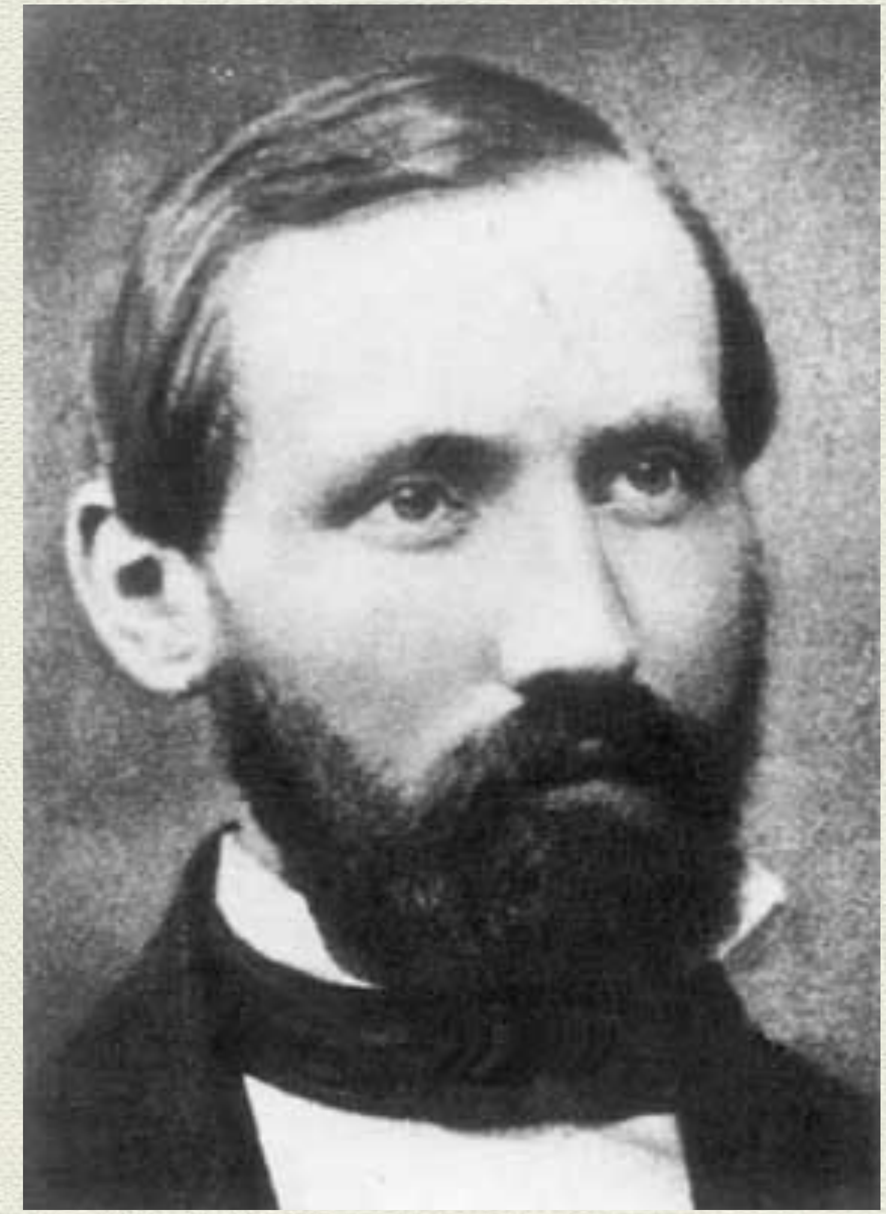

FOURIER OPTIMIZATON AND NUMBER THEORY I

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ICTP - International Centre for Theoretical Physics - Trieste, Italy

El Escorial, June 2022.

Objectives

- <u>Message:</u> Hidden in some number theory, there might be a problem in Fourier optimization wonderland for you.
- Some examples / current machinery available.
- Give you some ideas for discussion.
- Tell you a story or two…

Fourier optimization framework

1. Design ``a” Fourier optimization problem connected to your number theory/algebraic problem (proof of concept).
2. Evolve towards designing what should be ``the correct" Fourier optimization problem.
3. Solve the Fourier optimization problem (or at least try to find a good approximation for the solution).

Part 1: Zeros and Primes

“If I woke up from a 500-year sleep, the first thing I would ask is whether the Riemann hypothesis had been solved”

D.H.

Hilbert's list of 23 problems, ICM Paris 1900

task at once presents itself to penetrate further along the path here entered, as A. Hurwitz has already done in two interesting papers,* "Ueber arithmetische Eigenschaften gewisser transcendenter Funktionen." I should like, therefore, to sketch a class of problems which, in my opinion, should be attacked as here next in order. That certain special transcendental functions, important in analysis, take algebraic values for certain algebraic arguments, seems to us particularly remarkable and worthy of thorough investigation. Indeed, we expect transcendental functions to assume, in general, transcendental values for even algebraic arguments; and, although it is well known that there exist integral transcendental functions which even have rational values for all algebraic arguments, we shall still consider it highly probable that the exponential function $e^{i\pi z}$, for example, which evidently has algebraic values for all rational arguments z, will on the other hand always take transcendental values for irrational algebraic values of the argument z. We can also give this statement a geometrical form, as follows:

If, in an isosceles triangle, the ratio of the base angle to the angle at the vertex be algebraic but not rational, the ratio between base and side is always transcendental.

In spite of the simplicity of this statement and of its similarity to the problems solved by Hermite and Lindemann, I consider the proof of this theorem very difficult; as also the proof that

The expression α^β, for an algebraic base α and an irrational algebraic exponent β, e. g., the number $2^{\sqrt{2}}$ or $e^\pi = i^{-2i}$, always represents a transcendental or at least an irrational number.

It is certain that the solution of these and similar problems must lead us to entirely new methods and to a new insight into the nature of special irrational and transcendental numbers.

8. Problems of Prime Numbers.

Essential progress in the theory of the distribution of prime numbers has lately been made by Hadamard, de la Vallée-Poussin, Von Mangoldt and others. For the complete solution, however, of the problems set us by Riemann's paper "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse" it still remains to prove the correctness of an exceedingly important statement of Riemann, viz., *that the zero points of the function $\zeta(s)$ defined by the series*

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$$

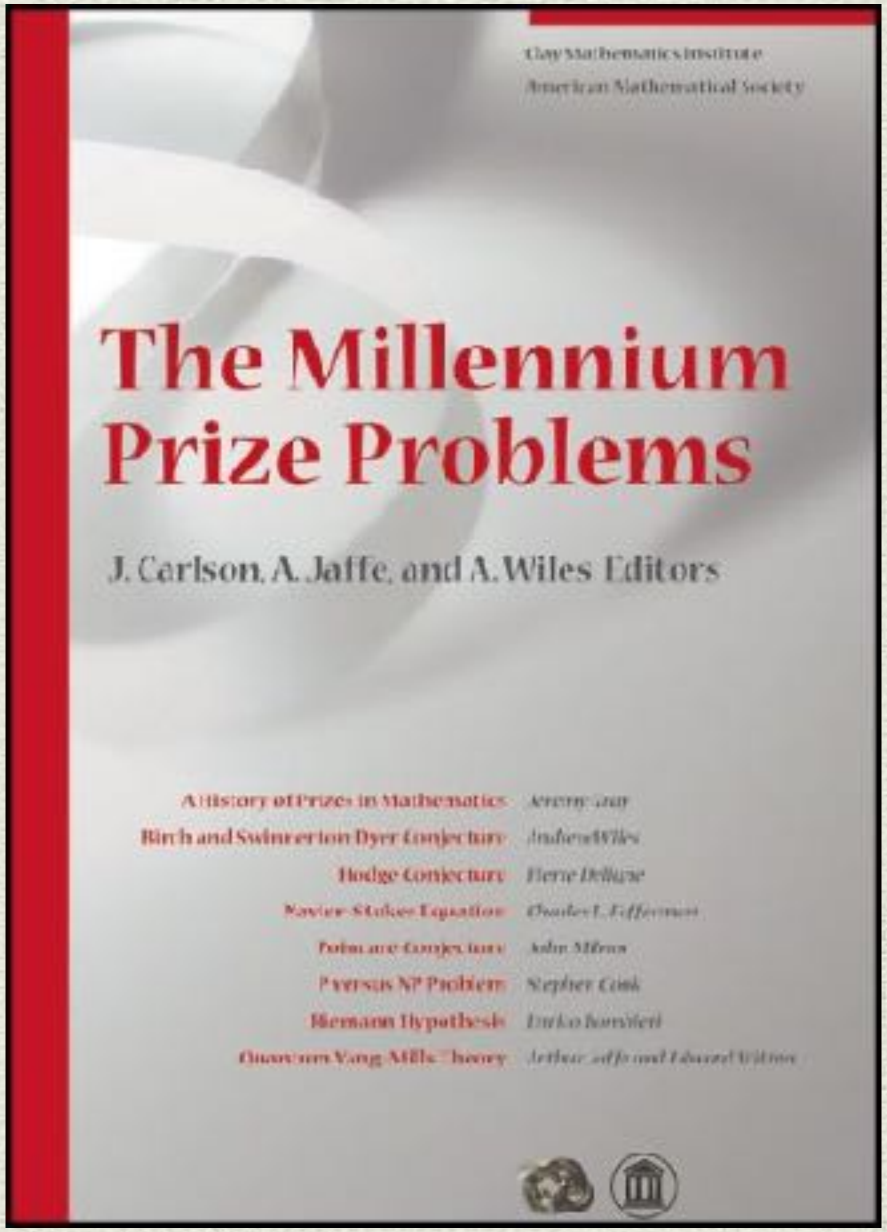
all have the real part $\frac{1}{2}$, except the well-known negative integral real zeros. As soon as this proof has been successfully established, the next problem would consist in testing more exactly Riemann's infinite series for the number of primes below a given number and, especially, *to decide whether the difference between the number of primes below a number x and the integral logarithm of x does in fact become infinite*

* Math. Annalen, vols. 22, 32 (1883, 1888).

David Hilbert (1862 - 1943)

The 7 millennium problems, Clay Math. Inst. 2000
The USD 1,000,000 list

- Riemann hypothesis
- Birch and Swinnerton-Dyer conjecture
- Hodge conjecture
- Navier-Stokes equation: existence and regularity
- Poincaré conjecture
- P vs NP
- Yang-Mills theory



Prime numbers

- The prime numbers are: 2,3,5,7,…

<u>Fundamental theorem of arithmetic</u>: Every natural number can be written in a unique way as a product of primes.

- Goldbach's conjecture: Every even integer (>2) is the sum of two primes.
- Twin prime conjecture: There exist infinitely many pairs of primes of the form p and p+2.

Counting the primes…

x	# primes <= x	Ratio	Difference
10	4	2.5	—
100	25	4	1.5
1,000	168	5.952	1.952
10,000	1,229	8.137	2.185
100,000	9,592	10.425	2.288
1,000,000	78,498	12.739	2.314
10,000,000	664,579	15.047	2.308
100,000,000	5,761,455	17.357	2.310
1,000,000,000	50,847,534	19.667	2.310

The number of primes up to 10^n should be

$$\frac{10^n}{2.3n} = \frac{10^n}{\log 10^n}$$

note that $2.3 \sim \log 10.$

Prime Number Theorem: Let $\pi(x)$ be the quantity of primes up to x. Then:

$$\pi(x) \sim \frac{x}{\log x}$$

- Conjectured by Gauss at the age of 16 (around 1792 - 1793) and proved only 100 later by Hadamard and de la Vallée Poussin.
- Probability that a random number x is prime is $1/\log x$. For example, the probability of a 7-digit number be a prime is:

