11th International Conference on Harmonic Analysis and Partial Differential Equations

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Minicourses

Carneiro, Emanuel (ICTP - The Abdus Salam International Centre for Theoretical Physics, Italy)

Fourier optimization and number theory

Abstract: Some problems in number theory are naturally related to different types of oscillatory structures. In such situations, certain Fourier optimization problems emerge, carrying important information about the original number theoretical entity and also being of intrinsic interest in analysis.

The aim of this minicourse is to present a variety of number theory problems which are amenable to a Fourier optimization framework. These include: (i) bounding objects related to the Riemann zeta-function; (ii) estimating the size of prime gaps; (iii) bounding the pair correlation of zeros of the Riemann zeta-function; (iv) bounding the height of low-lying zeros over families of L-functions; (v) and estimating the angular discrepancy of zeros of polynomials.

No previous research background in analytic number theory is required (at least to have an idea of what the minicourse is about); we will briefly review the main tools and definitions as we move along.

Cowling, Michael (University of New South Wales, Australia)

Harmonic analysis and group representations

Abstract: These three talks are based on a recent paper of Bruno, Cowling, Nicola and Tabacco (arXiv:1906.02060) and developments thereof. We look at how questions, problems and solutions in group representations turn into questions, problems and solutions in classical harmonic analysis, and vice versa.

The first is an introduction to the groups $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{Sp}(2n,\mathbb{R})$. We describe the unitary representations of the first group; this involves the link between convolution on the circle and on the line, and Riesz potential kernels. We examine the metaplectic representation of the second, which arises when we consider simultaneously modulations and translations on $L^2(\mathbb{R}^n)$.

In the second, we estimate the decay at infinity of matrix coefficients of representations of $\mathrm{SL}(2,\mathbb{R})$. This may be considered as an exercise in controlling special functions, and involves use of techniques such as van der Corput's lemma (in a new variation). We use these estimates to control dispersive equations on the two-dimensional hyperbolic space $\mathbb{H}^2$.

The third is about generalisations. We consider the metaplectic representation and the fractional Fourier transformation. We use estimates for the matrix coefficients of this unitary representation to provide a new kind of L^2–L^2 dispersive estimate for solutions of the Schrödinger equation. Time permitting, we discuss some related questions on eigenvalue problems and on hyperbolic equations in a negatively curved space-time.

Pipher, Jill (Brown University, US)

Elliptic equations and systems: the p- ellipticity condition

Abstract: We introduce the concept of p-ellipticity for complex matrices and systems of matrices, in particular in the context of boundary value problems for linear elliptic PDEs. The condition was formulated independently by Carbonaro-Dragicevic and Dindoš-Pipher in 2017. A related but weaker notion was introduced in seminal papers of Cialdea-Maz'ya in 2005-06, and was our motivation for exploring this new condition.

The De Giorgi - Nash - Moser theory of the 1950's-60s demonstrated the power of the structural condition of *ellipticity* in proving regularity of solutions to divergence form second order equations. Their work opened up the possibility of solving boundary value problems for classical equations (the Laplacian) in non-smooth domains, as well as for variable non-smooth coefficient elliptic equations more generally. In subsequent decades, there has been a remarkable development in the quantitative understanding of the relationship between smoothness of the coefficients, geometric properties of the domains, and local properties of the solutions to real coefficient elliptic divergence form equations. This development has relied on far-reaching extensions of classical ideas in harmonic analysis, from singular integrals to the theory of weights, and is based on the specific properties of solutions to elliptic equations in divergence form like strong maximum principles, boundary elliptic measure, and Hölder continuity of solutions.

For general complex coefficient operators in divergence form, and also for systems of equations, one does not expect to have maximum principles or continuity of solutions. The concept of p-ellipticity (agreeing with the familiar concept of classical ellipticity when $p = 2$) has proven to be a useful and natural assumption with consequences for solving boundary value problems in these settings. It allows for a limited version of the Moser iteration principle, which in turn implies some "higher regularity" of solutions.

I will discuss joint work with M. Dindoš where p-ellipticity is exploited to solve Dirichlet and Regularity boundary value problems, and joint work with Dindoš and J. Li on systems of equations and the various notions of p-ellipticity that are analogous to their classical elliptic counterparts. I aim to make these lectures accessible to students with a limited background in this field.

