Boundedness of the Hilbert transform in Marcinkiewicz spaces and applications

Kanat Tulenov
Institute of Mathematics and Mathematical Modeling
Almaty, Kazakhstan
Joint work with F. Sukochev and D. Zanin
(UNSW, Sydney, Australia)

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Overview

1. Boundedness of the Hilbert transform on L_p $(1 < p < \infty)$ space
2. Boundedness of the Hilbert transform on ℓ_p $(1 < p < \infty)$ space
3. Boundedness of the Hilbert transform from one symmetric space into itself
4. Boundedness of the Hilbert transform from one symmetric space into another
5. Optimal range of the Hilbert transform in Lorentz spaces
6. Optimal range of the Hilbert transform in Marcinkiewicz spaces
7. Connection with the Triangular truncation operator
8. Applications to operator Lipschitz estimates

What is the Hilbert transform?

Given some function f, its Hilbert transform, denoted by $H(f)$, is calculated through the integral

$$(\mathcal{H}f)(t) = p.v.\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(s)}{t-s}ds = \lim_{\varepsilon\to 0+} \frac{1}{\pi}\int_{|s-t|\geq\varepsilon} \frac{f(s)}{t-s}ds.$$

The Hilbert transform is named after David Hilbert (1862-1943). Its first use dates back to 1905 in Hilbert's work concerning analytical functions in connection to the Riemann problem. In 1928 it was proved by Marcel Riesz (1886-1969) that the Hilbert transform is a bounded linear operator on $L_p(\mathbb{R})$ for $1 < p < \infty$. This result was generalized for the Hilbert transform in several dimensions (and singular integral operators in general) by Antoni Zygmund (1900-1992) and Alberto Calderón (1920-1998). [1] [2] [3]

[1] M. Riesz, Sur les maxima des formes bilinéaires et sur les fonctionnelles linéaires, Acta Math. 49 (1927), 465–497.

[2] A. P. Caldeón and A. Zygmund, On the existence of certain singular integrals, Acta Math. 88 (1952), 85–139.

[3] King, Frederick W. (2009a), Hilbert Transforms, 1, Cambridge: Cambridge University Press, p. 898.

Boundedness of the Hilbert transform on L_p

Theorem

If $1 < p < \infty$, then the Hilbert transform on $L_p(\mathbb{R})$ is a bounded linear operator, meaning that there exists a constant $c_p > 0$ such that

$$\|\mathcal{H}f\|_{L_p(\mathbb{R})} \leq c_p \|f\|_{L_p(\mathbb{R})}$$

for all $f \in L_p(\mathbb{R})$.

This theorem is due to [1, M. Riesz (1927, VII)]; see also [4,Titchmarsh (1948, Theorem 101)]. The best constant c_p is given by $c_p = \tan(\frac{\pi}{2p})$ for $1 < p \leq 2$ and $c_p = \cot(\frac{\pi}{2p})$ for $2 \leq p < \infty$. This result is due to [5, Pichorides 1972]. [4] [5]

[4]Titchmarsh, E (1948), Introduction to the theory of Fourier integrals (2nd ed.), Oxford University: Clarendon Press (published 1986), ISBN 978-0-8284-0324-5.

[5]Pichorides, S. (1972), On the best value of the constants in the theorems of Riesz, Zygmund, and Kolmogorov, Studia Mathematica, 44: 165-179

Boundedness of the Hilbert transform on ℓ_p

The discrete Hilbert transform is an operator $\mathcal{H}_d$ which maps the sequence $\{x(n)\}_{n\in\mathbb{Z}}$ to the sequence $\{(\mathcal{H}_d x)(n)\}$ defined by

$$\big(\mathcal{H}_d x\big)(n) := \frac{1}{\pi} \sum_{\substack{k\in\mathbb{Z}\\ k\neq n}} \frac{x(k)}{n-k}.$$

Boundedness of the discrete Hilbert transform $\mathcal{H}_d$ on $\ell_p(\mathbb{Z})$ is due to [6, E. Titchmarsh (1927)]. Using a representation of the discrete Hilbert transform, [7, Rodrigo Banuelos, Mateusz Kwasnicki (2019)] proved that its ℓ^p norm, $1 < p < \infty$, is bounded above by the L_p-norm of the continuous Hilbert transform. Together with the already known lower bound, this resolves the long-standing conjecture that the norms of these operators are equal. [6] [7]

[6] E. C. Titchmarsh, Reciprocal formulae involving series and integrals, Math. Z. 26 (1927), 496.

