Renorming AM Spaces

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Historical Motivation

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- (Jarosz, '88) For **any** Banach space $(X, \|\cdot\|)$ there exists an equivalent renorming $|||\cdot|||$ such that the only isometries on $(X, |||\cdot|||)$ are $\{1, -1\}$.

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- Any AM space is a sublattice of a $C(K)$ space for some compact Hausdorff K.
- Any sublattice X of $C(K)$ is of the following form: there is an indexing set $\mathcal{I}$ and a set of tuples: $\{(s_i, t_i, \lambda_i) : i \in \mathcal{I}\}$ with each $s_i \neq t_i$ and $\lambda_i \geq 0$ such that $X = \{x \in C(K) : \forall i \in \mathcal{I}, x(t_i) = \lambda_i x(s_i)\}$

Main result

Theorem 1

- *Suppose* $(X, \|\cdot\|)$ *is a separable AM-space, and* $C > 1$. *Then* X *can be equipped with an equivalent lattice norm* $|||\cdot|||$ *so that* $\|\cdot\| \leq |||\cdot||| \leq C\|\cdot\|$, *and the identity map is the only lattice isometry on* $(X, |||\cdot|||)$.

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- To create new norm, add "weights" to elements in K to kill any extent isometries.
- Use dual space to show that only identity remains.

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- If $t \in K_m, s \in K_n$ with $x(t) = \lambda x(s)$ for all $x \in X$, then $\lambda = C^{n-m}$.
- X separates points in K_n for every n.
- Inspiration/application for construction: Benyamini's proof that G-spaces are linearly isomorphic to $C(K)$ spaces for some K uses 2-regular spaces.

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Proposition

Suppose X is a separable AM space and $C > 1$. Then there exists a $C-$regular space Y and a lattice isomorphism $U : X \to Y$ such that $\|U\| \cdot \|U^{-1}\| \leq C$.

Structure

Let X be a C-regular AM space, with associated underlying $K = \coprod K_n \cup \{\infty\}$. For $m \neq n \in \mathbb{N}$, let $D(m,n) :=$

$$\{t \in K_m : \exists s \in K_n \text{ such that } \forall x \in X, x(t) = C^{n-m}x(s)\}$$

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- $D(m,n)$ is homeomorphic to $D(n,m)$.
- $D(m,n)$ and $D(n,m)$ induce a map $\phi_{mn} : D(m,n) \to D(n,m)$ such that for all $f \in X$, $f(s) = C^{n-m} f(\phi_{mn}(s))$.

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Let $K'_n = K_n \backslash (\cup_{m<n} D(n,m))$, and let $K' = \cup_n K'_n$

- Atoms: characterized by "hereditarily" isolated points $k \in K'_n$ for some n, such that $\phi_{nm}(k)$ is also isolated whenever $k \in D(n,m)$. Induces $\theta_k \in X_+$, where $\theta_k(\phi_{nm}(s)) = C^{m-n}$, and $\theta_k(t) = 0$ otherwise.

Regular AM extensions

- Let X be a C-regular AM space with underlying sets K_n. A function $f \in C(K_M \cup ... \cup K_N)$ is **consistent** if $x(s) = C^{n-m}x(\phi_m n(s))$ for all $s \in K_m, N \leq m, n \leq M$.

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- If $x \in C(K_L \cup \ldots \cup K_N)$ is consistent, then there exists an extension $\tilde{x} \in X$ of x such that for all $j > N$, $\sup_{K_j} |\tilde{x}| \leq \max_{i \leq N} C_{i-j} \sup_{K_i} |x|$.

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- If, furthermore, $y \in X_+$ such that $0 \leq x \leq y$ on $K_L \cup \ldots \cup K_N$, then $\tilde{x}$ can be selected in such a way that $0 \leq \tilde{x} \leq y$.

Point separation in regular AM spaces

Lemma: Suppose $m \leq n$, $t \in K'_m$, $s \in K'_n$, and $U \subset K'_m$, $V \subset K'_n$ are disjoint open sets with the property that $t \in U \subset \overline{U} \subset K'_m$ and $s \in V \subset \overline{V} \subset K'_n$. Then for $\alpha, \beta \in [0, \infty)$, there exists $x \in X_+$ so that:

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1. For $j < m$, $x|_{K_j} = 0$.
2. $x(t) = \alpha$, $x(s) = \beta$, $x \leq \alpha$ on U, and $x \leq \beta$ on V.
3. If $m < n$, then $x|_{K_m \setminus U} = 0$.
4. If $m < n$, then for $m < j < n$, $0 \leq x|_{K_j} \leq C^{m-j}\alpha$.
5. On K_n, $0 \leq x \leq C^{m-n}\alpha \vee \beta$.
6. For $j > n$, $0 \leq x|_{K_j} \leq (C^{m-j}\alpha) \vee (C^{n-j}\beta)$.

