Noetherian solvability and explicit solution of a singular integral equation with weighted Carleman shift in Besov space

Nurlan Yerkinbayev

Institute of Mathematics and Mathematical Modeling,
Almaty, Kazakhstan

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WORKSHOP ON BANACH SPACES AND BANACH LATTICES II

Introduction

Let Γ be a closed Lyapunov contour from the class C^1_ν, $\frac{2}{p} - 1 < \nu \leq 1$, $1 < p < 2$.

Let D be a region of the complex plane E bounded by a closed Lyapunov contour Γ and let $D^+ := D$, $D^- := E - \overline{D^+}$. Throughout this work, we assume that $0 \in D^+$ and $z = \infty \in D^-$.

In the present work, we study a solvability of the following singular integral equation with a Carleman shift in the Besov space $B^{\frac{1}{p}}_{p,1}(\Gamma)$, $1 < p < 2$, $\theta = 1$ and $r = \frac{1}{p}$:

$$(M\varphi)(t) \equiv a(t)\varphi(t) + \frac{b(t)}{\pi i}\int\limits_\Gamma \frac{\varphi(\tau)}{\tau - t}d\tau + \frac{b(t)u(t)}{\pi i}\int\limits_\Gamma \frac{\varphi(\tau)}{\tau - \alpha(t)}d\tau = g(t), \tag{1}$$

where $a(t), b(t), u(t), g(t) \in B^{\frac{1}{p}}_{p,1}(\Gamma)$. Operator M is bounded on $B^{\frac{1}{p}}_{p,1}(\Gamma)$[1]

[1] N. Bliev, *Generalized analytic functions in fractional spaces.* Boston: Longman, Published

References

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Besov space

Let m be an integer and $r > 0$ is a real number satisfying $m > r > 0$. Recall, a function φ belongs to the Besov space $B^r_{p,\theta}(\Gamma), 1 \leq p, \theta \leq \infty$, if $\varphi \in L_p(\Gamma)$ and the following seminorm is finite

$$||\varphi||_{b^r_{p,\theta}(\Gamma)} = \Big(\int_\Gamma |h|^{-1-\theta r} ||\Delta^m_h \varphi||^\theta_{L_p(\Gamma)} dh \Big)^{\frac{1}{\theta}},$$

where $h \in \Gamma$ and $\Delta^m_h \varphi = \Delta_h(\Delta^{m-1}_h \varphi)$ $(\Delta_h \varphi(t) = \varphi(t+h) - \varphi(t))$ is the finite difference of order m. The space $B^r_{p,\theta}(\Gamma)$, $1 \leq p, \theta \leq \infty$ is a Banach space with the norm [2]

$$||\varphi||_{B^r_{p,\theta}(\Gamma)} = ||\varphi||_{L_p(\Gamma)} + ||\varphi||_{b^r_{p,\theta}(\Gamma)}.$$

[2] O.V. Besov, V.P. Il'in, and S.M. Nikol'skiĭ, *Integral representations of function and embedding theorems, 1, 2.*, John Willey, New York, 1978; 1979.

$B_2(\Gamma)$ and $B_{2\times 2}(\Gamma)$ spaces

Let $B_2(\Gamma)$ be a set of all 2-dimensional vectors with components from $B^{\frac{1}{p}}_{p,1}(\Gamma)$ and $B_{2\times 2}(\Gamma)$ be a set of all 2×2 square matrices with elements from $B^{\frac{1}{p}}_{p,1}(\Gamma)$. The set $B_2(\Gamma)$ can be supplied with the norm by taking as the norm of the vector $X = (x_1, x_2)$ and considering the sum of the norms of the individual components:

$$\|X\|_{B_2(\Gamma)} := \|x_1\|_{B^{\frac{1}{p}}_{p,1}(\Gamma)} + \|x_2\|_{B^{\frac{1}{p}}_{p,1}(\Gamma)}.$$

In this case, the norm of the matrix $A = \{a_{kj}\}_1^2 \in B_{2\times 2}(\Gamma)$ can be defined, for example, by

$$\|A\|_{B_{2\times 2}(\Gamma)} := 2 \max_{j,k} \|a_{jk}\|_{B^{\frac{1}{p}}_{p,1}(\Gamma)}.$$

Then the space $B_{2\times 2}(\Gamma)$ with this norm will also be a Banach algebra[3].