[7] Rodrigo Banuelos, Mateusz Kwasnicki (2019), On the ℓ^p-norm of the discrete Hilbert transform, Duke Math. J. 168(3): 471–504.

The Hilbert transform on L_1 and L_∞

If $f \in L_1(\mathbb{R})$, then it does not necessarily imply that $\mathcal{H}f \in L_1(\mathbb{R})$. For example, let $f = \chi_{[0,1]} \in L_1(\mathbb{R})$. Then its Hilbert transform is given by

$$(\mathcal{H}f)(t) = \frac{1}{\pi} \log \left| \frac{t}{t-1} \right|$$

and clearly $\mathcal{H}f$ does not belong to $L_1(\mathbb{R})$.
Similarly, the Hilbert transform is not bounded on $L_\infty(\mathbb{R})$. It may even not defined for all functions in $L_\infty(\mathbb{R})$.
Therefore, the Hilbert transform is bounded on $L_p(\mathbb{R})$ if and only if $1 < p < \infty$.

The Hilbert transform on L_p

Theorem

$$\mathcal{H} : L_p(\mathbb{R}) \to L_p(\mathbb{R}) \ \textit{is bounded} \ \Leftrightarrow 1 < p < \infty \ \Leftrightarrow L_p \textit{ is reflexive} \Rightarrow L_p \textit{ is separable.}$$

Therefore, it is natural to ask the following questions:
(1) Given a Banach space X of functions on $\mathbb{R}$, chosen from a suitably large class of such spaces containing L_p, is there a simple method for determining whether or nor $\mathcal{H}$ defines a continuous mapping from X into itself?
(2) Do the spaces X, for which $\mathcal{H} : X \to X$ is continuous, have any other, more intrinsic, property in common?
(3) In particular, is reflexivity/separability of X necessary and/or sufficient in order that $\mathcal{H} : X \to X$ is bounded?

Preliminaries

Let (I, μ) denote the measure space $I = \mathbb{R}_+, \mathbb{R}$, where $\mathbb{R}_+ := (0, \infty)$ and $\mathbb{R}$ is the set of real numbers, equipped with Lebesgue measure μ. Let $L(I, \mu)$ be the space of all measurable real-valued functions on I equipped with Lebesgue measure μ i.e. functions which coincide almost everywhere are considered identical. Define $L_0(I)$ to be the subset of $L(I, \mu)$ which consists of all functions f such that $\mu(\{t : |f(t)| > s\})$ is finite for some $s > 0$.

For $f \in L_0(I)$ (where $I = \mathbb{R}_+$ or $\mathbb{R}$), we denote by f^* the decreasing rearrangement of the function $|f|$. That is,

$$f^*(t) = \inf\{s \geq 0 : \ \mu(\{|f| > s\}) \leq t\}, \quad t > 0.$$

Symmetric spaces

Definition

We say that $(E, \|\cdot\|_E)$ is a symmetric (quasi-)Banach function space on I if the following holds:

(a) $E(I)$ is a subset of $L_0(I)$;

(b) $(E(I), \|\cdot\|_{E(I)})$ is a (quasi-)Banach space;

(c) If $f \in E(I)$ and if $g \in L_0(I)$ are such that $|g(t)| \leq |f(t)|$ for almost all $t > 0$, then $g \in E(I)$ and $\|g\|_{E(I)} \leq \|f\|_{E(I)}$;

(d) If $f \in E(I)$ and if $|f|$ and $|g|$ are equi-measurable, then implies that $g \in E(I)$ and $\|g\|_{E(I)} = \|f\|_{E(I)}$.