$$\frac{1}{2.3 \times 7} \sim \frac{1}{16}$$

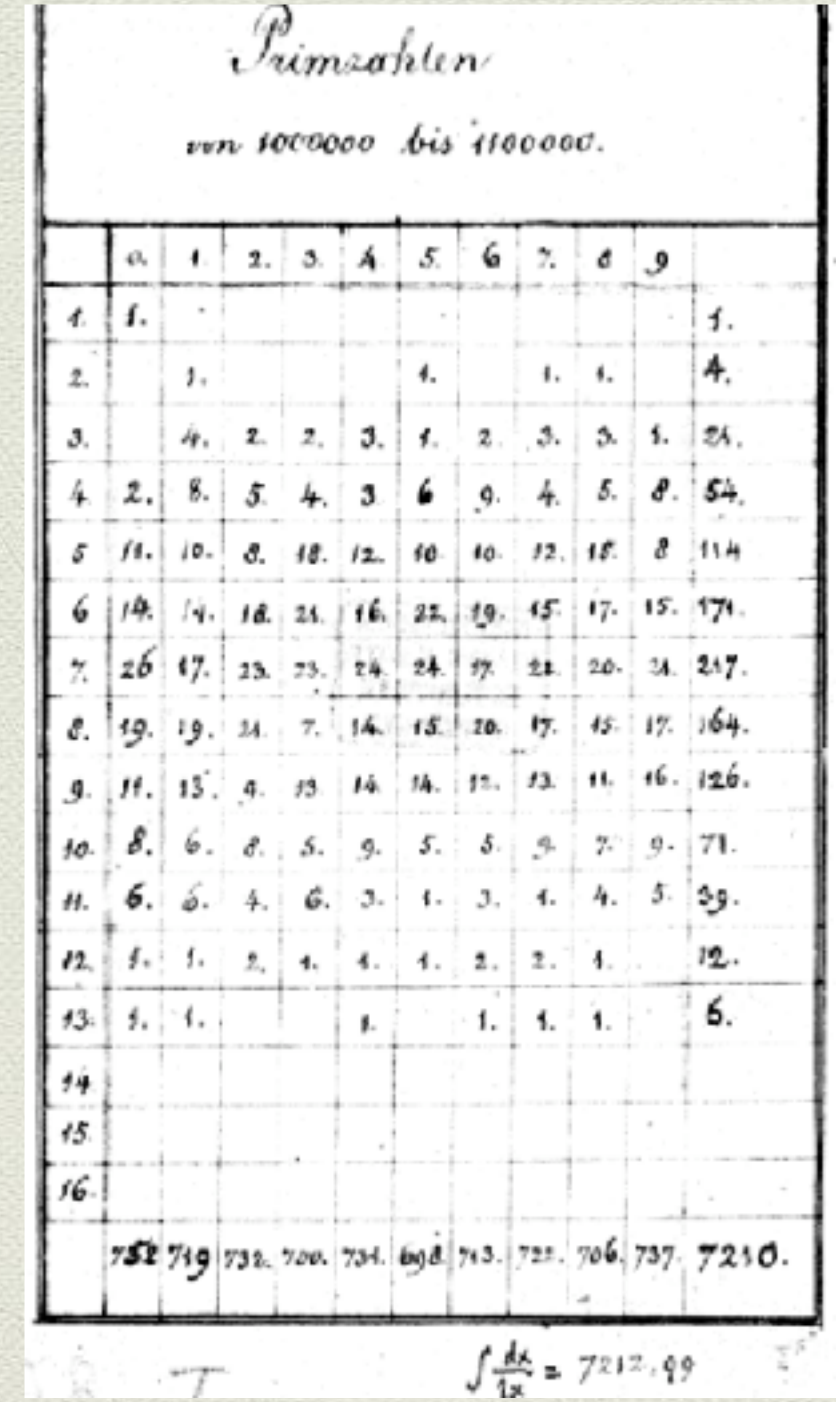
The advisor

Carl Friedrich Gauss (1777 - 1855)

Gauss’ chiliads

- A table where Gauss would keep his statistics on the prime numbers.

Source: American Institute of Mathematics and Göttingen’s Library.



Primzahlen

von 1000000 bis 1100000.

	0.	1.	2.	3.	4	5.	6	7.	8	9	
1.	1.										1.
2.		1.				1.		1.	1.		4.
3.		4.	2.	2.	3.	1.	2.	3.	3.	1.	21.
4	2.	8.	5.	4.	3	6	9.	4.	5.	8.	54.
5	11.	10.	8.	18.	12.	10	10.	12.	15.	8	114
6	14.	14.	18.	21.	16.	22.	19.	15.	17.	15.	171.
7.	26	17.	23.	23.	24.	24.	17.	22.	20.	21.	217.
8.	19.	19.	21.	7.	14.	15.	20.	17.	15.	17.	164.
9.	11.	13.	9.	13.	14.	14.	12.	13.	11.	16.	126.
10.	8.	6.	8.	5.	9.	5.	5.	9.	7.	9.	71.
11.	6.	6.	4.	6.	3.	1.	3.	1.	4.	5.	39.
12.	1.	1.	2.	1.	1.	1.	2.	2.	1.		12.
13.	1.	1.			1.		1.	1.	1.		6.
14											
15											
16.											
	752	719	732.	700.	734.	698.	713.	722.	706.	737.	7210.

$\int \frac{dx}{lx} = 7212{,}99$

The student

Georg Friedrich Bernhard Riemann (1826-1866)

- Riemann wrote only one 8-page paper in number theory. It was called “Über die Anzahl der Primzahlen unter finer gegebenen Grösse”, published in November, 1859. In this paper he considers the function of one complex variable, defined initially for $\mathrm{Re}(s) > 1$

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots .$$

$$= \left(1 + \frac{1}{2^s} + \frac{1}{4^s} + \dots\right)\left(1 + \frac{1}{3^s} + \frac{1}{9^s} + \dots\right)\dots = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} .$$

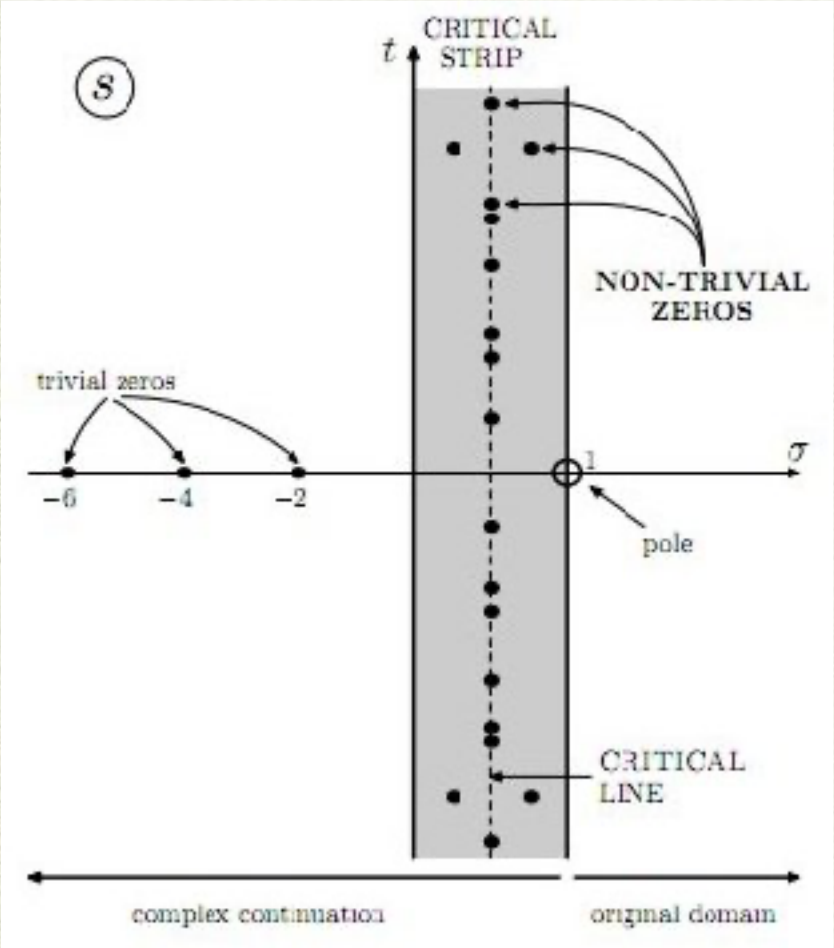
- Before, Euler had already considered similar sums

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{\pi^2}{6} .$$

- He presented some of its properties e.g.

$$\zeta_2(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \dots . \qquad (\text{Re(s)} > 0)$$

$$\zeta_2(s) = (1 - 2^{1-s})\zeta(s) . \quad \text{(pole at s=1)}$$



- Functional equation

$$\xi(s) = \tfrac{1}{2}s(s-1)\,\pi^{-s/2}\,\Gamma\left(\tfrac{s}{2}\right)\zeta(s)$$

$$\xi(1-s) = \xi(s)$$

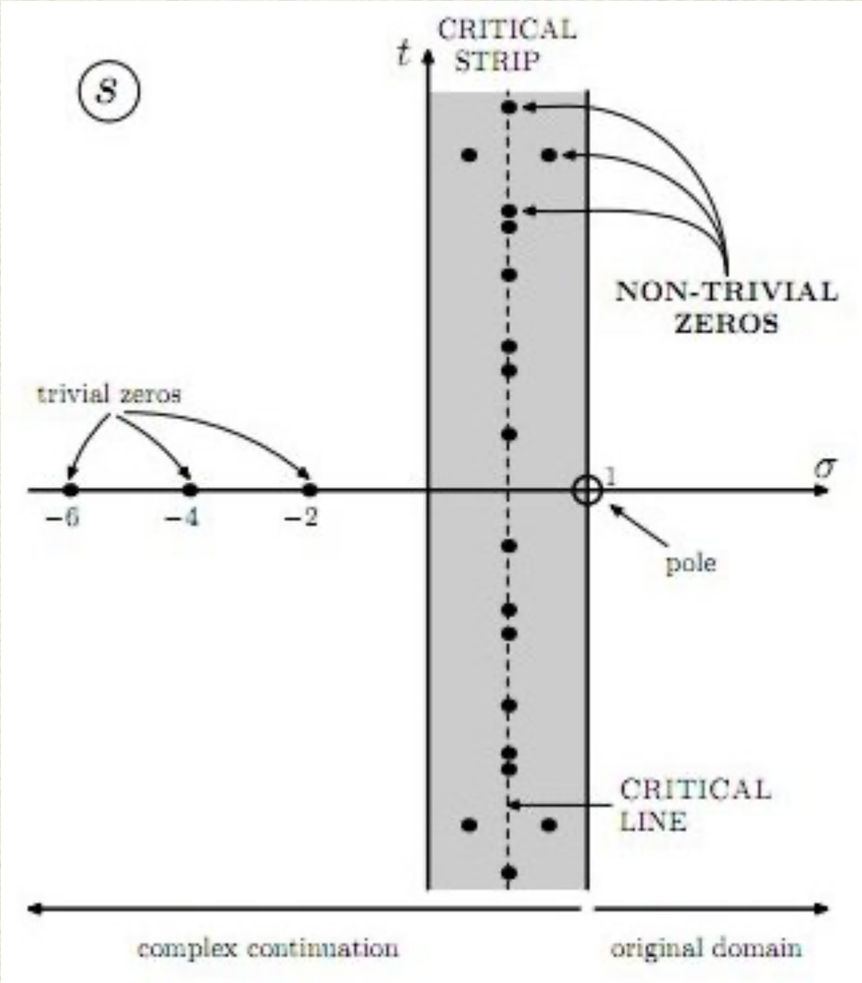
- From the product formula, $\zeta(s)$ has no zeros in Re(s) > 1. From the functional equation $\zeta(s)$ has the trivial zeros $s = -2, -4, \ldots$ All the other zeros are in the critical strip $0 \leq \text{Re(s)} \leq 1$.

- He already knew how to estimate the quantity of zeros up to a certain height T (i.e. with imaginary part between 0 and T):

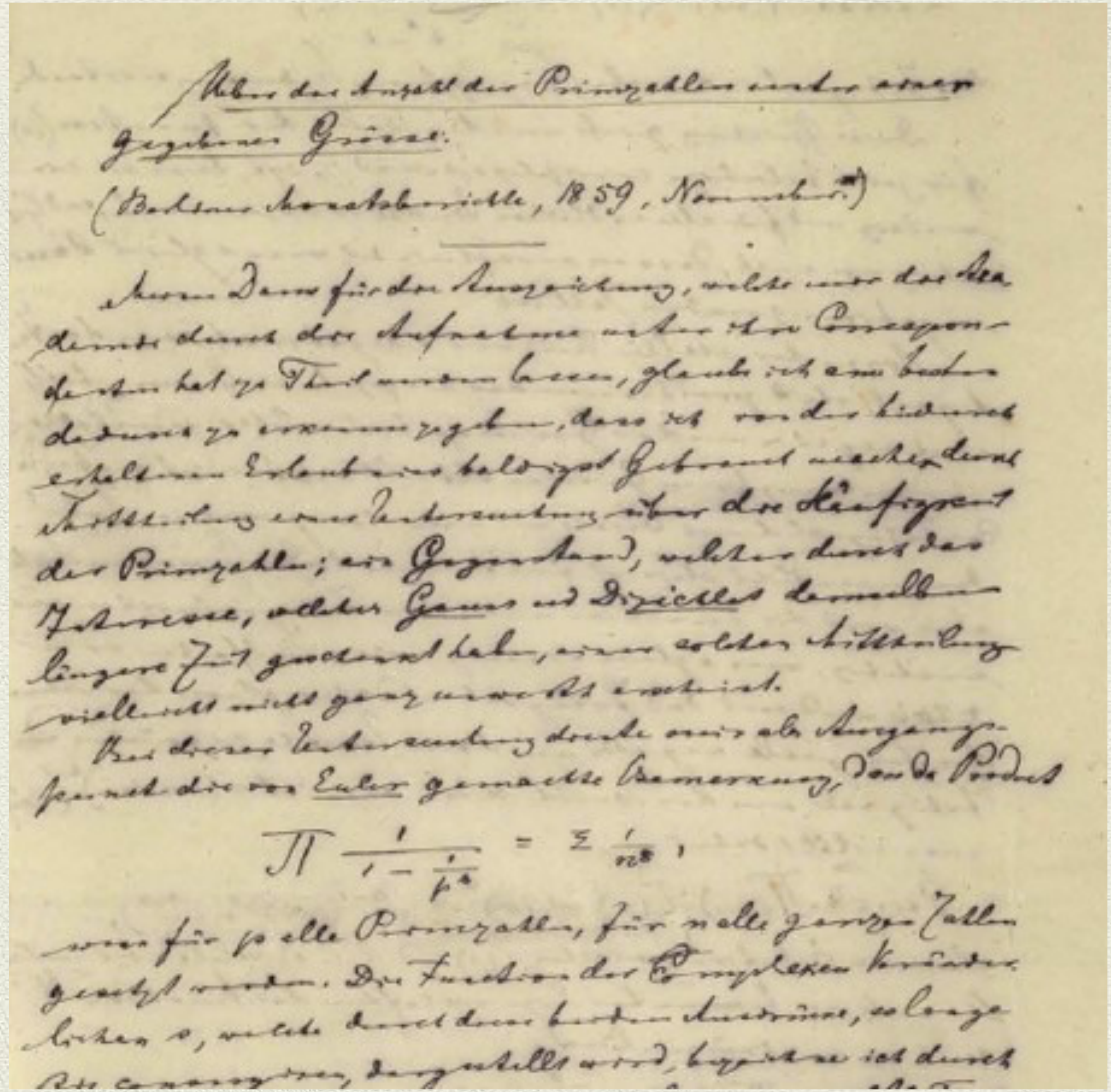
$$N(T) = \frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + O(\log T)$$

<u>Riemann Hypothesis (1859):</u>

All the non-trivial zeros of $\zeta(s)$ have real part 1/2.