[3] S. Prëssdorf, *Some Classes of Singular Equations*, North – Holland Publishing Company. New York, 1978.

Carleman shift and Hölder space

Let $\alpha(t)$ be a Carleman shift. In other words, it is a map which homeomorphically translates Γ onto itself with preservation or change of orientation on Γ, and satisfies the condition $\alpha\left[\alpha\left(t\right)\right] = t$.

$$\alpha'\left(\cdot\right) \in H_{\mu}\left(\Gamma\right) \ \ \textit{and} \ \ \alpha'(t) \neq 0, \ \forall t \in \Gamma.$$

Here, $H_{\mu}\left(\Gamma\right)$ is a space of functions defined on Γ satisfying the Hölder condition with the exponent $0 < \mu \leq 1$. The space $H_{\mu}\left(\Gamma\right)$ is Banach equipped with the norm [4]

$$\|\varphi\|_{H_{\mu}(\Gamma)} := \max_{t\in\Gamma} |\varphi(t)| + \sup_{t,\tau\in\Gamma} \frac{|\varphi(\tau) - \varphi(t)|}{|\tau - t|^{\mu}}.$$

[4] O.V. Besov, V.P. Il'in, and S.M. Nikol'skiĭ, *Integral representations of function and embedding theorems, 1, 2.*, John Willey, New York, 1978; 1979.

Noether operators

Let

$$\alpha(A) = \dim\ker A,\ \ \beta(A) = \dim\ \mathrm{coker} A.$$

The ordered pair of numbers $(\alpha(A),\ \beta(A))$ is called the d –characteristic of the operator A. If at least one of the numbers $\alpha(A)$ or $\beta(A)$ is finite, then the difference

$$IndA = \alpha(A) - \beta(A)$$

is called the index of the operator A.
The operator A is said to have a finite d-characteristic or a finite index if both of the numbers $\alpha(A)$ and $\beta(A)$ are finite.
A closed normally solvable operator A is called a Noether operator or F –operator if its d-characteristic is finite. It is called a semi-Noether operator if at least one of the numbers $\alpha(A)$ or $\beta(A)$ is finite. In the class of semi-Noether operators we shall in the sequel distinguish between F_+-operators $(\alpha(A) < \infty)$ and F_-–operators $(\beta(A) < \infty)$.

Operators

Here, S is a singular integral operator defined by

$$(S\varphi)(t) = \frac{1}{\pi i} \int\limits_{\Gamma} \frac{\varphi(\tau)}{\tau - t} d\tau, \ \ \forall \varphi \in B^{\frac{1}{p}}_{p,1}(\Gamma).$$

And, W is an operator on $B^{\frac{1}{p}}_{p,1}(\Gamma)$ by the following formula

$$(W\varphi)(t) = \varphi[\alpha(t)].$$

Also, we shall be concerned with the weighted Carleman shift operator on $B^{\frac{1}{p}}_{p,1}(\Gamma)$ which is defined by

$$(U\varphi)(t) = u(t)\varphi[\alpha(t)].$$

Note that $U^2 = I$ is equivalent to $u[\alpha(t)]u(t) = 1,\ t \in \Gamma$[5].

- $US = SU$ if α preserves the orientation of $\Gamma(\gamma = +1)$,
- $US = -SU$ if α changes the orientation of $\Gamma(\gamma = -1)$.

[5] V.G. Kravchenko, G.S. Litvinchuk, Introduction to the Theory of Singular Integral Operators with Shift, Kluwer Academic Publishers, Amsterdam, 1994.

Operators

Let us consider the following operators

$$J_1 := \frac{1}{2}(I - U) \text{ and } J_2 := \frac{1}{2}(I + U).$$

Next, we obtain an analogue of Lemma 2[6] in the space $B^{\frac{1}{p}}_{p,1}(\Gamma)$.

Lemma

Let $\varphi \in B^{\frac{1}{p}}_{p,1}(\Gamma)$. We have

$$(J_k S\varphi)(t) = \begin{cases} (SJ_k\varphi)(t), & \text{if} \quad US = SU, \\ (SJ_{3-k}\varphi)(t), & \text{if} \quad US = -SU, \qquad t \in \Gamma. \end{cases}$$

[6] L.P. Castro, E.M. Rojas, *Explicit solutions of Cauchy singular integral equations with weighted Carleman shift*, J. Math. Anal. Appl. **371** (2010), 128–133.