Invited Speakers

Astala, Kari (University of Helsinki, Finland)

The Burkholder functional, restricted quasiconvexity and energy integrals in non-linear elasticity

Abstract: The Burkholder functional B_p arises naturally in various contexts, from singular integrals and martingales to vector valued calculus of variations. It gives a particularly interesting candidate to test Morrey's conjecture in two dimensions, i.e. whether every rank-one convex functional in $\mathbb{R}^{2\times 2}$ is quasiconvex.

In this talk on our joint work with D. Faraco (UAM), A. Guerra (IAS) and J. Kristensen (Oxford), we show that B_p is quasiconvex at matrices $A \in \mathbb{R}^{2\times 2}$ with $B_p(A) < 0$, when tested with Sobolev maps with $B_p(Df) \leq 0$.

This has several immediate corollaries on quasiconvexity of energy functionals in non-linear elasticity where, to avoid cavitation and interpenetration of matter, the natural minimisers are Sobolev homeomorphisms.

Bez, Neal (Saitama University, Japan)

The nonlinear Brascamp–Lieb inequality

Abstract: The classical version of the Brascamp–Lieb inequality

$$\int_{\mathbb{R}^n} \prod_{j=1}^m (f_j \circ L_j)^{c_j} \leq \mathrm{B} \prod_{j=1}^m \left(\int_{\mathbb{R}^{n_j}} f_j \right)^{c_j}$$

is a unification of fundamental inequalities such as the multilinear Hölder, Young convolution and Loomis–Whitney inequalities. Here, the mappings $L_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$ are linear surjections. The well-advanced theory for this inequality includes Lieb's theorem which reveals a special role played by gaussians – the Brascamp–Lieb constant $\mathrm{B} = \mathrm{B}((L_j),(c_j))$ is exhausted by centred gaussian inputs. In 2005, Bennett, Carbery and Wright obtained a nonlinear (local) version of the Loomis–Whitney inequality in which the associated linear mappings L_j are replaced by nonlinear perturbations. Here we present some joint work with Jon Bennett, Stefan Buschenhenke, Michael Cowling and Taryn Flock in which we provide a nonlinear version of the Brascamp–Lieb inequality in full generality. Our proof is built upon a tight induction-on-scales argument incorporating gaussian near-extremisers. A significant component of our proof is deriving an effective version of Lieb's theorem which guarantees the existence of gaussian near-extremisers with suitable control on their eccentricity.

Cruz-Uribe, David (The University of Alabama, United States)

Jones factorization and Rubio de Francia extrapolation for matrix weights

Abstract: In the 1990s, Nazarov, Treil and Volberg introduced a generalization of the scalar Muckenhoupt A_p condition to matrix weights. Let W be a $d \times d$ symmetric, positive definite matrix weight function. For $1 < p < \infty$, we say W is in matrix A_p if

$$[W]_{A_p} = \sup_Q \frac{1}{|Q|} \int_Q \left(\frac{1}{|Q|} \int_Q |W^{1/p}(x) W^{-1/p}(y)|_{op}^{p'}\, dy \right)^{p/p'} dx < \infty,$$

where the supremum is taken over all cubes in $\mathbb{R}^n$. They showed that the Hilbert transform is bounded on $L^p(W)$, the space of vector-valued functions with norm

$$\|f\|_{L^p(W)} = \left(\int_{\mathbb{R}^n} |W^{1/p}(x) f(x)|^p\, dx \right)^{1/p}.$$

Later, Christ and Goldberg extended this result to all Calderón-Zygmund singular integrals.

In the 1990s, Nazarov, Treil and Volberg posed two related problems: extend the Jones factorization theorem and the Rubio de Francia theory of extrapolation, to matrix A_p weights. Joint with Marcin Bownik, we have proved both results. In this talk we will describe our approach, which is based on the theory of convex-set valued functions.

de la Salle, Mikael (CNRS, Université de Lyon, France)