The original manuscript



Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.

(Berliner Monatsberichte, 1859, November.)

Meinen Dank für die Auszeichnung, welche mir die Akademie durch die Aufnahme unter ihre Correspondenten hat zu Theil werden lassen, glaube ich am besten dadurch zu erkennen zu geben, dass ich von der hierdurch erhaltenen Erlaubniss baldigst Gebrauch mache durch Mittheilung einer Untersuchung über die Häufigkeit der Primzahlen; ein Gegenstand, welcher durch das Interesse, welches Gauss und Dirichlet demselben längere Zeit geschenkt haben, einer solchen Mittheilung vielleicht nicht ganz unwerth erscheint.

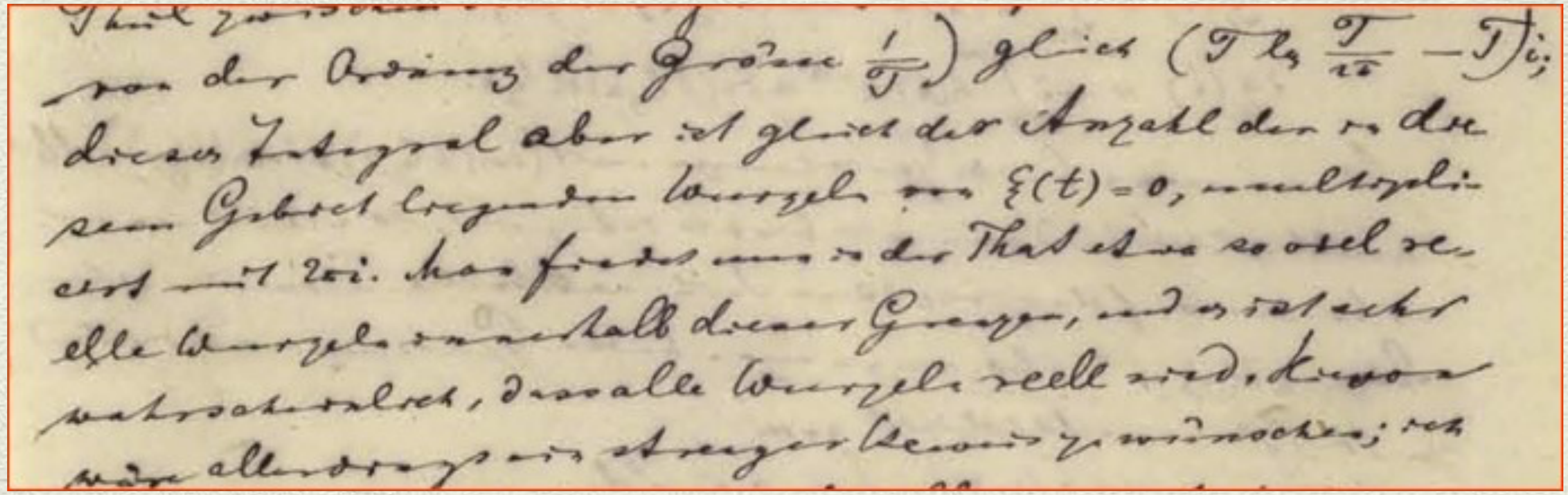
Bei dieser Untersuchung diente mir als Ausgangspunkt die von Euler gemachte Bemerkung, dass das Product

$$\prod \frac{1}{1-\frac{1}{p^s}} = \sum \frac{1}{n^s},$$

wenn für p alle Primzahlen, für n alle ganzen Zahlen gesetzt werden. Die Function der complexen Veränderlichen s, welche durch diese beiden Ausdrücke, so lange sie convergiren, dargestellt wird, bezeichne ich durch

Source: American Institute of Mathematics and Göttingen's Library

The Riemann hypothesis



Source: American Institute of Mathematics and Göttingen's Library

"One now finds indeed approximately this number of roots within these limits, and it is very probable that all roots are real. Certainly one would wish for a stricter proof here; I have meanwhile temporarily put aside the search for this, after some fleeting futile attempts, as it appears unnecessary for the next objective of my investigation.…"

The published version in the *Monatsberichte der Berliner Akademie*, November of 1859.

Source: American Institute of Mathematics and Göttingen's Library

Bericht

über die

zur Bekanntmachung geeigneten Verhandlungen der Königl. Preuſs. Akademie der Wissenschaften zu Berlin

im Monat November 1859.

Vorsitzender Sekretar: Hr. Encke.

3. Nov. Gesammtsitzung der Akademie.

Hr. Steiner las über einige allgemeine Bestimmungsarten der Curven und Flächen zweiter Ordnung und daraus folgenden Sätzen.

Hierauf trug Hr. Kummer folgende von Hrn. Riemann, Correspondenten der Akademie, mittelst eines an den Sekretar Hrn. Encke gerichteten Schreibens vom 19. October d. J. eingesandte Mittheilung „über die Anzahl der Primzahlen unter einer gegebenen Gröſse" vor:

Meinen Dank für die Auszeichnung, welche mir die Akademie durch die Aufnahme unter ihre Correspondenten hat zu Theil werden lassen, glaube ich am besten dadurch zu erkennen zu geben, daſs ich von der hiedurch erhaltenen Erlaubniſs baldigst Gebrauch mache durch Mittheilung einer Untersuchung über die Häufigkeit der Primzahlen; ein Gegenstand, welcher durch das Interesse, welches Gauss und Dirichlet demselben längere Zeit geschenkt haben, einer solchen Mittheilung vielleicht nicht ganz unwerth erscheint.

Bei dieser Untersuchung diente mir als Ausgangspunkt die von Euler gemachte Bemerkung, daſs das Product

$$\Pi \frac{1}{1-\frac{1}{p^s}} = \Sigma \frac{1}{n^s},$$

Some facts:

- The fact that $\zeta(1+it) \neq 0$ was proved by Hadamard and de la Vallée Poussin in 1896 and implies the prime number theorem.
- G. H. Hardy and J. E. Littlewood liked to suggest the Riemann hypothesis (disguised) as a thesis project for their Ph.D. students. Hardy was the first to prove that there are infinitely many zeros on the critical line.

$$\pi(x) := \sum_{p \leq x} 1 = \int_2^x \frac{du}{\log u} + O\left(x^{1/2} \log x\right)$$

- A. Selberg (1942) proved that a positive proportion of the zeros is on the critical line. B. Conrey (1989) proved that at least 40% of the zeros are there.
- The first 10,000,000,000,000 of zeros are there…

Part 2: Oscillations

Our guest of honor

Eiffel Tower - Paris
72 names de French scientists
(curiosity: 21 mathematicians)

Jean-Baptiste Joseph Fourier
(1768 - 1830)

The Fourier transform

$f \in L^1(\mathbb{R})$

- For each $t \in \mathbb{R}$ we define: $$\hat{f}(t) = \int_{-\infty}^{\infty} e^{-2\pi itx} f(x)\, dx$$
- Fourier inversion: $$f(x) = \int_{-\infty}^{\infty} e^{2\pi itx} \hat{f}(t)\, dt$$
- (Plancherel) If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ then:

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |\hat{f}(t)|^2\, dt$$

- "Band-limited" function: $\hat{f}$ has compact support

Learning Analysis

Real Analysis

Complex Analysis

Harmonic Analysis

Analytic Number Theory

Functional Analysis

PDEs

…

Focus on fundamentals

Sr. Miyagi teaching Analysis

Part 3: Let us practice!

$$\xi(s) = \tfrac{1}{2}s(s-1)\,\pi^{-s/2}\,\Gamma\!\left(\tfrac{s}{2}\right)\zeta(s)$$

$$\xi(1-s) = \xi(s) \quad \text{and} \quad \overline{\xi(\bar{s})} = \xi(s)$$

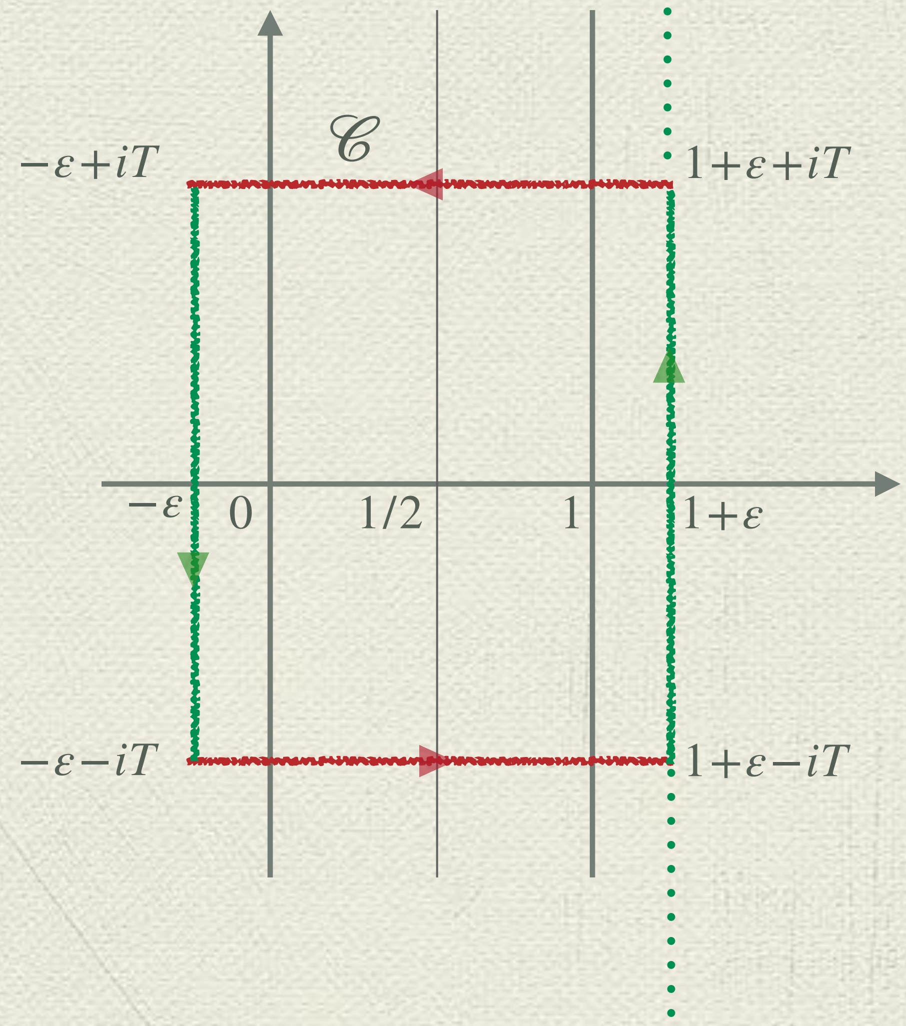
$$\frac{\xi'(s)}{\xi(s)} = \frac{1}{s} + \frac{1}{s-1} - \frac{\log \pi}{2} + \frac{1}{2}\frac{\Gamma'(s/2)}{\Gamma(s/2)} + \frac{\zeta'(s)}{\zeta(s)}$$

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{n\geq 1}\frac{\Lambda(n)}{n^s}; \qquad \Lambda(n) = \log p \quad \text{when } n = p^k\,.$$

....

Let h be a good function

$$\sum_{\rho\;;\;\zeta(\rho)=0} h\!\left(\frac{\rho-\frac{1}{2}}{i}\right) = \frac{1}{2\pi i}\int_{\mathscr{C}} h\!\left(\frac{s-\frac{1}{2}}{i}\right)\frac{\xi'(s)}{\xi(s)}\,ds$$



Theme 1: Explicit formula

$$\sum_{\rho\ ;\ \zeta(\rho)=0} h\left(\frac{\rho-\frac{1}{2}}{i}\right) = h\left(\frac{1}{2i}\right) + h\left(-\frac{1}{2i}\right) - \frac{1}{2\pi}\widehat{h}(0)\log\pi$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty} h(u)\,\mathrm{Re}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{iu}{2}\right)\,du$$

$$-\frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\left(\widehat{h}\left(\frac{\log n}{2\pi}\right)+\widehat{h}\left(\frac{-\log n}{2\pi}\right)\right)$$

Zeros, primes and the Fourier transform

Under RH… $\rho = \frac{1}{2} + i\gamma \quad (\gamma \text{ real!!})$

Theme 1: Explicit formula (on RH)

$$\sum_{\gamma} h(\gamma) = h\left(\frac{1}{2i}\right) + h\left(-\frac{1}{2i}\right) - \frac{1}{2\pi}\hat{h}(0)\log\pi$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty} h(u)\,\mathrm{Re}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{iu}{2}\right)\,du$$

$$-\frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\left(\hat{h}\left(\frac{\log n}{2\pi}\right)+\hat{h}\left(\frac{-\log n}{2\pi}\right)\right)$$

Note: h must be good…and to try to estimate, the right-hand side must also be something reasonable, e.g. h bandlimited.