Hence, multiplying by a_α and applying the projection operator J_k to both sides of the equation (1), we have

$$\begin{cases} aa_\alpha\varphi_2 + (a_\alpha b + ab_\alpha)(S\varphi_2) = (a_\alpha g + uag_\alpha), & \text{if} \quad US = SU, \\ aa_\alpha\varphi_1 + (a_\alpha b - ab_\alpha)(S\varphi_1) = (a_\alpha g - uag_\alpha), & \text{if} \quad US = -SU. \end{cases} \tag{2}$$

where $a(t) = a$ and $a[\alpha(t)] = a_\alpha$, $b(t) = b$ and $b[\alpha(t)] = b_\alpha$, $\varphi_k = J_k\varphi$, $k = 1, 2$. We obtain a corresponding system of two SIEs with a Cauchy kernel relatively unknown vector $\rho\left(t\right) = \left\{\rho_1\left(t\right), \rho_2\left(t\right)\right\}$

$$\begin{cases} a\rho_1 + \frac{b}{\pi i}\int\limits_\Gamma \frac{\rho_1(\tau)}{\tau - t}d\tau + \frac{\gamma bu}{\pi i}\int\limits_\Gamma \frac{\alpha'(\tau)\rho_2(\tau)}{\alpha(\tau) - \alpha(t)}d\tau = g, \\ a_\alpha\rho_2 + \frac{b_\alpha u_\alpha}{\pi i}\int\limits_\Gamma \frac{\rho_1(\tau)}{\tau - t}d\tau + \frac{\gamma b_\alpha}{\pi i}\int\limits_\Gamma \frac{\alpha'(\tau)\rho_2(\tau)}{\alpha(\tau) - \alpha(t)}d\tau = g_\alpha, \end{cases} \tag{3}$$

where the coefficient γ takes the value 1 or -1, if respectively $\alpha(t)$ maps Γ onto itself with preservation or change of orientation on Γ.

An operator, applying of which to the vector (ρ_1, ρ_2) gives the left hand side of (3), can be written as

$$L \equiv Q_1 I + Q_2 S, \tag{4}$$

$$Q_1 = \begin{pmatrix} a & 0 \\ 0 & a_\alpha \end{pmatrix}, \quad Q_2 = \begin{pmatrix} b & \gamma bu \\ b_\alpha u_\alpha & \gamma b_\alpha \end{pmatrix},$$

I is an identity operator in $B_2(\Gamma)$. Operator L in (4) acts to the space $B^{\frac{1}{p}}_{p,1}(\Gamma)$[7]. Moreover, it belongs to $L(B_2)$ and can be written as

$$L = CP_1 + DP_2, \tag{5}$$

$$C = Q_1 + Q_2, \quad D = Q_1 - Q_2, \quad \text{and} \quad P_1 = \frac{1}{2}(I_1 + S_1), \quad P_2 = \frac{1}{2}(I_1 - S_1),$$

$$S_1 = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}, \quad I_1 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

Operators as L generates an algebra.

[7] N. Bliev, Generalized analytic functions in fractional spaces. Boston, MA: Longman; 1997. Published.

We introduce an additional SIE with a Carleman shift corresponding to (1)

$$K\chi \equiv a\chi + b(S\chi) - bu(WS\chi) = 0. \tag{6}$$

It is easy to see that if $\rho_1(t) = \chi(t)$, $\rho_2(t) = -\chi[\alpha(t)]$, then using the procedure described above, you can obtain the corresponding system (4) from (6). Allied operators are understood in the sense of the following identities

$$\int_\Gamma (M\varphi)(t)\,\psi(t)\,dt = \int_\Gamma \varphi(t)\left(M'\psi\right)(t)\,dt.$$

The mentioned union equations have the form

$$M'\psi \equiv a\psi - \frac{1}{\pi i}\int_\Gamma \frac{b(\tau)\,\psi(\tau)}{\tau - t}d\tau - \frac{\gamma}{\pi i}\int_\Gamma \frac{b[\alpha(\tau)]u[\alpha(\tau)]\alpha'(\tau)\,\psi[\alpha(\tau)]}{\tau - t}d\tau, \tag{7}$$

$$K'\omega \equiv a\omega - \frac{1}{\pi i}\int_\Gamma \frac{b(\tau)\,\omega(\tau)}{\tau - t}d\tau + \frac{\gamma}{\pi i}\int_\Gamma \frac{b[\alpha(\tau)]\,u[\alpha(\tau)]\alpha'(\tau)\,\omega[\alpha(\tau)]}{\tau - t}d\tau. \tag{8}$$