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- *Suppose X is a regular AM space. Then X^* is lattice isometric to $M(K')$.*
- *Denote by $\mathcal{A}_1$ the set of normalized atoms in X^*. Equip $\mathcal{A}_1$ with the weak* topology. Then the map $\mathrm{j} : K' \to \mathcal{A}_l$, with $t \mapsto \delta_t$, is a topological homeomorphism.*

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- Denote the atoms in X by $(\theta_i)_{i\in I}$. each θ_i corresponds to "hereditarily isolated" point $a_i \in K'$. Given $C < 2$, let $c < \sqrt[3]{C}$,

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- Let $A = \{n \in \mathbb{N}, K'_n \text{ is infinite }\}$, and $B = \{n \in \mathbb{N}, K'_n \text{ is non-empty finite }\}$.
- For $n \in B$, list elements of $K'_n = \{t_{1n}, ..., t_{p_n n}\}$, and for $n \in A$, pick countable dense subset of distinct elements $t_{1n}, t_{2n}, ...$
- for each $n \in A$, pick decreasing $c > \lambda_{1n} > \lambda_{2n} > ...$, with $\lim_k \lambda_{kn} = 1$, For $n \in B$, pick decreasing $c > \lambda_{1n} > \lambda_{2n} > ... > 1$.

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- For $t \in K'$, let $\mu(t) = \lambda_{in}$ if $t = t_{in}$, and let $\mu(t) = 1$ otherwise.

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- Case $|I| = 0$: For $x \in X$, let

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- Case $|I| = 1$: Let θ_a be the atom in X determined $a \in K'$ P_1 be projection mapping x to the part of x disjoint from a and any $\phi_{mn}(a)$, and let $X_1 = \ker(P_1)$.. Now let

$$|||x||| = \max(|||(I - P_1)(x)|||_1, \|P_1(x)\|).$$

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- For $(i,j) \in \mathcal{P}$, let $\|\cdot\|_{i,j}$ be the norm on $\mathbb{R}^2$ whose unit ball is the octagon with vertices

$$\left(\pm\left(1 - \frac{c-1}{c(2\pi(i,j)+1)}\right), \pm 1\right) \text{ and } \left(\pm 1, \pm\left(1 - \frac{c-1}{2c\pi(i,j)}\right)\right)$$

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- Denote atoms by $(\theta_{a_i})_{i \in I}$, can now generate renorming $|||\cdot|||$, with

$$|||x||| = \max\left\{\sup_{t \in K'} \mu(t)|x(t)|, \sup_{(i,j)\in\mathcal{P}} \left\|\left(\mu(a_i)x(a_i), \mu(a_j)x(a_j)\right)\right\|_{i,j}\right\}$$

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- NB: the formal identity map $(\mathbb{R}^2, \|\cdot\|_{i_1,j_1}) \to (\mathbb{R}^2, \|\cdot\|_{i_2,j_2})$ is an isometry iff $i_1 = i_2$ and $j_1 = j_2$.

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    1. (1) $m \geq n$, and $s = \phi_{nm}(t)$.
    2. (2) $\mathrm{w}^* - \lim_i \mu(t_i)\delta_{t_i} = \alpha\mu(t)\delta_t$ for some $\alpha > 0$.

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- Suppose we are given $t \in K'_n$ and a sequence $(t_i) \subset K' \backslash \{t\}$. Then the following are equivalent:
  1. (1) There exists $m \geq n$ so that for i large enough, $t_i \in K'_m$. Furthermore, (t_i) converges to $s = \phi_{nm}(t)$.
  2. (2) $\mu(t_i)\delta_{t_i} \overset{w*}{\to} \alpha\mu(t)\delta_t$ for some $\alpha > 0$.

  Moreover, if (1) holds, then, in (2), $\alpha = C^{n-m}/\mu(t)$.

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- If $k = t_{in}$, the above implies that $\psi(k) = k$.
- For $k \neq t_{ni}$ use density of k_{ni} and w^* continuity of T^*.

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2. Let G be a closed subgroup of S_∞. Then there exists a reonorming $|||\cdot|||$ of c_0 so that $G \times \{-1, 1\}$ is topologically isomorphic to $ISO(c_0, |||\cdot|||)$.
3. For any Polish group G, there exists a separable Banach space B such that $\{-1, 1\} \times G$ is topologically isomorphic to $ISO(B, |||\cdot|||)$.

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- Can we do the same (or something approaching it) for AM spaces in general?