Theme 2: Estimating zeta

$$N(t) = \frac{t}{2\pi}\log\left(\frac{t}{2\pi}\right) - \frac{t}{2\pi} + \frac{7}{8} + S(t) + O\left(\frac{1}{t}\right)$$

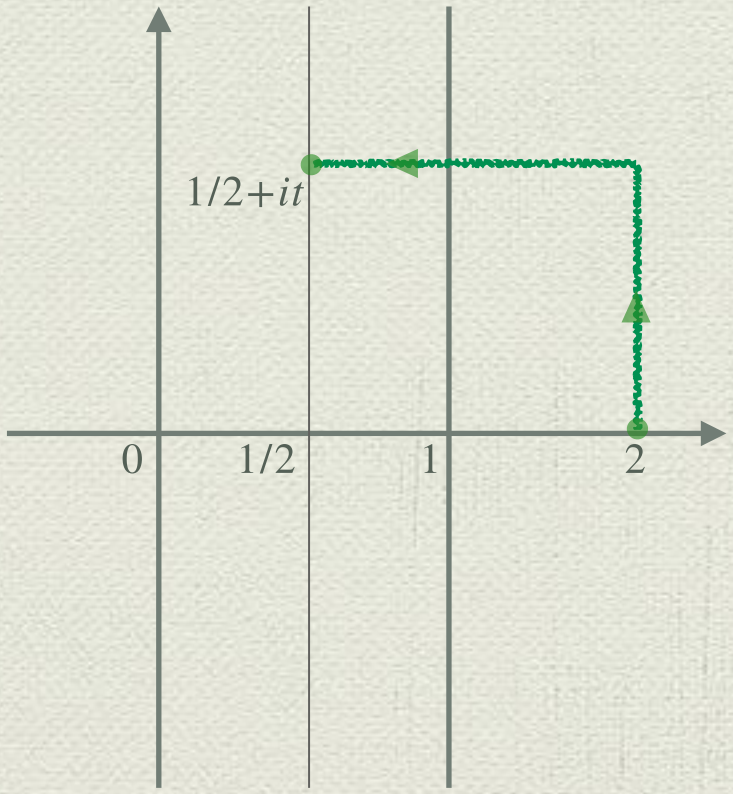
$$S(t) := \frac{1}{\pi}\arg\zeta\left(\tfrac{1}{2}+it\right) \qquad \left(\text{recall}: \quad z = |z|e^{i\theta}\right)$$

Unconditionally:

$$S(t) = O(\log t)$$

Under RH, Littlewood showed in 1924 that:

$$\log|\zeta(\tfrac{1}{2}+it)| = O\left(\frac{\log t}{\log\log t}\right) \quad ; \quad S(t) = O\left(\frac{\log t}{\log\log t}\right)$$



Theorem (Chandee & Soundararajan - 2011) Under RH:

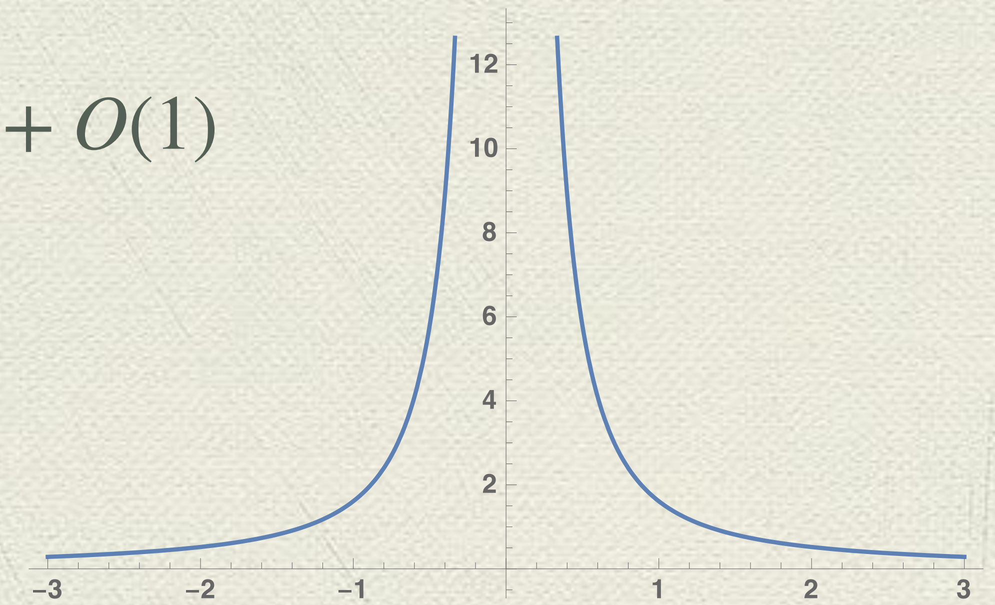
$$\log\left|\zeta(\tfrac{1}{2}+it)\right| \leq \left(\frac{\log 2}{2} + o(1)\right)\frac{\log t}{\log\log t}$$

Previous: Soundararajan (2009): C=0.372…

Idea:

$$\log\left|\zeta(\tfrac{1}{2}+it)\right| = \log t - \frac{1}{2}\sum_{\gamma} f(t-\gamma) + O(1)$$

$$f(x) = \log\left(\frac{4+x^2}{x^2}\right)$$



$$\xi(s) = \tfrac{1}{2}s(s-1)\,\pi^{-s/2}\,\Gamma\left(\tfrac{s}{2}\right)\zeta(s) = e^{A+Bs}\prod_\rho\left(1-\tfrac{s}{\rho}\right)e^{s/\rho}$$

$$B = -\sum_\rho \mathrm{Re}(1/\rho)$$

$$\left|\frac{\xi(1/2+it)}{\xi(5/2-it)}\right| = \left|\frac{\xi(1/2+it)}{\xi(-3/2+it)}\right| = \prod_{\rho=1/2+i\gamma}\left(\frac{(t-\gamma)^2}{4+(t-\gamma)^2}\right)^{\frac{1}{2}}.$$

Using Stirling's formula

$$\log\Gamma(z) = \frac{1}{2}\log 2\pi - z + \left(z-\tfrac{1}{2}\right)\log z + O\left(|z|^{-1}\right),$$

we get

$$\log\left|\zeta(\tfrac{1}{2}+it)\right| = \log t - \frac{1}{2}\sum_\gamma \log\left(\frac{4+(t-\gamma)^2}{(t-\gamma)^2}\right) + O(1)$$

Theorem (C. , Chandee & Milinovich - 2013) Under RH:

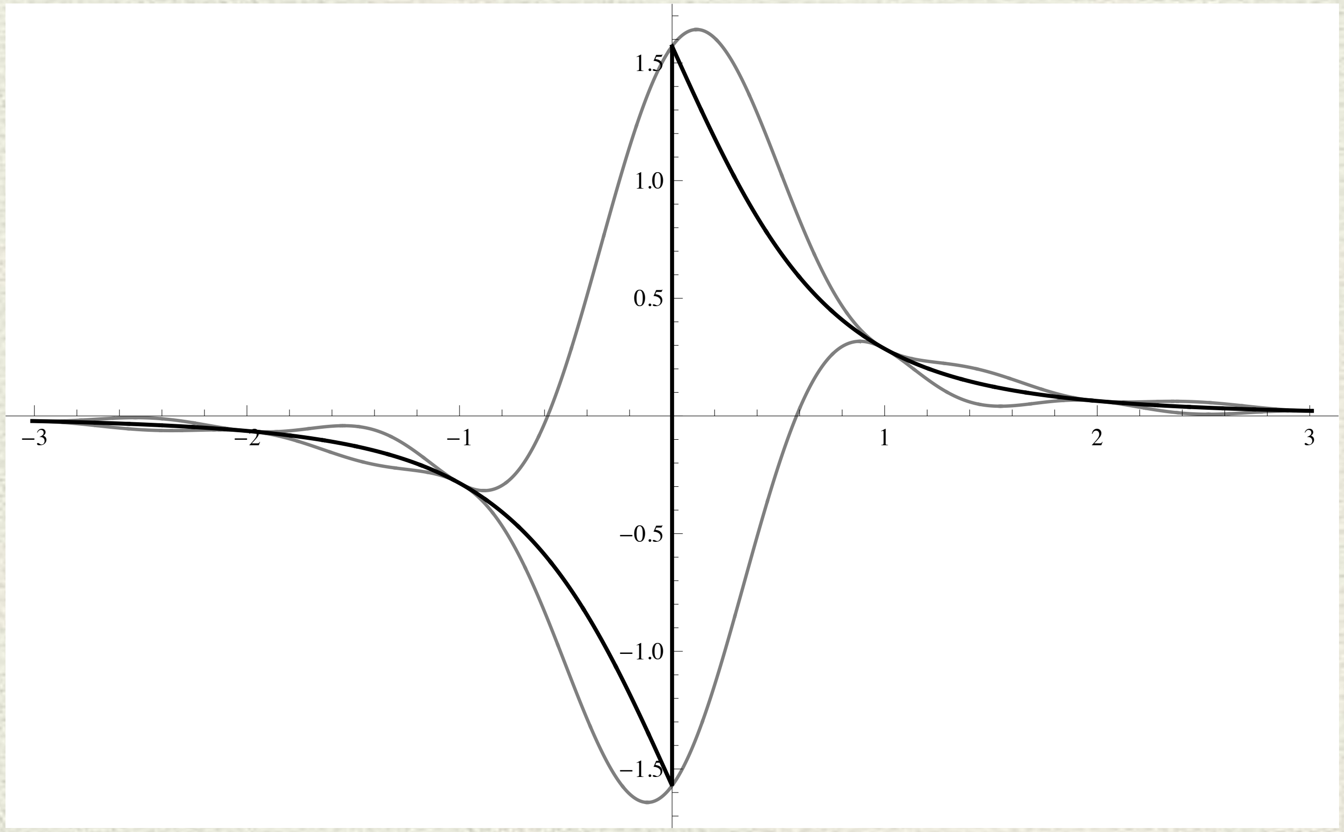
$$|S(t)| \leq \left(\frac{1}{4} + o(1)\right) \frac{\log t}{\log \log t}$$

Previous work: Goldston & Gonek (2007): C=1/2

Idea:

$$S(t) = \frac{1}{\pi} \sum_{\gamma} f(t - \gamma) + O(1)$$

$$f(x) = \arctan\left(\frac{1}{x}\right) - \frac{x}{1 + x^2}$$



We have written our objects as:

$$\text{Object}(t) = K(t) + \sum_{\gamma} g(t-\gamma) + O(1)\,.$$

From the explicit formula it would be "very nice" if we could choose a special function M such that:

(i) $g \le M$

(ii) $\widehat{M}$ has compact support, say in $[-\Delta, \Delta]$.

(iii) $\widehat{M}(0)$ is as close as possible to $\widehat{g}(0)$, i.e.

$$\int_{-\infty}^{\infty} \{M(x) - g(x)\}\,\mathrm{d}x$$

is minimal.

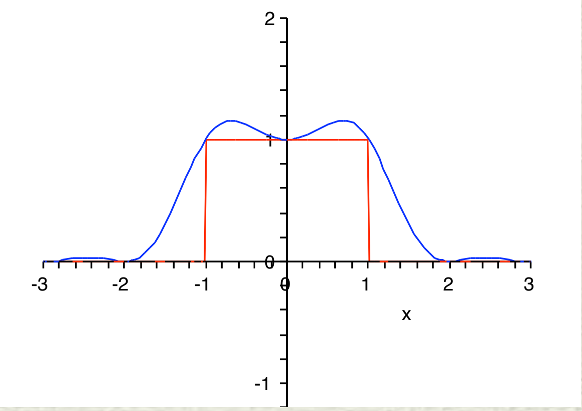
Theme 3: Bandlimited approximations

Given $f:\mathbb{R}\to\mathbb{R}$, find $L:\mathbb{R}\to\mathbb{R}$ and $M:\mathbb{R}\to\mathbb{R}$ such that:

$i)$ $L \leq f \leq M$.

$ii)$ $\mathrm{supp}(\widehat{L}),\ \mathrm{supp}(\widehat{M}) \subset [-\Delta, \Delta]$.

$iii)$ Minimizing $\|f-L\|_1$ and $\|f-M\|_1$.