Fourier analysis with non commutative groups

Abstract: It is well-known (this follows immediately from the boundness of the Hilbert transform) that the Fourier series of every function $L_p(\mathbb{R}/\mathbb{Z})$ converges in L_p when $1 < p < \infty$. This is not true for $p = 1$ or $p = \infty$, but there are more clever *summation methods*, for example Fejér's method that allows to obtain convergence in L_1, and even L_∞ if the function is continuous. All this remains true on the torus $(\mathbb{R}/\mathbb{Z})^d$ of arbitrary dimension, and more generally on the Pontryagin dual $\hat{\Gamma}$ of every discrete abelian group. My talk will be a survey about more or less recent investigations of such questions for non abelian groups, motivated by open problems in Banach space theory or operator algebras. I will in particular explain that for groups such as $\mathrm{SL}_n(\mathbb{Z})$ and suitable p, there is, in some precise sense, no L_p Fourier summation method at all. I will also relate this to harmonic analysis questions on euclidean spheres. Based on joint works with Tim de Laat, Vincent Lafforgue, Javier Parcet and Éric Ricard.

De Mari, Filippo (Università di Genova, Italy)

Unitarization of the Radon transform

Abstract: We consider the Radon transform associated to pairs (X, Ξ), a variant of Helgason's notion of dual pair, where $X = G/K$ and $\Xi = G/H$, G being a locally compact group and K and H closed subgroups thereof. Under some technical assumptions, we prove that if the quasi regular representations of G acting on $L^2(X)$ and $L^2(\Xi)$ are irreducible, then the Radon transform admits a unitarization intertwining the two representations. If, in addition, the representations are square integrable, we provide an inversion formula for the Radon transform based on the voice transform associated to these representations. The general assumptions (in particular irreducibility and square integrability of the representations) fail in the case when X is either a noncompact symmetric space or a homogeneous tree and Ξ is the corresponding space of horocycles. Nonetheless, a unitarization theorem holds true in both cases and the outcoming unitary operator does intertwine the quasi regular representations.

This is joint work with G. Alberti, F. Bartolucci, E. De Vito, M. Monti and F. Odone.

Di Plinio, Francesco (Università di Napoli "Federico II", Italy)

Maximal and singular operators in codimension 1 and higher

Abstract: This talk surveys recent developments in the analysis of maximal and multiplier operators whose singularity is a variable subspace of codimension 1 or higher. A first set of results concerns maximal operators associated to singular averages along finite subsets of the (d, n) Grassmannian of d-dimensional subspaces. We discuss algebraic almost orthogonality principles and essentially sharp L^2 bounds without any assumption on the structure of the subset. In the codimension 1 case, that is $n = d + 1$, we prove the precise critical weak $(2, 2)$-bound. Secondly, we describe a new approach to quantitative square function estimates for conical multipliers based on directional time-frequency analysis and embedding theorems for directional Carleson sequences. A suitable combination of these estimates yields an improvement on the logarithmic bound for the Fourier restriction to an N-gon due to A. Córdoba. Finally, we present new estimates for maximal multipliers along variable codimension 1 subspaces subsuming the Carleson-Sjölin theorem.

This is joint work with Ioannis Parissis (U Basque Country) and partly with N. Accomazzo (U British Columbia), O. Bakas, L. Roncal (BCAM) and P. Hagelstein (Baylor U).

Gressman, Philip T. (University of Pennsylvania, USA)

Testing conditions for multilinear Radon-Brascamp-Lieb inequalities

Abstract: We will discuss a new necessary and sufficient testing condition for L^p-boundedness of a class of multilinear functionals which includes both the Brascamp-Lieb inequalities and generalized Radon transforms associated to algebraic incidence relations. The testing condition involves bounding the average of an inverse power of certain Jacobian-type quantities along fibers of associated projections and covers many widely-studied special cases, including convolution with measures on nondegenerate hypersurfaces or on nondegenerate curves.

Hofmann, Steve (University of Missouri, USA)

Quantitative absolute continuity of caloric measure

Abstract: For an open set $\Omega \subset \mathbb{R}^d$ with an Ahlfors regular boundary, solvability of the Dirichlet problem for Laplace's equation, with boundary data in L^p for some $p < \infty$, is equivalent to quantitative, scale invariant absolute continuity (more precisely, the weak-A_∞ property) of harmonic measure with respect to surface measure on $\partial\Omega$. A similar statement is true in the caloric setting. Thus, it is of interest to find geometric criteria which characterize the open sets for which such absolute continuity (hence also solvability) holds. Recently, this has been done in the harmonic case, but in the caloric setting, matters are still at a more primitive stage. On the other hand, we now have a definitive characterization at least in the case that the boundary is known in advance to be given (locally) as a Lip(1,1/2) graph, and in turn, this leads to a result of free boundary type in which, more generally, quantitative absolute continuity of caloric measure, with respect to "surface measure" on the parabolic Ahlfors regular (lateral) boundary Σ, implies parabolic uniform rectifiability of Σ.