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[8] C. Bennett and R. Sharpley, *Interpolation of Operators*, Pure and Applied Mathematics, **129**. Academic Press, 1988.

[9] S. Krein, Y. Petunin, and E. Semenov, *Interpolation of linear operators*, Amer. Math. Soc., Providence, R.I., (1982).

Boyd indices

It is well known that $L_p(I)$, $(0 < p \le \infty)$ is a basic example of (quasi-)Banach symmetric spaces of functions. Let E be a symmetric Banach function space. For $s > 0$, the dilation operator $D_s : E \to E$ is defined by setting

$$D_s f(t) = f(t/s), \quad t > 0, f \in E.$$

The upper and lower Boyd indices of E are numbers $\overline{\beta}_E$ and $\underline{\beta}_E$ defined by

$$\overline{\beta}_E := \lim_{s \downarrow 0+} \frac{\log \|D_s\|_{E\to E}}{\log s}, \ \underline{\beta}_E := \lim_{s\to\infty} \frac{\log \|D_s\|_{E\to E}}{\log s}.$$

Moreover, they satisfy $0 \le \underline{\beta}_E \le \overline{\beta}_E \le 1$ (see [8, Definition III.5.12 and Proposition III.5.13, p. 149]). As is easily checked, if $E = L_p$, $1 \le p \le \infty$, then $\underline{\beta}_E = \overline{\beta}_E = 1/p$.

Lorentz spaces

Let $1 \leq p < \infty$ and let φ be an increasing concave function $\varphi : [0, \infty) \to [0, \infty)$ for which $\lim_{t \to 0+} \varphi(t) = 0$ (or simply $\varphi(0+) = 0$). For the function φ, the Lorentz space $\Lambda_{\varphi,p}(I)$ is defined by setting

$$\Lambda_{\varphi,p}(I) := \left\{ f \in L_0(I) : \int_{\mathbb{R}_+} f^*(s)^p d\varphi(s) < \infty \right\}$$

equipped with the norm

$$\|f\|_{\Lambda_{\varphi,p}(I)} := \int_{\mathbb{R}_+} f^*(s)^p d\varphi(s). \tag{1}$$

$\Lambda_{\varphi,1}(I) = \Lambda_{\varphi}(I)$. For more details on Lorentz spaces, we refer the reader to [8, Chapter II.5] and [9, Chapter II.5].
In particular, if $\varphi(t) = \log(1 + t)$, $t > 0$, then we denote $\Lambda_{\varphi}(I)$ by $\Lambda_{\log}(I)$.

Marcinkiewicz spaces

Let $\phi : [0, \infty) \to [0, \infty)$ be an increasing concave function for which $\lim_{t \to 0+} \varphi(t) = 0$ (or simply $\varphi(0+) = 0$). Define the Marcinkiewicz space $M_\phi(I)$ as follows

$$M_\phi(I) := \left\{ f \in L_0(I) : \sup_{t>0} \frac{1}{\phi(t)} \int_0^t f^*(s)ds < \infty \right\}$$

with the norm

$$\|f\|_{M_\phi(I)} := \sup_{t>0} \frac{1}{\phi(t)} \int_0^t f^*(s)ds.$$

These spaces are examples of symmetric Banach function spaces.

Weak L_1-space

Further, define the space $L_{1,\infty}(\mathbb{R}_+)$ (resp. $L_{1,\infty}(\mathbb{R})$) by setting

$$L_{1,\infty}(\mathbb{R}_+) = \{f \in L_0(\mathbb{R}_+) : \sup_{t>0} tf^*(t) < \infty\},$$

and equip $L_1(\mathbb{R}_+)$ with the functional $\|\cdot\|_{L_{1,\infty}}$ defined by the formula

$$\|f\|_{L_{1,\infty}} := \sup_{t>0} tf^*(t).$$

It is well-known that this space is a quasi-Banach symmetric space and is called weak-L_1 space.