Beurling-Selberg 1938-1950

Theorem (C., Littmann & Vaaler, 2010-11) Solution for f as:

$$f(x) = \int_0^\infty e^{-\lambda\pi x^2}\,d\mu(\lambda) \quad \text{and} \quad f(x) = \mathrm{sgn}(x)\int_0^\infty e^{-\lambda\pi x^2}\,d\mu(\lambda)$$

$$f(x) = arctan\left(\frac{1}{x}\right) - \frac{x}{1+x^2} = \mathrm{sgn}(x)\int_0^\infty e^{-\pi\lambda x^2}\left(\int_0^\infty \frac{t}{2\sqrt{\pi\lambda^3}}\, e^{-\frac{t^2}{4\lambda}}\left(\frac{1}{t}\sin(\sqrt{\pi}t) - \sqrt{\pi}\cos(\sqrt{\pi}t)\right)dt\right)d\lambda$$

Theme 4: Paley-Wiener theorem

For $f \in L^2(\mathbb{R})$ the following are equivalent:

i) $\operatorname{supp}(\hat{f}) \subset [-\Delta, \Delta]$.

ii) f can be extended to an entire function of order 1 and

$$|f(z)| \leq C_\varepsilon \, e^{(2\pi\Delta+\varepsilon)|z|} \quad \text{for all } \varepsilon > 0.$$

Raymond Paley (1907-1933)

Norbert Wiener (1894 - 1964)

Application:

Let $\Delta > 0$ and $f(x) = \arctan\left(\frac{1}{x}\right) - \frac{x}{1+x^2}$.

There is a unique entire function $M_\Delta \geq g$ such that

i) $$|M_\Delta(x+iy)| \ll \frac{\Delta^2}{1+\Delta|x+iy|}e^{2\pi\Delta|y|}.$$

ii) The Fourier transform $\widehat{M_\Delta}$ is supported on $[-\Delta, \Delta]$ and

$$|\widehat{M_\Delta}(\xi)| \ll 1$$

iii) The L^1- distance between M_Δ and f equals to

$$\int_{-\infty}^{\infty} \{M_\Delta(x) - g(x)\}\, \mathrm{d}x = \frac{\pi}{2\Delta}.$$

$$f(x) = \arctan\left(\frac{1}{x}\right) - \frac{x}{1+x^2}$$

$$h(z) := M_\Delta(t-z)$$

$$\widehat{h}(\xi) = \widehat{M_\Delta}(-\xi)e^{-2\pi i\xi t}$$

$$S(t) = \frac{1}{\pi}\sum_\gamma f(t-\gamma) + O(1) \leq \frac{1}{\pi}\sum_\gamma M_\Delta(t-\gamma) + O(1)$$

$$= \frac{1}{\pi}\Bigg(\underbrace{h\left(\frac{1}{2i}\right) + h\left(-\frac{1}{2i}\right)}_{1} - \underbrace{\frac{1}{2\pi}\widehat{h}(0)\log\pi}_{2}$$

$$\underbrace{+\frac{1}{2\pi}\int_{-\infty}^{\infty} h(u)\,\mathrm{Re}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{iu}{2}\right)du}_{3}$$

$$\underbrace{-\frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\left(\widehat{h}\left(\frac{\log n}{2\pi}\right) + \widehat{h}\left(\frac{-\log n}{2\pi}\right)\right)}_{4}\Bigg) + O(1)$$

1 $\ll \dfrac{\Delta^2 e^{\pi\Delta}}{1+\Delta t}$; 2 $\ll 1$; 4 $\ll \displaystyle\sum_{n=2}^{e^{2\pi\Delta}} \frac{\Lambda(n)}{\sqrt{n}} \ll e^{\pi\Delta}$

3

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} h^{\pm}(u)\,\mathrm{Re}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{iu}{2}\right)\,\mathrm{d}u \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} M_{\Delta}^{\pm}(u)\big(\log t + O(\log(2+|u|))\big)\,\mathrm{d}u \\
&= \pm\frac{\log t}{4\Delta} + O(1)\,.
\end{aligned}
$$

Hence

$$
S(t) \le \frac{\log t}{4\pi\Delta} + O\left(e^{\pi\Delta}+1\right) + O\left(\frac{\Delta^2 e^{\pi\Delta}}{1+\Delta t}\right).
$$

Choose $\pi\Delta = \log\log t - 3\log\log\log t$

$$
S(t) \le \left(\frac{1}{4}+o(1)\right)\frac{\log t}{\log\log t}\,.
$$

Fourier optimization and number theory II

Emanuel Carneiro

ICTP - International Centre for Theoretical Physics - Trieste, Italy

El Escorial, June 2022

Last night...

in "El Rey de Copas" in San Lorenzo de El Escorial

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in "El Rey de Copas" in San Lorenzo de El Escorial

Our hero

Last night...

in "El Rey de Copas" in San Lorenzo de El Escorial

Matt

Last night...

in "El Rey de Copas" in San Lorenzo de El Escorial

Mistery girl

Matt

Last night...

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Matt: How about I buy you a beer?

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Goal: Help Matt buy Catherine Zeta-Jones a beer tonight!

A classical problem

1. For $f : \mathbb{R} \to \mathbb{R}$, our normalization for Fourier transform is

$$\widehat{f}(t) = \int_{\mathbb{R}} e^{-2\pi i x t} f(x)\, \mathrm{d}x.$$

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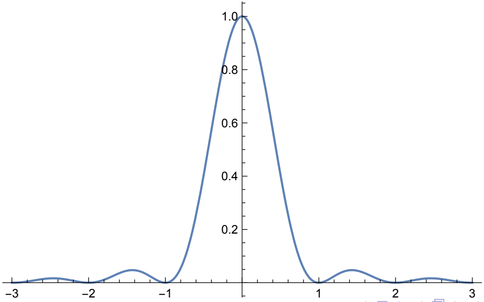

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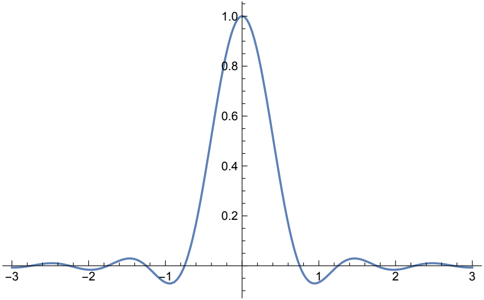

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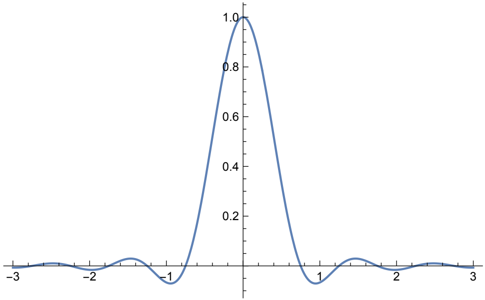

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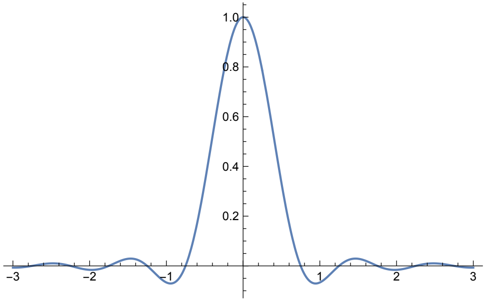

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4. There exists a unique extremizer.

Prime gaps

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3. Hoheisel (1930): There is always a prime in $[x, x + x^{\theta}]$ for some $0 < \theta < 1$, and x large.
4. Baker - Harman - Pintz (2001): There is always a prime in $[x, x + x^{0.525}]$ for x large.

Prime gaps on RH

Cramér's bounds (1920)

1. $$p_{n+1} - p_n = O(\sqrt{p_n}\log p_n),$$
   i.e. every interval $[x, x + c\sqrt{x}\log x]$, for some $c > 0$, contains a prime when x is large.

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2. Historic progress:
   - Goldston '83: $c = 4$.
   - Ramaré and Saouter '03: $c = 8/5$
   - Dudek '15: $c = 1 + o(1)$.

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3. Non-asymptotic version - best result due to Dudek, Grenié, Molteni '16: for $x \geq 4$, every interval
   $$[x, x + c\sqrt{x}\log x]$$
   contains a prime. Here $c = 1 + \frac{4}{\log x}$.
   - Ramaré and Saouter '03 ($c = 8/5$)

Improved estimates

joint with M. Milinovich and K. Soundararajan '19

Theorem (Asymptotic version)

Assume RH. For x large, every interval

$$\left[x, x + \frac{21}{25}\sqrt{x}\log x\right]$$

contains a prime.

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Theorem (Asymptotic version)

Assume RH. For x large, every interval

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contains a prime.

Theorem (Non-asymptotic version)

Assume RH. For $x \geq 4$, every interval

$$\left[x, x + \frac{22}{25}\sqrt{x}\log x\right]$$

contains a prime.

Strategy

1. Explicit formula connecting zeros of $\zeta(s)$ and primes.
2. Fourier optimization problems
3. Brun-Titchmarsh inequality.

Explicit formula

Lemma (Guinand-Weil Explicit Formula)

Let $h(s)$ be analytic in the strip $|\mathrm{Im}(s)| \leq 1/2 + \varepsilon$ for some $\varepsilon > 0$, and such that $|h(s)| \ll (1 + |s|)^{-(1+\delta)}$ when $|\mathrm{Re}(s)| \to \infty$.

$$\sum_{\rho} h(\gamma) = h\Big(\frac{1}{2i}\Big) + h\Big(-\frac{1}{2i}\Big) - \frac{1}{2\pi}\widehat{h}(0)\log\pi + \frac{1}{2\pi}\int_{-\infty}^{\infty} h(u)\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4} + \frac{iu}{2}\Big)\,\mathrm{d}u - \frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\Big(\widehat{h}\Big(\frac{\log n}{2\pi}\Big) + \widehat{h}\Big(\frac{-\log n}{2\pi}\Big)\Big),$$

where $\rho = \frac{1}{2} + i\gamma$ are the non-trivial zeros of $\zeta(s)$.

Idea: to use this with $\widehat{h}\Big(\pm\frac{\log\cdot}{2\pi}\Big)$ localized in an interval without primes.

Setup

For this let f be a smooth function such that $\mathrm{supp}(\widehat{f}) \subset [-1, 1]$, let $0 < \Delta \leq 1$, let $1 < a$, and set

$$g(z) = \Delta f(\Delta z) \quad ; \quad h(z) = g(z)a^{iz}$$

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Then

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Then

$$\sum_{\rho} g(\gamma)a^{i\gamma} = g\Big(\frac{1}{2i}\Big)a^{1/2} + g\Big(-\frac{1}{2i}\Big)a^{-1/2} - \frac{1}{2\pi}\widehat{g}\Big(-\frac{\log a}{2\pi}\Big)\log \pi$$
$$+ \frac{1}{2\pi}\int_{-\infty}^{\infty} g(u)\, a^{iu}\, \mathrm{Re}\, \frac{\Gamma'}{\Gamma}\Big(\frac{1}{4} + \frac{iu}{2}\Big)\, \mathrm{d}u$$
$$- \frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\Big(\widehat{g}\Big(\frac{\log(n/a)}{2\pi}\Big) + \widehat{g}\Big(\frac{-\log na}{2\pi}\Big)\Big),$$

Setup III

For this let f be a smooth function such that $\mathrm{supp}(\widehat{f}) \subset [-1,1]$, let $0 < \Delta \leq 1$, let $1 < a$, and set

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Setup IV

We get

$$
\begin{aligned}
&\left| g\left(\frac{1}{2i}\right) a^{1/2} + g\left(-\frac{1}{2i}\right) a^{-1/2} \right| \\
&\qquad \leq \sum_{\gamma} |g(\gamma)| + \frac{1}{2\pi}\left|\widehat{g}\left(-\frac{\log a}{2\pi}\right)\right| \log \pi \\
&\qquad + \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} g(u)\, a^{iu}\, \mathrm{Re}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{iu}{2}\right) \mathrm{d}u \right| \\
&\qquad + \left| \frac{1}{2\pi} \sum_{n \geq 2} \frac{\Lambda(n)}{\sqrt{n}} \left(\widehat{g}\left(\frac{\log(n/a)}{2\pi}\right) + \widehat{g}\left(-\frac{\log na}{2\pi}\right)\right)\right|.
\end{aligned}
$$

Recall that

$$
g(z) = \Delta f(\Delta z) \quad ; \quad h(z) = g(z) a^{iz}
$$

Localization

Assume that for a certain $c > 0$ there is an infinite sequence of $x \to \infty$ such that $[x, x + c\sqrt{x}\log x]$ contains no primes. Choose

$$[x, x + c\sqrt{x}\log x] = \left[a\, e^{-2\pi\Delta}, a\, e^{2\pi\Delta}\right]$$

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Then

$$4\pi\Delta = \log\left(1 + c\frac{\log x}{\sqrt{x}}\right) = c\frac{\log x}{\sqrt{x}} + O\left(\frac{\log^2 x}{x}\right)$$

and

$$a = x\left(1 + c\frac{\log x}{\sqrt{x}}\right)^{1/2} = x + O(\sqrt{x}\log x).$$

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Idea: Perform an asymptotic analysis as $x \to \infty$.

Asymptotic analysis: error terms

Since $\mathrm{supp}(\widehat{g}) \subset [-\Delta, \Delta]$, and the interval $\left[a\, e^{-2\pi\Delta}, a\, e^{2\pi\Delta}\right]$ has no primes, one can verify that

1.
$$\left| \frac{1}{2\pi} \sum_{n \geq 2} \frac{\Lambda(n)}{\sqrt{n}} \left(\widehat{g}\left(\frac{\log(n/a)}{2\pi} \right) + \widehat{g}\left(-\frac{\log na}{2\pi} \right) \right) \right| = O(1).$$

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2. As $a \to \infty$ we have $\widehat{g}\left(-\frac{\log a}{2\pi}\right) = 0$.