This is joint work with S. Bortz, J. M. Martell and K. Nyström.

Labate, Demetrio (University of Houston, USA)

Provable approximations on smooth manifolds using deep neural networks

Abstract: The expressive power of deep neural networks (DNNs) is illustrated by their remarkable ability to approximate high-dimensional functions in a way that appears to overcome the classical curse of dimensionality. This ability is exemplified by their success in solving high-dimensional PDE problems where classical numerical solvers fail due to their inability to accurately represent complicated high-dimensional structures. To provide a theoretical framework to explain this phenomenon, we examine the approximation of functions f defined on a d-dimensional smooth manifold $\mathcal{M} \subset \mathbb{R}^D$ using DNNs, where $d < D$. We prove that the convergence estimates of the approximation and generalization errors by DNNs do not depend on the nominal high dimensionality D of the f but only on its lower intrinsic dimension d.

Lev, Nir (Bar-Ilan University, Israel)

Fuglede's tiling-spectrality conjecture for convex domains

Abstract: Which domains in Euclidean space admit an orthogonal basis consisting of exponential functions? For example, the cube is such a domain, but the ball is not. In 1974, Fuglede made a fascinating conjecture that these domains could be characterized geometrically as the domains which can tile the space by translations. While this conjecture was disproved for general sets, in a recent paper with Máté Matolcsi we did prove that Fuglede's conjecture is true for convex domains in all dimensions. I will survey the subject and discuss this result.

Lie, Victor (Purdue University, USA)

The LGC-method

Abstract: In this talk we will discuss in the context provided by some relevant problems/applications a versatile method developed by the author in order to analyze the boundedness properties of large classes of (sub-)linear or multi-linear operators. This so-called *LGC-method* consists of three key steps:

- phase *linearization*: the time/frequency plane is discretized in regions within which the phase of the operator's multiplier oscillates at the linear level;
- *Gabor* frame discretization: within each of the regions obtained at the first item, one performs an adapted Gabor frame decomposition of the input functions;
- *cancelation* via *time-frequency correlation*: the resulting discretized operator is now analyzed at the L^2 level via a TT^* argument exploiting the size distribution of the Gabor coefficients via the structure of the time-frequency correlation level sets.

As a consequence of this methodology one can provide a unified approach to three main themes in Harmonic Analysis:

- The Linear Hilbert Transform and Maximal Operator along variable curves;
- Carleson Type operators in the presence of curvature;
- The bilinear Hilbert transform and maximal operator along variable curves.

More recent applications of this method to new classes of *hybrid* operators (i.e. having *both zero and non-zero curvature* features)–including but not restricted to the Bilinear Hilbert Carleson operator–will also be discussed.

Mateu, Joan (Universitat Autònoma de Barcelona)

Global minimisers of energies related to dislocations

Abstract: We will discuss the characterization of the minimisers for a class of non-local perturbations of the Coulomb energy. It is shown that this minimisers are given by the characteristic function of an ellipsoid. We will see how tecniques on Calderón-Zygmund operators are used to obtain this type of results.

Zahl, Joshua (The University of British Columbia, Canada)

The Kakeya conjecture and sticky Kakeya sets

Abstract: A Kakeya set is a compact subset of $\mathbb{R}^n$ that contains a unit line segment pointing in every direction, and the Kakeya conjecture asserts that such sets must have Hausdorff dimension n. I will discuss some progress towards this conjecture for a special class of Kakeya sets, called sticky Kakeya sets. This is joint work with Hong Wang.

Zhang, Ruixiang (University of California, Berkeley)

Local smoothing for the wave equation in $2+1$ dimensions

Abstract: Sogge's local smoothing conjecture for the wave equation predicts that the local L^p space-time estimate gains a fractional derivative of order almost $1/p$ compared to the fixed time L^p estimates, when $p > 2n/(n-1)$. Jointly with Larry Guth and Hong Wang, we recently proved the conjecture in $\mathbb{R}^{2+1}$. I will talk about our proof and explain several important ingredients such as induction on scales and an incidence type theorem.