Calderón operator and Hilbert transform

For each $f \in \Lambda_{\log}(\mathbb{R}_+)$, define the operator $S : \Lambda_{\log}(\mathbb{R}_+) \to (L_{1,\infty} + L_\infty)(\mathbb{R}_+)$ by

$$(Sf)(t) := \frac{1}{t}\int_0^t f(s)ds + \int_t^\infty f(s)\frac{ds}{s}, \ \ f \in \Lambda_{\log}(\mathbb{R}_+). \tag{2}$$

It is obvious that S is linear operator and $(Sf)(t)$ is a decreasing function of t.
If $f \in \Lambda_{\log}(\mathbb{R})$, then the classical Hilbert transform $\mathcal{H}$ is defined by the principal-value integral

$$(\mathcal{H}f)(t) = p.v.\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(s)}{t-s}ds, \quad f \in \Lambda_{\log}(\mathbb{R}).$$

Boyd's results (Necessary condition)

In his thesis [10, Chapter II] (see also [11]) Boyd posed a problem of finding necessary and sufficient conditions for the Hilbert transformation ($\mathcal{H}$ to be bounded from one symmetric Banach space E into another F.

Theorem (Boyd, 1966, Necessary condition)

If the Hilbert transform $\mathcal{H} : E(\mathbb{R}) \to F(\mathbb{R})$ is bounded, then $E \subseteq F$.

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[10] D.W. Boyd, The Hilbert Transformation on rearrangement invariant Banach spaces, Thesis, University of Toronto (1966).

[11] D. Boyd, The Hilbert transform on rearrangement-invariant spaces, Can. J. Math. 19 (1967), 599-616.

Boyd's results (Necessary and Sufficient condition)

Theorem (Boyd, 1966, Necessary and Sufficient condition, but $E = F$)

The Hilbert transform $\mathcal{H} : E(\mathbb{R}) \to E(\mathbb{R})$ is bounded if and only if

$$0 < \underline{\beta}_E \leq \overline{\beta}_E < 1.$$

Theorem (Boyd, 1966, Necessary and Sufficient condition)

The Hilbert transform $\mathcal{H} : E(\mathbb{R}) \to F(\mathbb{R})$ is bounded if and only if $S : E(\mathbb{R}_+) \to F(\mathbb{R}_+)$.

Here, operator S is defined by formula (2).

It is shown in [12, I. Halperin] that, for $p > 1$, $\Lambda_{\varphi,p}(\mathbb{R})$ is is not always reflexive, and that $\Lambda_\phi(\mathbb{R})$ and $M_\phi(\mathbb{R})$ are never reflexive. In fact, $M_\phi(\mathbb{R})$ is non-separable. If we take $\phi(t) = \frac{t^\alpha}{\alpha}$ for $0 < \alpha < 1, t > 0$, then $\mathcal{H} : \Lambda_\phi(\mathbb{R}) \to \Lambda_\phi(\mathbb{R})$ and $\mathcal{H} : M_\phi(\mathbb{R}) \to M_\phi(\mathbb{R})$ are bounded. Therefore, neither reflexivity nor separability of X is necessary for boundedness $\mathcal{H}$ from X into itself. Taking $\varphi(t) = \log(1 + t), t > 0$ for $p > 1$, $\Lambda_{\varphi,p}(\mathbb{R})$, we obtain that $\mathcal{H}$ is not bounded from $\Lambda_{\varphi,p}(\mathbb{R})$ into itself. Hence, reflexivity is insufficient for that $\mathcal{H} : X \to X$ is bounded.

Theorem (Boyd, 1966)

If $1 < p < \infty$, then $\mathcal{H} : \Lambda_{\varphi,p}(\mathbb{R}) \to \Lambda_{\varphi,p}(\mathbb{R})$ is bounded if and only if $\Lambda_{\varphi,p}(\mathbb{R})$ is uniformly convex.