It is clear that equation (8) is accompanying to the union equation (7). The system of SIE without shift corresponding to the equations (7) and (8) have forms

$$\begin{cases} a\omega_1 - \frac{1}{\pi i}\int\limits_{\Gamma} \frac{b(\tau)\omega_1(\tau)}{\tau - t} d\tau - \frac{1}{\pi i}\int\limits_{\Gamma} \frac{b[\alpha(\tau)]u[\alpha(\tau)]\omega_2(\tau)}{\tau - t} d\tau = 0, \\ a_\alpha\omega_2 - \frac{\gamma\alpha'(t)}{\pi i}\int\limits_{\Gamma} \frac{b(\tau)u(\tau)\omega_1(\tau)}{\alpha(\tau)-\alpha(t)} d\tau - \frac{\gamma\alpha'(t)}{\pi i}\int\limits_{\Gamma} \frac{b[\alpha(\tau)]\omega_2(\tau)}{\alpha(\tau)-\alpha(t)} d\tau = 0. \end{cases} \tag{9}$$

It is easy to verify that the system (9) is union with the corresponding system of equations (3). The following result is necessary.

Lemma

1. *If a Carleman shift* α *preserves orientation on* Γ*, then the operators* M *in* (1) *and* K *in* (6) *have the following connection.*

$$\vartheta M - K\vartheta = T_1,$$

where $\vartheta(t) = \alpha(t) - t \neq 0$*,* T_1 *is a completely continuous operator.*

2. *If* α *changes orientation on* Γ*, then*

$$SM - KS = T_2,$$

where T_2 *is a completely continuous operator.*

Lemma

Operators M *and* K *is simultaneously Noetherian or non-Noetherian. In the case both are Noetherian operators, we have*

$$IndM = IndK.$$

Theorem

Operator $L = CP_1 + DP_2$ is F_+ (or F_-)-operator if and only if

$$\det C \neq 0, \det D \neq 0, \ t \in \Gamma.$$

Theorem

In order for SIE (1) with a Carleman shift to be Noetherian, it suffices to satisfy the following conditions

1. $aa_\alpha - a_\alpha b - ab_\alpha \neq 0$, $aa_\alpha + a_\alpha b + ab_\alpha \neq 0$, *if* $\alpha(t)$ *is an orientation-preserving shift;*
2. $aa_\alpha - a_\alpha b + ab_\alpha \neq 0$, $aa_\alpha + a_\alpha b - ab_\alpha \neq 0$, *if* $\alpha(t)$ *is an orientation-changing shift.*

Theorem

If the Noetherian conditions are satisfied, then the index for SIE (1) with a Carleman shift is calculated by the following formulas:

1. *If $\alpha(t)$ is an orientation-preserving shift, then*

$$IndM = \frac{1}{4\pi}\left\{\arg\left[\frac{aa_\alpha - a_\alpha b - ab_\alpha}{aa_\alpha + a_\alpha b + ab_\alpha}\right]\right\}_\Gamma;$$

2. *If $\alpha(t)$ is an orientation-changing shift, then*

$$IndM = \frac{1}{2\pi}\left\{\arg\left[aa_\alpha - a_\alpha b + ab_\alpha\right]\right\}_\Gamma.$$

Explicit solutions of the singular integral equation

Let us introduce the following functions

$$G(t)=\begin{cases}\frac{aa_\alpha-a_\alpha b-ab_\alpha}{aa_\alpha+a_\alpha b+ab_\alpha}, & \text{if} \quad US=SU,\\ \frac{aa_\alpha-a_\alpha b+ab_\alpha}{aa_\alpha+a_\alpha b-ab_\alpha}, & \text{if} \quad US=-SU,\end{cases}$$

and

$$H(t)=\begin{cases}\frac{a_\alpha g+uag_\alpha}{aa_\alpha+a_\alpha b+ab_\alpha}, & \text{if} \quad US=SU,\\ \frac{a_\alpha g-uag_\alpha}{aa_\alpha+a_\alpha b-ab_\alpha}, & \text{if} \quad US=-SU.\end{cases}$$

$$\Phi_k(z)=\frac{1}{2\pi i}\int\limits_\Gamma \frac{\varphi_k(\tau)}{\tau-z}d\tau\in B^{\frac{2}{p}}_{p,1}(D^\pm)\hookrightarrow C(\overline{D^\pm})^8,$$

where $\varphi_k(t)=J_k\varphi(t)$ $(k=1,2.)$, $\varphi(t)\in B^{\frac{1}{p}}_{p,1}(\Gamma)$.