Asymptotic analysis: error terms II

3. Also, using Stirling's formula $\frac{\Gamma'}{\Gamma}(s) = \log s + O(|s|^{-1})$ we get

$$\int_{-\infty}^{\infty} g(u)\, a^{iu}\, \mathrm{Re}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{iu}{2}\right) \mathrm{d}u$$

$$= \int_{-\infty}^{\infty} f(y)\, e^{2\pi i y \left(\frac{\log a}{2\pi\Delta}\right)} \log\left|\frac{1}{4} + \frac{iy}{2\Delta}\right| \mathrm{d}y + O(1)$$

$$= \int_{-\infty}^{\infty} f(y)\, e^{2\pi i y \left(\frac{\log a}{2\pi\Delta}\right)} \left(\frac{1}{2}\log(\Delta^2 + 4y^2) + \log\left(\frac{1}{4\Delta}\right)\right) \mathrm{d}y + O(1)$$

$$= \log\left(\frac{1}{4\Delta}\right) \widehat{f}\left(-\frac{\log a}{2\pi\Delta}\right) + O(1) = O(1).$$

Main competition

Matters are reduced to

$$\left| g\left(\frac{1}{2i}\right) a^{1/2} + g\left(-\frac{1}{2i}\right) a^{-1/2} \right| \leq \sum_{\gamma} |g(\gamma)| + O(1).$$

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Observe that

$$\begin{aligned} g\left(\frac{1}{2i}\right) &= \Delta f\left(\frac{\Delta}{2i}\right) = \Delta \int_{-1}^{1} e^{\pi t \Delta}\, \widehat{f}(t)\, \mathrm{d}t \\ &= \Delta \int_{-1}^{1} \widehat{f}(t)\, \mathrm{d}t + \Delta \int_{-1}^{1} \left(e^{\pi t \Delta} - 1\right) \widehat{f}(t)\, \mathrm{d}t \\ &= \Delta f(0) + O(\Delta^2). \end{aligned}$$

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We may similarly estimate $g(-\frac{1}{2i})$ and, hence, the (LHS) above is

$$g\left(\frac{1}{2i}\right)a^{1/2}+g\left(-\frac{1}{2i}\right)a^{-1/2}=\Delta f(0)\big(a^{1/2}+a^{-1/2}\big)+O(\Delta^2a^{1/2}).$$

Sum over zeros

Let $N(x)$ denote the number of zeros with $0 < \gamma \leq x$. Using the fact that $N(x) = \frac{x}{2\pi} \log \frac{x}{2\pi} - \frac{x}{2\pi} + O(\log x)$, we evaluate the sum $\sum_{\gamma} |g(\gamma)|$ using summation by parts to get

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$$\sum_\gamma |g(\gamma)| = \frac{1}{2\pi} \int_{-\infty}^{\infty} |g(x)| \, \log^+ \frac{|x|}{2\pi} \, \mathrm{d}x + O\big(\|g\|_\infty + \|g'(x) \, \log^+ |x|\|_1\big),$$

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where $\log^+ x = \max\{\log x, 0\}$ for $x > 0$.

Recalling that $g(x) = \Delta f(\Delta x)$,

$$\begin{aligned}\sum_\gamma |g(\gamma)| &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |f(y)|\, \log^+|y/2\pi\Delta|\,\mathrm{d}y + O(1)\\ &= \frac{\log(1/2\pi\Delta)}{2\pi}\|f\|_1 + O(1).\end{aligned}$$

Conclusion

We get

$$\Delta f(0)\, a^{1/2} + O(\Delta^2 a^{1/2}) \leq \frac{\log(1/2\pi\Delta)}{2\pi}\|f\|_1 + O(1).$$

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$$\frac{c\, f(0)}{4\pi} \log x \leq \frac{\|f\|_1}{4\pi} \log x + O(1)$$

along this sequence of $x \to \infty$. This is only possible if

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$$\frac{c\, f(0)}{4\pi}\log x \leq \frac{\|f\|_1}{4\pi}\log x + O(1)$$

along this sequence of $x \to \infty$. This is only possible if

$$c \leq \frac{\|f\|_1}{f(0)} \leq 0.9259...$$

as we wanted to show.

This leads to...

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3. Here **B** is the Brun-Titchmarsh constant in our desired scale
$$\mathbf{B} := \limsup_{x \to \infty} \frac{\pi(x + \sqrt{x}) - \pi(x)}{\sqrt{x}/\log x}.$$

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$$\mathbf{B} := \limsup_{x\to\infty} \frac{\pi(x+\sqrt{x}) - \pi(x)}{\sqrt{x}/\log x}.$$
4. By the PNT (on the left) and work of Iwaniec (on the right):
$$1 \leq \mathbf{B} \leq \frac{36}{11}.$$

Extremal problems

1. **Problem 1:** Given $1 \le A \le \infty$, find

$$\mathcal{C}(A) := \sup_{\substack{F \in \mathcal{A} \\ F \neq 0}} \frac{1}{\|F\|_1} \left(|F(0)| - A \int_{[-1,1]^c} \big|\widehat{F}(t)\big| \, \mathrm{d}t \right),$$

   where the supremum is taken over the class $\mathcal{A}$ of continuous functions $F : \mathbb{R} \to \mathbb{C}$, with $F \in L^1(\mathbb{R})$.

2. **Problem 2:** Given $1 \le A \le \infty$, find

$$\mathcal{C}^+(A) := \sup_{\substack{F \in \mathcal{A}^+ \\ F \neq 0}} \frac{1}{\|F\|_1} \left(F(0) - A \int_{[-1,1]^c} \big(\widehat{F}(t)\big)_+ \, \mathrm{d}t \right),$$

   where the supremum is taken over the class $\mathcal{A}^+$ of even and continuous functions $F : \mathbb{R} \to \mathbb{R}$, with $F \in L^1(\mathbb{R})$.

What we can prove

1. Existence of extremizers for $A > 1$. No extremizers for $A = 1$.
2. Uniqueness in the bandlimited problem (+ variational condition).
3. Good upper and lower bounds for all of these problems.
4. $A \mapsto \mathcal{C}(A)$ and $A \mapsto \mathcal{C}^+(A)$ are monotone decreasing with

$$2 = \mathcal{C}^+(1) = \mathcal{C}(1) \geq \mathcal{C}^+(A) \geq \mathcal{C}(A) \geq \mathcal{C}^+(\infty) \geq \mathcal{C}(\infty) \geq 1.0799...$$

5. $$\mathcal{C}^+\left(\frac{36}{11}\right) > \frac{25}{21} = 1.1948...$$

$$F(x) = -4.8\,x^2 e^{-3.3x^2} + 1.5\,x^2 e^{-7.4x^2} + 520\,x^{24} e^{-9.7x^2} + 1.3\,e^{-2.8x^2} + 0.18\,e^{-2x^2}$$

Conclusion 1

Theorem (Asymptotic version)

Assume RH. Then

$$\limsup_{n\to\infty} \frac{p_{n+1} - p_n}{\sqrt{p_n}\log p_n} \leq \frac{1}{\mathcal{C}^+(\mathbf{B})} \leq \frac{1}{\mathcal{C}^+(36/11)} \leq \frac{21}{25}.$$

Conclusion 2 - non-asymptotic version

1. Use version of the Brun-Titchmarsh inequality due to Montgomery and Vaughan.
$$\pi(x+y)-\pi(x) < \frac{2y}{\log y},$$
for all $x, y > 1$. Relevant range is $y \sim \sqrt{x}$, which corresponds to $A = 4$ in the extremal problem.
2. With $H(x) = \frac{\cos(2\pi x)}{1-16x^2}$ and $\lambda = 0.9$ we use $F(x) = H(x/\lambda)$. Then
$$J(F) = \frac{F(0) - 4\int_{[-1,1]^c} \left|\widehat{F}(t)\right| \mathrm{d}t}{\|F\|_1} = 1.1405... > \frac{25}{22}.$$
3. Work the previous argument to make all error terms effective (Mellin transform approach slightly simpler).

Fourier optimization and number theory III

Emanuel Carneiro

ICTP

El Escorial, June 2022

Last night...

in "Alaska" in San Lorenzo de El Escorial

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After a few successful beers...

Matt: I would like to invite you to our conference dinner on Thursday.

Girl: I need to know more about your passion for number theory and analysis. How about proving to me that, under GRH, at least 69% of the Dirichlet $L-$functions modulo a prime q do not vanish at the point

$$\frac{1}{2} + \frac{\pi i}{4 \log q} \quad ?$$

Themes

For a family of L-functions, under GRH:

Themes

For a family of L-functions, under GRH:

1. Estimate the order of vanishing at the central point.

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For a family of L-functions, under GRH:

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2. Estimate the order of vanishing at any given height.

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1. Estimate the order of vanishing at the central point.
2. Estimate the order of vanishing at any given height.
3. Estimate the height of the first zero.

Key takeway: It is possible to approach these problems via a unified framework of Hilbert spaces.

(joint work with A. Chirre and M. Milinovich).

L-functions

- A Dirichlet character χ modulo q is a (non-identically zero) function $\chi : \mathbb{N} \to \mathbb{C}$, periodic of period q, such that

$$\chi(mn) = \chi(m) \,.\, \chi(n)$$

$$\chi(n) = 0 \;\text{ if }\; \gcd(n, q) > 1.$$

- We then have a Dirichlet L-function:

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)\, n^{-s}.$$

- More generally

$$L(s, f) = \sum_{n=1}^{\infty} \lambda_f(n)\, n^{-s}$$

Theorem

Let q be prime and assume GRH for Dirichlet L-functions modulo q. Then, for any fixed $t > 0$, we have

$$\frac{1}{q-2} \sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi \neq \chi_0}} \operatorname*{ord}_{s=\frac{1}{2}} L\left(s + \frac{2\pi i t}{\log q}, \chi\right) \leqslant \frac{1}{2}\left(1 + \left|\frac{\sin 4\pi t}{4\pi t}\right|\right)^{-1} + O\left(\frac{1}{\log q}\right),$$

where χ_0 denotes the principal character $(\mathrm{mod}\ q)$. Hence, for any $\varepsilon > 0$, the proportion of primitive Dirichlet characters $\chi \ (\mathrm{mod}\ q)$ for which $L\left(\frac{1}{2} + \frac{2\pi i t}{\log q}, \chi\right) \neq 0$ is at least

$$1 - \frac{1}{2}\left(1 + \left|\frac{\sin 4\pi t}{4\pi t}\right|\right)^{-1} - \varepsilon$$

when q is large.

Murty '89: $L(\frac{1}{2}, \chi) \neq 0$ for a proportion of at least $\frac{1}{2} - \varepsilon$.

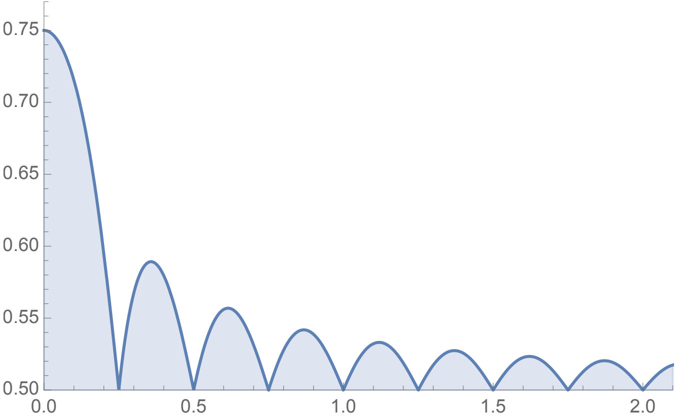

Figure: A plot of the function $t \mapsto 1 - \frac{1}{2}\left(1 + \left|\frac{\sin 4\pi t}{4\pi t}\right|\right)^{-1}$ for small $t > 0$.

Example: for $t = 1/8$, the proportion of non-vanishing of $L\left(\frac{1}{2} + \frac{\pi i}{4 \log q}, \chi\right) \neq 0$ is at least $\frac{4+\pi}{4+2\pi} - \varepsilon = 0.69449\ldots$

Setup

Let $\mathcal{F}$ be a family of (automorphic objects). For each $f \in \mathcal{F}$, let

$$L(s,f) = \sum_{n=1}^{\infty} \lambda_f(n)\, n^{-s}$$

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1. $L(s,f)$ admits an analytic continuation to an entire function.
2. GRH holds for this family (zeros $\rho_f = \frac{1}{2} + i\gamma_f$).
3. Functional equation

$$\Lambda(s,f) = \varepsilon_f\,\Lambda\big(1-s,\overline{f}\,\big),$$

with $\Lambda(s,f) = L_\infty(s,f)\,L(s,f)$, $|\varepsilon_f| = 1$ and $L\big(s,\overline{f}\,\big)$ is the dual L-function with coefficients $\lambda_{\overline{f}}(n) = \overline{\lambda_f(n)}$.

   - if $f \in \mathcal{F}$, then $\overline{f} \in \mathcal{F}$;
   - $L_\infty(s,f)$ has no zeros on the critical line.