Lemma (13, Halperin, 1954)

The space $\Lambda_{\varphi,p}(\mathbb{R})$ is uniformly convex if and only if $\sup\limits_{t>0} \frac{\varphi(t)}{\varphi(2t)} < 1$.

12 13

[12] I. Halperin, Reflexivity in the L^λ function spaces, Duke Math. J., 21 (1954), 205-208.

[13] I. Halperin, Uniform convexity in function spaces, Duke Math. J., 21 (1954), 195-204.

Andersen's result

Let $E(\mathbb{Z}_+)$ be a symmetric Banach space of sequences. In 1976, K.F. Andersen obtained Boyd's results in discrete case.

Theorem (Andersen, App.Anal.,1976)

The Hilbert transform $\mathcal{H}_d : E(\mathbb{Z}) \to E(\mathbb{Z})$ is bounded if and only if

$$0 < \underline{\beta}_E \leq \overline{\beta}_E < 1.$$

Here, $\mathcal{H}_d$ is defined by

$$(\mathcal{H}_d x)(n) := \frac{1}{\pi} \sum_{\substack{k \in \mathbb{Z} \\ k \neq n}} \frac{x(k)}{n-k}, \quad x \in \Lambda_{\log}(\mathbb{Z}_+). \tag{3}$$

Kolmogorov's result

In 1928, A. Kolmogorov obtained the following result [14, Theorem III.4.9 (b), p. 139].

Theorem (Kolmogorov, Fundamenta Math., 1928)

The Hilbert transform

$$\mathcal{H} : L_1(\mathbb{R}) \to L_{1,\infty}(\mathbb{R})$$

is bounded.

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[14] C. Bennett and R. Sharpley, *Interpolation of Operators,* Pure and Applied Mathematics, **129**. Academic Press, 1988.

Komori's result

In 2001 Y. Komori obtained a discrete version of the Kolmogorov's result for the discrete Hilbert transform.

Theorem (Komori, Far East J. Math. Sci., 2001)

The discrete Hilbert transform

$$\mathcal{H}_d : \ell_1(\mathbb{Z}) \to \ell_{1,\infty}(\mathbb{Z})$$

is bounded.

Problems

Problem (1)

Given a symmetric quasi-Banach function space $E = E(\mathbb{R})$, determine the least symmetric quasi-Banach function space $F = F(\mathbb{R})$ such that $\mathcal{H} : E(\mathbb{R}) \to F(\mathbb{R})$ is bounded.

Problem (2)

Given a symmetric quasi-Banach sequence space $E = E(\mathbb{Z})$, determine the least symmetric quasi-Banach sequence space $F = F(\mathbb{Z})$ such that $\mathcal{H}_d : E(\mathbb{Z}) \to F(\mathbb{Z})$ is bounded.

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[15] J. Soria, P. Tradacete, *Optimal rearrangement invariant range for Hardy-type operators* , Proc. of the Royal Soc. of Edinburgh, **146A** (2016), 865–893.

Optimal range of the Calderón operator

Definition

Let E be a quasi-Banach symmetric space on $\mathbb{R}_+$. Let $E(\mathbb{R}_+) \subset \Lambda_{\log}(\mathbb{R}_+)$ and let S be the operator defined in (2). Define

$$F(\mathbb{R}_+) := \{f \in (L_{1,\infty} + L_\infty)(\mathbb{R}_+) : \exists g \in E(\mathbb{R}_+),\ f^* \leq Sg^*\} \tag{4}$$

such that

$$\|f\|_{F(\mathbb{R}_+)} := \inf\{\|g\|_{E(\mathbb{R}_+)} : f^* \leq Sg^*\} < \infty.$$

Optimal range of the Hilbert transform. Resolutions of Problems 1 and 2

The following result provides solution to the Problem 1.