[8] N. Bliev, Generalized analytic functions in fractional spaces. Boston, MA: Longman; 1997. Published.

Explicit solutions of the singular integral equation

Since Sokhotski-Plemelj formula

$$\varphi_k(t) = \Phi_k^+(t) - \Phi_k^-(t),$$
$$(S\varphi_k)(t) = \Phi_k^+(t) + \Phi_k^-(t),$$

holds for the boundary value on Γ of such Cauchy type integrals, we can rewrite the equation (1) in the form of a boundary value problem:

$$\Phi^+(t) = G(t)\Phi^-(t) + H(t). \tag{10}$$

The index of $G(t)$ by Γ will be denoted by $\varkappa$.

$$\varkappa = IndG(t) = \frac{1}{2\pi}\int\limits_{\Gamma} d[\arg G(t)] = \frac{1}{2\pi i}\int\limits_{\Gamma} d[\log G(t)].$$

We can easily obtain an expression for the boundary values of the canonical function:

$$\chi(z) = \begin{cases} \chi^+(z) = e^{\Pi(z)}, & \text{if} \quad z \in D^+, \\ \chi^-(z) = z^{-\varkappa}e^{\Pi(z)}, & \text{if} \quad z \in D^-, \end{cases}$$

where

$$\Pi(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{\log[\tau^{-\varkappa} G(\tau)]}{\tau - z} d\tau,$$

Theorem

Equation (1) has a solution in $B^{\frac{1}{p}}_{p,1}(\Gamma)$ if and only if equation (2) has a solution. Moreover, if φ_k $(k = 1, 2)$ is a solution of (2), then equation (1) has a solution and it is given by the formula

$$\varphi(t) = \begin{cases} \frac{g(t) - 2b(t)(S_\Gamma \varphi_2)(t)}{a(t)}, & \text{if} \quad US = SU. \\ \frac{g(t) - 2b(t)(S_\Gamma \varphi_1)(t)}{a(t)}, & \text{if} \quad US = -SU. \end{cases}$$

Theorem

Equation (1) has solutions and they are given by

$$\varphi(t) = \frac{g(t) - 2b(t)(S_\Gamma \varphi_k)(t)}{a(t)}, \quad k = 1, 2,$$

$$(S\varphi_k)(t) = \Phi_k^+(t) + \Phi_k^-(t)$$

where $k = 1$ *in case of* $SU = US$*, and* $k = 2$ *if* $SU = -US$*.*
Case $\varkappa \geq 0$. *In this case solutions are given by*

$$\Phi^\pm(z) = \chi^\pm(z)\Psi^\pm(z) + \chi^\pm(z)P_{\varkappa-1}(z), \tag{11}$$

$$\Psi(z) = \frac{1}{\pi i}\int\limits_\Gamma \frac{H(\tau)}{\chi^+(\tau)}\frac{d\tau}{\tau - z} \in B^{\frac{1}{p}}_{p,1}(D^\pm),$$

and $P_{\varkappa-1} \equiv 0$ *if* $\varkappa = 0$*, and* $P_{\varkappa-1}(z)$ *is a polynomial of degree no greater than* $n - 1$ *with arbitrary complex coefficients* $c_0, c_1, ..., c_{\varkappa-1}$ *for* $\varkappa > 0$*. If* $\varkappa = 0$*, then problem (1) has a unique solution.*

Theorem

Case $\varkappa < 0$. *For this case, we assume:*

$$\int_{\Gamma} \frac{H(\tau)t^{\varkappa-1}}{\chi^{+}(\tau)} d\tau = 0, \varkappa = 1, 2, ..., \varkappa$$

Then, we have that $P_{\varkappa-1}(z) \equiv 0$ in equality (11).

Thank you for your attention!