One-level density

Katz and Sarnak (90's): conjecture that for each natural family $\{L(s,f),\ f \in \mathcal{F}\}$ of L-functions there is an associated symmetry group $G = G(\mathcal{F})$, where G is either:

1. unitary U,
2. symplectic Sp,
3. orthogonal O,
4. even orthogonal $\mathrm{SO(even)}$,
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If $\phi : \mathbb{R} \to \mathbb{R}$ is a smooth function whose Fourier transform has compact support, they conjecture that

$$\lim_{\#\mathcal{F}\to\infty} \frac{1}{\#\mathcal{F}} \sum_{f\in\mathcal{F}} \sum_{\gamma_f} \phi\left(\gamma_f \frac{\log c_f}{2\pi}\right) = \int_{\mathbb{R}} \phi(x)\, W_G(x)\, \mathrm{d}x,$$

c_f is the analytic conductor of $L(s,f)$, and W_G is a density function.

For these five symmetry groups, Katz and Sarnak determined the density functions:

$$\begin{aligned}
W_{\mathrm{U}}(x) &= 1\,; \\
W_{\mathrm{Sp}}(x) &= 1 - \frac{\sin 2\pi x}{2\pi x}\,; \\
W_{\mathrm{O}}(x) &= 1 + \tfrac{1}{2}\delta_0(x)\,; \\
W_{\mathrm{SO(even)}}(x) &= 1 + \frac{\sin 2\pi x}{2\pi x}\,; \\
W_{\mathrm{SO(odd)}}(x) &= 1 - \frac{\sin 2\pi x}{2\pi x} + \delta_0(x)\,,
\end{aligned}$$

where $\delta_0(x)$ is the Dirac distribution at $x = 0$.

A classical reference

PUBLICATIONS MATHÉMATIQUES DE L'I.H.É.S.

HENRYK IWANIEC
WENZHI LUO
PETER SARNAK

Low lying zeros of families of *L*-functions

Publications mathématiques de l'I.H.É.S., tome 91 (2000), p. 55-131

<http://www.numdam.org/item?id=PMIHES_2000__91__55_0>

Many examples in the literature where this has been proved $(\mathrm{supp}(\hat{\phi}) \subset (-\Delta, \Delta))$, e.g.

1. unitary symmetry:
    - primitive Dirichlet L-functions modulo a prime q (Hughes and Rudnick' 03) with $\Delta = 2$
    - further average of q in a range (Drappeau, Pratt, and Radziwiłł'20) $\Delta = 2 + \frac{50}{1093}$

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3. symplectic symmetry:
   - Iwaniec, Luo and Sarnak '00, Fouvry and Iwaniec '03, Özlük and Snyder '99.

and many more...

Question

Assume that

$$\lim_{\#\mathcal{F}\to\infty} \frac{1}{\#\mathcal{F}} \sum_{f\in\mathcal{F}} \sum_{\gamma_f} \phi\left(\gamma_f \frac{\log c_f}{2\pi}\right) = \int_{\mathbb{R}} \phi(x)\, W_G(x)\, \mathrm{d}x,$$

holds for smooth $\phi : \mathbb{R} \to \mathbb{R}$ with $\mathrm{supp}(\hat{\phi}) \subset (-\Delta, \Delta)$.

What can you say about the order of vanishing at a given height?

At the central point

Let ϕ be smooth with $\mathrm{supp}(\hat{\phi}) \subset (-\Delta, \Delta)$. Let $\phi \geqslant 0$ and $\phi(0) \geqslant 1$.

$$\frac{1}{\#\mathcal{F}} \sum_{f \in \mathcal{F}} \operatorname*{ord}_{s=\frac{1}{2}} L\,(s\,,\; f) \leqslant \frac{1}{\#\mathcal{F}} \sum_{f \in \mathcal{F}} \sum_{\gamma_f} \phi\left(\gamma_f \frac{\log c_f}{2\pi}\right).$$

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Hence

$$\limsup_{\#\mathcal{F} \to \infty} \frac{1}{\#\mathcal{F}} \sum_{f \in \mathcal{F}} \underset{s=\frac{1}{2}}{\mathrm{ord}}\, L(s, f) \leqslant \limsup_{\#\mathcal{F} \to \infty} \frac{1}{\#\mathcal{F}} \sum_{f \in \mathcal{F}} \sum_{\gamma_f} \phi\left(\gamma_f \frac{\log c_f}{2\pi}\right)$$

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$$= \int_{\mathbb{R}} \phi(x)\, W_G(x)\, \mathrm{d}x.$$

Your problem is then to minimize the (RHS).

At a given height t

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$$
\begin{aligned}
&\frac{2}{\#\mathcal{F}} \sum_{f \in \mathcal{F}} \operatorname*{ord}_{s=\frac{1}{2}} L\left(s + \frac{2\pi i t}{\log c_f}, f\right) \\
&= \frac{1}{\#\mathcal{F}} \sum_{f \in \mathcal{F}} \left\{ \operatorname*{ord}_{s=\frac{1}{2}} L\left(s + \frac{2\pi i t}{\log c_f}, f\right) + \operatorname*{ord}_{s=\frac{1}{2}} L\left(s - \frac{2\pi i t}{\log c_f}, \bar{f}\right) \right\} \\
&= \frac{1}{\#\mathcal{F}} \sum_{f \in \mathcal{F}} \left\{ \operatorname*{ord}_{s=\frac{1}{2}} L\left(s + \frac{2\pi i t}{\log c_f}, f\right) + \operatorname*{ord}_{s=\frac{1}{2}} L\left(s - \frac{2\pi i t}{\log c_f}, f\right) \right\} \\
&\leqslant \frac{1}{\#\mathcal{F}} \sum_{f \in \mathcal{F}} \sum_{\gamma_f} \phi\left(\gamma_f \frac{\log c_f}{2\pi}\right).
\end{aligned}
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At a given height t

Let ϕ be smooth with $\mathrm{supp}(\hat{\phi}) \subset (-\Delta, \Delta)$. Let $\phi \geqslant 0$ and $\phi(\pm t) \geqslant 1$.

$$\begin{aligned}
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&\quad \leqslant \frac{1}{\#\mathcal{F}} \sum_{f \in \mathcal{F}} \sum_{\gamma_f} \phi\left(\gamma_f \frac{\log c_f}{2\pi}\right).
\end{aligned}$$

Hence

$$\limsup_{\#\mathcal{F} \to \infty} \frac{2}{\#\mathcal{F}} \sum_{f \in \mathcal{F}} \operatorname*{ord}_{s=\frac{1}{2}} L\left(s + \frac{2\pi i t}{\log c_f}, f\right) \leqslant \int_{\mathbb{R}} \phi(x)\, W_G(x)\, \mathrm{d}x,$$

Analysis problems

Consider the class

$$\mathcal{A}_{2\pi\Delta}(t) := \left\{\phi \in L^1(\mathbb{R}) \cap C(\mathbb{R})\ ;\ \mathrm{supp}(\widehat{\phi}) \subset [-\Delta, \Delta]\ ;\ \phi \geqslant 0\ ; \phi(\pm t) \geqslant 1\right\}.$$

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1. *One-delta extremal problem.* For $G \in \{\mathrm{U}, \mathrm{Sp}, \mathrm{O}, \mathrm{SO(even)}, \mathrm{SO(odd)}\}$ and $\Delta > 0$, find:

$$\inf_{\phi \in \mathcal{A}_{2\pi\Delta}(0)} \int_{\mathbb{R}} \phi(x) \, W_G(x) \, \mathrm{d}x.$$

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2. *Two-delta extremal problem.* For $G \in \{\mathrm{U}, \mathrm{Sp}, \mathrm{O}, \mathrm{SO(even)}, \mathrm{SO(odd)}\}$, $\Delta > 0$ and $t > 0$, find:

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Problem (1) above solved by Iwaniec, Luo and Sarnak ('00) for $\Delta = 2$ and Freeman and Miller for small Δ (via Fredholm operators).

Theorem

Precise answers to these problems.

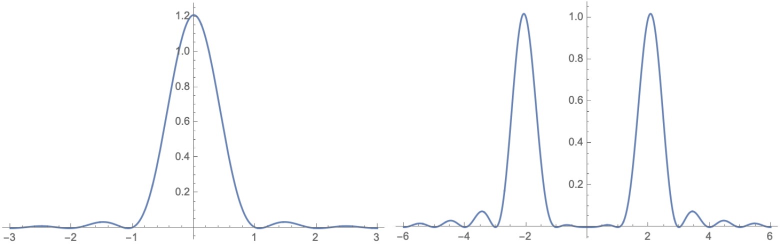

Figure: Plots of the extremal function ϕ when $\Delta = 2$ and $t = 1/4$ (on the left) and when $t = 2$ (on the right).

Orthogonal

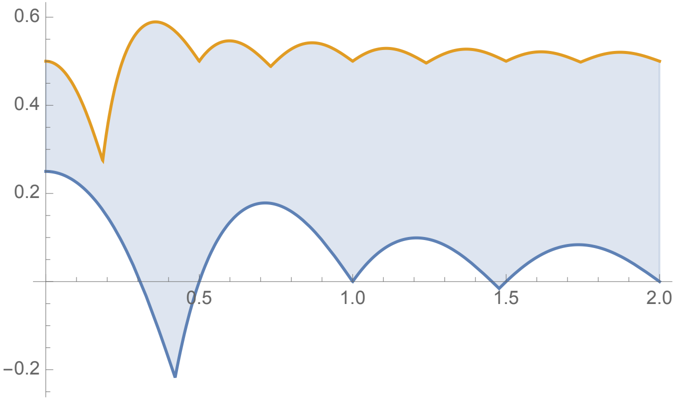

Figure: Plots of $\mathcal{P}(t)$, the lower bound for the proportion of non-vanishing, when $G = \mathrm{O}$ and $\Delta \in \{1, 2\}$ (blue and orange, respectively), for small $t > 0$. When $\Delta = 2$ the maximum value is $0.5892\ldots$ attained at $t = 0.3575\ldots$, and the limit as $t \to \infty$ is $\frac{1}{2}$.

Even orthogonal

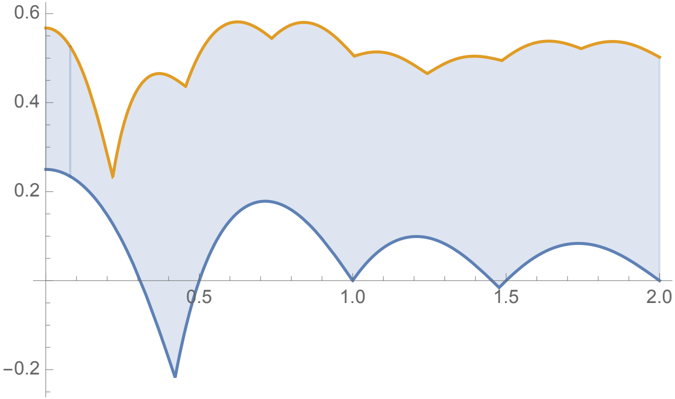

Figure: Plots of $\mathcal{P}(t)$, the lower bound for the proportion of non-vanishing, when $G = \mathrm{SO}(\mathrm{even})$ and $\Delta \in \{1, 2\}$ (blue and orange, respectively), for small $t > 0$. When $\Delta = 2$ the maximum value is $0.5814\ldots$ attained at $t = 0.6247\ldots$, and the limit as $t \to \infty$ is $\frac{1}{2}$.

Odd orthogonal

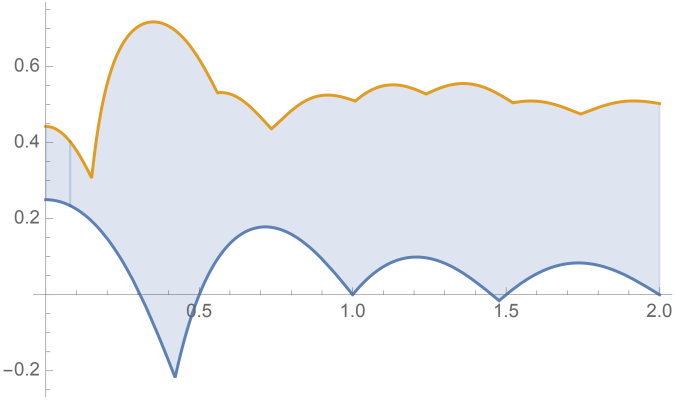

Figure: Plots of $\mathcal{P}(t)$, the lower bound for the proportion of non-vanishing, when $G = \mathrm{SO}(\mathrm{odd})$ and $\Delta \in \{1, 2\}$ (blue and orange, respectively), for small $t > 0$. When $\Delta = 2$ the maximum value is $0.7175\ldots$ attained at $t = 0.3505\ldots$, and the limit as $t \to \infty$ is $\frac{1}{2}$.