Theorem (Sukochev/T/Zanin, JFA, 2019)

Let E be a quasi-Banach symmetric space on $\mathbb{R}$. If $E(\mathbb{R}_+) \subset \Lambda_{\log}(\mathbb{R}_+)$, then

(i) $(F(\mathbb{R}), \|\cdot\|_{F(\mathbb{R})})$ *is a quasi-Banach symmetric space.*

(ii) *Moreover,* $(F(\mathbb{R}), \|\cdot\|_{F(\mathbb{R})})$ *is the least (smallest) symmetric quasi-Banach space (range) such that the Hilbert transform* $\mathcal{H} : E(\mathbb{R}) \to F(\mathbb{R})$ *is bounded.*

The discrete version of this result was obtained in [T, Arch. Math., 2019]. [16]

[16] D. E. Edmunds, Z. Mihula, V. Musil, and L. Pick, *Boundedness of classical operators on rearrangement-invariant spaces*, J. Funct. Anal. **278:4** (2020), 108341.

Corollary (Sukochev/T/Zanin, JFA, 2019)

Let E be a quasi-Banach symmetric space on $\mathbb{R}$. If $E(\mathbb{R}_+) \subset \Lambda_{\log}(\mathbb{R}_+)$, then the Hilbert transform

$$\mathcal{H} : E(\mathbb{R}) \to G(\mathbb{R})$$

is bounded if and only if $F(\mathbb{R}) \subseteq G(\mathbb{R})$, where $F(\mathbb{R})$ is the space defined in (4).

An example with trivial Boyd indices

Let us denote

$$(L_{1,\infty}(\mathbb{R}))_0 := \{f \in L_{1,\infty}(\mathbb{R}) : \lim_{t\to 0+} tf^*(t) = 0\}.$$

The following result shows that the optimal range for the Hilbert transform on $L_1(\mathbb{R})$ is, in fact, $(L_{1,\infty}(\mathbb{R}))_0$. In particular, this result refines the classical Kolmogorov's theorem [17, Theorem III.4.9 (b), p. 139].

Corollary (Sukochev/T/Zanin, JFA, 2019)

If $E(\mathbb{R}) = L_1(\mathbb{R})$, *then*

$$F(\mathbb{R}) = (L_{1,\infty}(\mathbb{R}))_0.$$

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[17] C. Bennett and R. Sharpley, *Interpolation of Operators,* Pure and Applied Mathematics, **129**. Academic Press, 1988.

[18] J. Soria, P. Tradacete, *Characterization of the restricted type spaces* $R(X)$, Math. Ineq. Applic. **18** (2015), 295–319.

Lorentz case

Theorem (Sukochev/T/Zanin, Bull. de Sci. Math., 2021)

If $E(\mathbb{R}) = \Lambda_\phi(\mathbb{R})$, *then* $F(\mathbb{R}) = \Lambda_\psi(\mathbb{R})$ *holds for the function* $\psi(t) := \inf_{s>1} \frac{\phi(st)}{1+\log(s)}$ *satisfying* $\lim_{t\to\infty} \frac{\psi(t)}{t} = 0$ *and* $S\left(\frac{\psi(t)}{t}\right) \leq c_{\phi,\psi} \frac{\phi(t)}{t}, \quad t > 0.$
In other words,

$$\mathcal{H} : \Lambda_\phi(\mathbb{R}) \to F(\mathbb{R})$$

is bounded if and only if $\Lambda_\psi(\mathbb{R}) \subseteq F(\mathbb{R})$.

Marcinkiewicz case

Question. Given a Marcinkiewicz space $E = M_\varphi(\mathbb{R}) \subseteq \Lambda_{\log}(\mathbb{R})$, determine or characterize the least (smallest) Marcinkiweicz space $F = M_\psi(\mathbb{R})$ such that $\mathcal{H} : M_\varphi(\mathbb{R}) \to M_\psi(\mathbb{R})$ is bounded.

