left\| \mathcal{H}^\alpha f_2 \right\|_2. \tag{16}$$

Recall that $\omega(\bar{n}_b)$ is just the Schrödinger propagator, so we have proved weak L^2 dispersion estimates for the solution of the Schrödinger equation.

The methods here extend to the oscillator representation of the *metaplectic group* $\mathrm{Mp}(n)$, and hence give dispersion estimates for the Schrödinger equation in $\mathbb{R}^n$, without much effort. See [BCNT].

Earlier, Cauli, Nicola and Tabacco [CNT19] proved estimates like (15) and (16) in $\mathbb{R}^n$. They imposed the condition that the functions f_1 and f_2 belong to Feichtinger's modulation space $\mathsf{M}^1(\mathbb{R}^n)$.

3.3. **Why does representation theory come in?** First, our theorem is tying together two commutative groups of operators that do not commute mutually, the Schrödinger propagators and the fractional Fourier transform. It is a happy accident that these are both inside a Lie group of operators, and the natural place to study both at the same time is the Lie group. The language of representation theory lets us do this.

Second, the Lie group in question is a noncompact semisimple Lie group. These are rather unusual, in that they present some harmonic analytic behaviour that is not seen in classical harmonic analysis (the Kunze–Stein phenomenon, for instance). In particular, we can study decay at infinity in a new and different way. This is what underlies the exotic C^* algebras that encapsulate the curious interaction between the two groups of operators.

3.4. **Open problems.** There are some notable open problems related to $\mathrm{SL}(2,\mathbb{R})$.

First, the one-sheeted hyperboloid

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 1\}$$

is a homogeneous space of $\mathrm{SL}(2,\mathbb{R})$. On this space, there is a natural pseudo-Riemannian metric and associated hyperbolic operator, which is a toy model (in a 1+1-dimensional space) for the wave operator in a negatively curved universe. Strichartz [Str73] began a program of harmonic analysis on hyperboloids, that would have led to some understanding of the effect of negative curvature on the behaviour of waves, but as far as I know, this program has not been carried to completion.

Second, there are other manifolds related to $\mathrm{SL}(2,\mathbb{R})$ where analysis and spectral theory are of interest. A notable example is a conjecture of Selberg, which, in the language of representation theory, says that complementary series representations do not appear in the decomposition of the (right) regular representation of $\mathrm{SL}(2,\mathbb{R})$ on the space $\mathsf{L}^2(\mathrm{SL}(2,N\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R}))$ for any positive integer N. For more details of this conjecture, see, for instance, [Sa95].

4. Afterthoughts

It should be pointed out that in [BCNT], two analogues of (15) in $\mathbb{R}^n$ are proved. One involves an integration over the maximal compact subgroup of the metaplectic group $\mathrm{Mp}(2n,\mathbb{R})$; this can also be shown by the more abstract methods here. The second involves an integration over the maximal torus T^n of the same maximal compact subgroup, and seems to need some of the phase space ideas from mathematical physics that are used in [BCNT]. The use of a Sobolev embedding argument like that above then gives weak L^2 estimates for solutions of the Schrödinger equation for functions f_1 and f_2 in the Hermite–Sobolev space $\{f \in L^2(\mathbb{R}^n) : \mathcal{H}^\alpha f \in \mathsf{L}^2(\mathbb{R}^n)\}$ when $\alpha > n/2$.

If $f \in \mathsf{L}^2(\mathbb{R}^n)$ and $\mathcal{H}^\alpha f \in \mathsf{L}^2(\mathbb{R}^n)$, where α is a nonnegative integer, then $f \in \mathsf{L}^2(\mathbb{R}^n, w_\alpha)$, the weighted space of all functions f such that $(1+|\cdot|)^{2\alpha} f \in L^2(\mathbb{R}^n)$; this follows from the L^2 boundedness of all the higher Riesz transforms associated to the Hermite operator, which can be seen very easily from their action on the basis vectors. An easy complex interpolation argument extends this to all nonnegative real α. If follows that if $f \in \mathsf{L}^2(\mathbb{R}^n)$ and $\mathcal{H}^\alpha f \in \mathsf{L}^2(\mathbb{R}^n)$, where $\alpha > n/4$, then $f \in \mathsf{L}^1(\mathbb{R}^n)$. For $f_1, f_2 \in \mathsf{L}^1(\mathbb{R}^n)$, estimates of the form

$$|\langle \omega(\bar{n}_b) f_1, f_2\rangle| \lesssim |b|^{-1/2} \left\|f_1\right\|_1 \left\|f_2\right\|_1$$

follow immediately from the explicit formula for the convolution kernel of the Schrödinger propagator. Hence it is possible to prove sharper versions of (15) and (16) by considerably easier means. However, we do not know how to prove (14) easily.

The Hermite–Riesz transforms were first defined by Thangavelu [Th90]. Thangavelu [Th09] showed that if u belongs to the Sobolev space $W^{n+\delta,1}(\mathbb{R}^n)$, then u also belongs to Feichtinger's modulation space $\mathsf{M}^1(\mathbb{R}^n)$. Together with the results of [CNT19, BCNT], this suggests that it may be interesting to further explore the relation between Hermite–Sobolev spaces, the modulation spaces and the existence of weak L^2 decay estimates for the Schrödinger propagator along the lines discussed here.

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