Symplectic

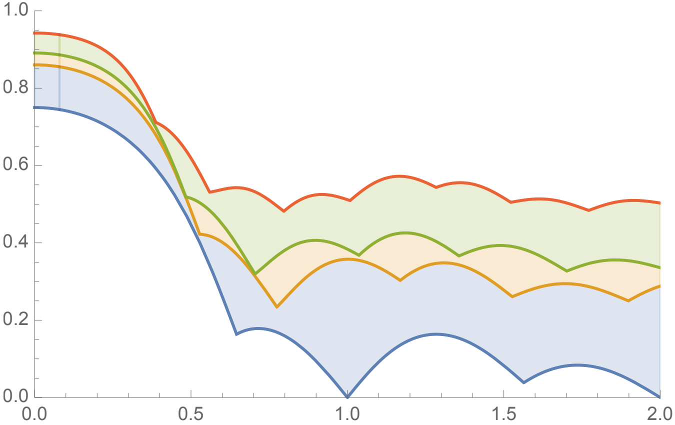

Figure: Plots of $\mathcal{P}(t)$, the lower bound for the proportion of non-vanishing, when $G = \mathrm{Sp}$ and $\Delta \in \{1, \frac{4}{3}, \frac{3}{2}, 2\}$ (blue, orange, green, and red, respectively), for small $t > 0$. The maximum values are attained when $t \to 0^+$ and can be explicitly computed. In particular, $\mathcal{P}(0^+)$ is equal to $\frac{3}{4}$ (for $\Delta = 1$), $0.8604\ldots$ (for $\Delta = \frac{4}{3}$), $0.8910\ldots$ (for $\Delta = \frac{3}{2}$), and $0.9427\ldots$ (for $\Delta = 2$). From Theorem

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- Recall the Paley-Wiener theorem: if $f \in L^2(\mathbb{R})$ then:

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Proposition (Equivalence of norms)

$$\|F\|_{\mathcal{H}_{G,\pi\Delta}} \sim \|F\|_{\mathcal{H}_{\pi\Delta}}$$

Theme 2: Riesz - Fejér decompositions

1. If a polynomial $P : \mathbb{R} \to \mathbb{C}$ is non-negative on $\mathbb{R}$ then

$$P(x) = |Q(x)|^2 \quad (i.e. \;\; P(z) = Q(z)\overline{Q(\overline{z})})$$

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3. Similar for entire functions: if $\phi : \mathbb{C} \to \mathbb{C}$ is of exp. type $2\pi\Delta$ and is non-negative on $\mathbb{R}$, then

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F(w) = \langle F, K_{G,\pi\Delta}(w,\cdot) \rangle_{\mathcal{H}_{G,\pi\Delta}} = \int_{\mathbb{R}} F(x)\, \overline{K_{G,\pi\Delta}(w,x)}\, W_G(x)\, \mathrm{d}x
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Solution: one-delta problem

$$\mathcal{A}_{2\pi\Delta}(0) := \Big\{\phi \in L^1(\mathbb{R}) \cap C(\mathbb{R})\ ;\ \mathrm{supp}(\widehat{\phi}) \subset [-\Delta, \Delta]\ ;\ \phi \geqslant 0\ ; \phi(0) \geqslant 1\Big\}.$$

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Hence

$$\frac{1}{K(0,0)} \leqslant \|F\|^2_{\mathcal{H}}\,,$$

with equality if and only if $F(z) = c\,K(0,z)$.

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Lemma

Let H be a Hilbert space and $0 \neq v_1, v_2 \in H$ be two vectors such that $\|v_1\| = \|v_2\|$ and define $\mathcal{J} = \big\{x \in H;\ |\langle x, v_1\rangle| \geqslant 1 \text{ and } |\langle x, v_2\rangle| \geqslant 1\big\}$.

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Finding the reproducing kernels

$$F(w) = \int_{\mathbb{R}} F(x) \left(k_w(x) \left(1 + \frac{\sin 2\pi x}{2\pi x} \right) \right) \mathrm{d}x = \int_I \check{F}(y)\, e^{-2\pi i w y}\, \mathrm{d}y$$

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$$u_w(y) + \frac{1}{2}\int_{y-1}^{y+1} u_w(s)\, \mathrm{d}s = e^{-2\pi i w y} \quad \text{for a.e. } \; y \in \left[-\tfrac{\Delta}{2}, \tfrac{\Delta}{2}\right].$$

with $u_w \in L^2(\mathbb{R})$, with $\mathrm{supp}\,(u_w) \subset \left[-\tfrac{\Delta}{2}, \tfrac{\Delta}{2}\right]$.

Theorem ((Paley-Wiener) Reproducing kernel: unitary symmetry)

For any $\Delta > 0$, we have

$$K_{\mathrm{U},\pi\Delta}(w,z) = \frac{\sin \pi\Delta(z-\overline{w})}{\pi(z-\overline{w})}.$$

Theorem (Reproducing kernel: orthogonal symmetry)

For any $\Delta > 0$, we have

$$K_{\mathrm{O},\pi\Delta}(w,z) = \frac{\sin \pi\Delta(z-\overline{w})}{\pi(z-\overline{w})} - \frac{1}{(2+\Delta)}\left(\frac{\sin \pi\Delta z}{\pi z}\right)\left(\frac{\sin \pi\Delta \overline{w}}{\pi \overline{w}}\right).$$

Theorem (Reproducing kernel: symplectic symmetry)

(i) *For* $0 < \Delta \leqslant 1$ *we have*

$$K_{\mathrm{Sp},\pi\Delta}(w,z) = \frac{\sin \pi\Delta(z-\overline{w})}{\pi(z-\overline{w})} + \frac{1}{(2-\Delta)}\left(\frac{\sin \pi\Delta z}{\pi z}\right)\left(\frac{\sin \pi\Delta \overline{w}}{\pi \overline{w}}\right).$$

(ii) *For* $1 < \Delta \leqslant 2$*, define the constants*

$$\begin{aligned} \tau &:= e^{(2-\Delta)i/4} - ie^{\Delta i/4}\,; \\ a &:= e^{\Delta i/4} - ie^{(2-\Delta)i/4} + ie^{\Delta i/4} - \left(\tfrac{2-\Delta}{4}\right)\tau\,; \\ b &:= e^{\Delta i/4} - ie^{(2-\Delta)i/4} - e^{(2-\Delta)i/4} - \left(\tfrac{2-\Delta}{4}\right)\tau\,, \end{aligned}$$

and functions of a complex variable w*,*

$$C(w) := \frac{-16\pi^2w^2 + 4\pi i w\, e^{2\pi i w}}{1-16\pi^2 w^2}\ ;\quad F(w) := \frac{2\cos(\pi(2-\Delta)w) + 8\pi w \sin(\pi\Delta w)}{1-16\pi^2w^2}\,;$$

$$G(w) := \frac{2\cos(\pi\Delta w) - 4\pi i w\, e^{-\pi\Delta i w} - e^{\pi(2-\Delta)iw}}{1-16\pi^2w^2} + \frac{\sin(\pi(2-\Delta)w)}{2\pi w(1-16\pi^2w^2)} - \left(\frac{2-\Delta}{4}\right)F(w)\,;$$

$$A(w) := \frac{\overline{a}\,\overline{G(\overline{w})} - \overline{b}\,G(w)}{\overline{a}b - a\overline{b}}\ ;\quad B(w) := \frac{b\,G(w) - a\,\overline{G(\overline{w})}}{\overline{a}b - a\overline{b}}\ ;\quad D(w) := \frac{1}{2}\big(\tau A(w) + \overline{\tau}B(w) - F(w)\big).$$

Theorem (cont.)

Then

$$K_{\mathrm{Sp},\pi\Delta}(w,z) = \frac{\overline{A(w)}\big(-ie^{2\pi iz}-1\big)\big(e^{-\Delta(\pi iz+i/4)}-e^{-(2-\Delta)(\pi iz+i/4)}\big)}{2\pi iz+i/2}$$
$$+\frac{\overline{B(w)}\big(ie^{2\pi iz}-1\big)\big(e^{-\Delta(\pi iz-i/4)}-e^{-(2-\Delta)(\pi iz-i/4)}\big)}{2\pi iz-i/2}$$
$$+\frac{C(\overline{w})\big(e^{\pi\Delta i(z-\overline{w})}-e^{\pi(2-\Delta)i(z-\overline{w})}\big)+\overline{C(w)}\big(-e^{-\pi\Delta i(z-\overline{w})}+e^{-\pi(2-\Delta)i(z-\overline{w})}\big)}{2\pi i(z-\overline{w})}$$
$$+\frac{\overline{D(w)}\,\sin\big(\pi(2-\Delta)z\big)}{\pi z}+\frac{\sin\big(\pi(2-\Delta)(z-\overline{w})\big)}{\pi(z-\overline{w})}.$$

Appendix

A complementary problem

Hughes and Rudnick '03 consider the problem of establishing the *existence* of low-lying zeros in the family of primitive Dirichlet L-functions modulo q . They prove that, under GRH, if the support is $[-\Delta, \Delta]$ in the one-level density theorem then

$$\limsup_{q\to\infty} \min_{\substack{\gamma_\chi \\ \chi\neq\chi_0}} \left| \frac{\gamma_\chi \, \log q}{2\pi} \right| \leqslant \frac{1}{2\Delta} \, ,$$

With $\Delta = 2$ this becomes $1/4$.

Given a family $\mathcal{F}$, we seek an upper bound for

$$\limsup_{\#\mathcal{F}\to\infty} \min_{\substack{\gamma_f \\ f\in\mathcal{F}}} \left| \frac{\gamma_f \log c_f}{2\pi} \right| ,$$

Slight modification

For an orthogonal (resp. odd orthogonal) family, we expect half (resp. all) of the L-functions in the family to trivially vanish at the central point due to the sign of the functional equation.

When $G = \mathrm{O}$ or $G = \mathrm{SO(odd)}$, we are going to assume that there exists a subset $\bigcup_{f\in\mathcal{F}} \mathcal{Z}(f)$ of the zeros at the central point such that

$$\lim_{\#\mathcal{F}\to\infty} \frac{1}{\#\mathcal{F}} \sum_{f\in\mathcal{F}} \sum_{\gamma_f\in\mathcal{Z}(f)} 1 = \frac{1}{2} \quad (\text{when } G = \mathrm{O})$$

$$\lim_{\#\mathcal{F}\to\infty} \frac{1}{\#\mathcal{F}} \sum_{f\in\mathcal{F}} \sum_{\gamma_f\in\mathcal{Z}(f)} 1 = 1 \quad (\text{when } G = \mathrm{SO(odd)}).$$

When $G \in \{\mathrm{U}, \mathrm{Sp}, \mathrm{SO(even)}\}$ we may simply regard $\mathcal{Z}(f) = \emptyset$.

Slight modification - part 2

With this understanding, the one-level density becomes

$$\lim_{\#\mathcal{F}\to\infty} \frac{1}{\#\mathcal{F}} \sum_{f\in\mathcal{F}} \sum_{\gamma_f\notin\mathcal{Z}(f)} \phi\left(\gamma_f \frac{\log c_f}{2\pi}\right) = \int_{\mathbb{R}} \phi(x)\, W_{G^\sharp}(x)\, \mathrm{d}x,$$

where we denote

$$G^\sharp := G \ \text{ if } \ G \in \{\mathrm{U}, \mathrm{Sp}, \mathrm{SO(even)}\} \quad ; \quad \mathrm{O}^\sharp := \mathrm{U} \quad ; \quad \mathrm{SO(odd)}^\sharp := \mathrm{Sp}.$$

We aim to bound

$$\beta(\mathcal{F}) := \limsup_{\#\mathcal{F}\to\infty} \min_{\substack{\gamma_f\notin\mathcal{Z}(f)\\ f\in\mathcal{F}}} \left|\frac{\gamma_f \log c_f}{2\pi}\right|.$$

Let $a > 0$ and let $\phi(x) = (x^2 - a^2)\, g(x)$, with g even, Schwartz, non-negative and $\mathrm{supp}(\widehat{g}) \subset (-\Delta, \Delta)$. If

$$0 > \lim_{\#\mathcal{F}\to\infty} \frac{1}{\#\mathcal{F}} \sum_{f\in\mathcal{F}} \sum_{\gamma_f \notin \mathcal{Z}(f)} \phi\left(\gamma_f \frac{\log c_f}{2\pi}\right) \tag{1}$$

we can certainly conclude that $\beta(\mathcal{F}) \leqslant a$. Note that the (RHS) above is

$$= \int_{\mathbb{R}} \phi(x)\, W_{G^\sharp}(x)\, \mathrm{d}x = \int_{\mathbb{R}} (x^2 - a^2)\, g(x)\, W_{G^\sharp}(x)\, \mathrm{d}x,$$

Hence (1) is equivalent to

$$a > \frac{\left(\int_{\mathbb{R}} x^2\, g(x)\, W_{G^\sharp}(x)\, \mathrm{d}x\right)^{1/2}}{\left(\int_{\mathbb{R}} g(x)\, W_{G^\sharp}(x)\, \mathrm{d}x\right)^{1/2}}.$$

The problem is the minimize the (RHS).

Theorem

Let $K = K_{G^\sharp,\pi\Delta}$ *be the reproducing kernel of the Hilbert space* $\mathcal{H}_{G^\sharp,\pi\Delta}$. *Let* ξ_0 *be the smallest positive real zero of the function* $x \mapsto \mathrm{Re}\big((1 - ix)K(i,x)\big)$. *Then*

$$\liminf_{\#\mathcal{F}\to\infty} \min_{\substack{\gamma_f \notin \mathcal{Z}(f)\\ f\in\mathcal{F}}} \left| \frac{\gamma_f \ \log c_f}{2\pi} \right| \leqslant \xi_0.$$