Theorem (Bekbayev/Sukochev/T/Zanin, Unpublished)

If $E(\mathbb{R}) = M_\varphi(\mathbb{R})$, then

$$F(\mathbb{R}) = M_\psi(\mathbb{R}),$$

where, $\psi'(t) := (S\varphi')(t), t > 0$.
In other words,

$$\mathcal{H} : M_\varphi(\mathbb{R}) \to F(\mathbb{R})$$

is bounded if and only if $M_\psi(\mathbb{R}) \subseteq F(\mathbb{R})$.

Example with trivial Boyd indices

Example If $\varphi(t) = \log(1+t)$, $t > 0$, then $\varphi'(t) = \frac{1}{1+t}$, $\; t > 0$. We have

$$(S\phi')(t) = \frac{1}{t}\int_0^t \frac{1}{1+s}ds + \int_t^\infty \frac{1}{s(1+s)}ds = \frac{\log(1+t)}{t} + \log(1+\frac{1}{t}).$$

Hence, $\psi(t) \asymp t\log(\frac{e}{t})$ on $(0,1]$ and $\psi(t) \asymp t\log(e+t)$ on $(1,\infty)$. Therefore, by the above theorem we obtain that the Hilbert transform

$$\mathcal{H} : M_{1,\infty}(\mathbb{R}) \to M_\psi(\mathbb{R}),$$

where ψ is as above, is bounded.

Triangular truncation operator

Define the triangular truncation (with respect to a continuous chain) as usual: if the operator A is an integral operator on the Hilbert space $L_2(a,b)$, $-\infty \le a < b \le \infty$, with the integral kernel K, i.e.

$$(Af)(t) = \int_a^b K(t,s)f(s)ds, \quad t > 0, \quad f \in L_2(a,b),$$

then $T(A)$ is an integral operator with truncated integral kernel

$$((T(A))f)(t) = \int_a^b \mathrm{sgn}(t-s)K(t,s)f(s)ds, \quad t \in (a,b), \quad f \in L_2(a,b).$$

Strictly speaking, it is the operator $P := \frac{\mathrm{id}-T}{2}$ which should be called (upper) triangular truncation operator (and the operator $\mathrm{id} - P = \frac{\mathrm{id}+T}{2}$ is lower triangular truncation operator). However, we ignore this difference and call both T and P the triangular truncation operators. [19]

[19] I.C. Gohberg and M.G. Kreĭn, *Theory and Applications of Volterra operators on Hilbert spaces*, Transl. Math. Monogr., **24**, Amer. Math. Soc., Providence, R.I., 1970.

Discrete Calderón operator

Define the discrete version of the operator $S_d : \Lambda_{\log}(\mathbb{Z}_+) \to \ell_\infty(\mathbb{Z}_+)$ by

$$(S_d x)(n) := \frac{1}{n+1} \sum_{k=0}^{n} x(k) + \sum_{k=n+1}^{+\infty} \frac{x(k)}{k}, \quad x \in \Lambda_{\log}(\mathbb{Z}_+). \tag{5}$$

For an operator superficially similar to T, J. Arazy (see [20,Theorem 4.1]) characterized all ideals $\mathcal{E}$ matching completely the above Boyd's results for the case when the Boyd indices of $\mathcal{E}$ are not trivial. However, even if $\mathcal{E}$ is an interpolation space for the couple $(\mathcal{L}_p, \mathcal{L}_q)$, $1 < p < q < \infty$, the techniques employed by Arazy, Gohberg and Krein, and in many other relevant papers, do not yield insight into whether $\mathcal{E}(H)$ is the optimal range space for the operator T restricted to $\mathcal{E}(H)$. Thus, finding the optimal range space for the operator T is of particular interest even in the simplest case, namely when the ideal $\mathcal{E}$ has non-trivial Boyd indices. [20]

[20] J. Arazy, *Some remarks on interpolation theorems and the boundedness of the triangular projection in unitary matrix spaces*, Integr. Equ. Oper. theory, **1**:4 (1978), 453–495

Problem

A classical Macaev's result (see [21]) and the Gohberg-Krein Theorem suggest that the natural receptacle for the operator T restricted to the trace class $\mathcal{L}_1(H)$ is the quasi-Banach ideal $\mathcal{L}_{1,\infty}(H)$ and this motivates our consideration of quasi-Banach ideals.

Problem

Given a symmetric quasi-Banach sequence space $E = E(\mathbb{Z})$, determine the least symmetric quasi-Banach sequence space $F = F(\mathbb{Z})$ such that $T : \mathcal{E}(H) \to \mathcal{F}(H)$.

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[21] V.I. Macaev, *Volterra operators obtained from self-adjoint operators by perturbation*, Dokl. Akad. Nauk SSSR, **139** (1961), 810–813 (in Russian).

Let H denote a fixed separable Hilbert space and let $B(H)$ be the algebra of all linear bounded operators on H. Let us denote by $K(H)$ the ideal of compact operators on H and $s(A) := \left\{s(n, A)\right\}_{n\in\mathbb{Z}_+}$ is the sequence of singular values of a compact operator A, i.e. $s(n, A) := \lambda(n, |A|)$, $n \in \mathbb{Z}_+$, $(|A| := (A^*A)^{1/2})$.

Definition

Let X be a linear subset in $B(H)$ equipped with a complete norm $\|\cdot\|_X$. We say that X is a symmetric operator space (in $B(H)$) if for $A \in X$ and for every $B \in B(H)$ with $s(n, B) \leq s(n, A)$, $n \in \mathbb{Z}_+$, we have $B \in X$ and $\|B\|_X \leq \|A\|_X$.

Recall the construction of a symmetric operator space (or non-commutative symmetric space) $\mathcal{E}(H)$.

Let E be a symmetric (quasi-)Banach sequence space on $\mathbb{Z}_+$. Set

$$\mathcal{E}(H) = \Big\{ A \in K(H) : \ s(A) \in E(\mathbb{Z}_+) \Big\}.$$

We equip $\mathcal{E}(H)$ with a natural quasi-norm

$$\|A\|_{\mathcal{E}(H)} = \|s(A)\|_{E(\mathbb{Z}_+)}, \quad A \in \mathcal{E}(H).$$

The following fundamental theorem was proved in the following theorem. It shows that the (quasi-)norm introduced above is, in fact, a (quasi-)norm.

Theorem (Kalton, Sukochev, J. Reine Angew. Math., 2008)

Let E be a symmetric sequence space on $\mathbb{Z}_+$. Set

$$\mathcal{E}(H) = \Big\{ A \in K(H) : \ s(A) \in E(\mathbb{Z}_+) \Big\}.$$

Then, so defined $(\mathcal{E}(H), \|\cdot\|_{\mathcal{E}(H)})$ is a symmetric operator space.

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[22] F. Sukochev, *Completeness of quasi-normed symmetric operator spaces*, Indag. Math. (N.S.) **25**:2 (2014), 376–388.

If $E=\ell_p(\mathbb{Z}_+)$, $1\le p<\infty$, then we obtain

$$\mathcal{L}_p(H):=\Big\{A\in K(H):\ s(A)\in\ell_p(\mathbb{Z}_+)\Big\},$$

which is so called Schatten-von Neumann class of all compact operators $A:H\to H$ with finite norm

$$\|A\|_{\mathcal{L}_p(H)}:=\Big(\sum_{k=1}^{\infty}s(k,A)^p\Big)^{1/p}.$$

If $p=\infty$, then we set $\mathcal{L}_\infty(H):=B(H)$ with the uniform norm, i.e. $\|A\|_{\mathcal{L}_\infty(H)}:=\|A\|$, $A\in B(H)$. Moreover, when $p=2$ the space $\mathcal{L}_2(H)$ (usually it is called Hilbert-Schmidt class) becomes Hilbert space with the inner product

$$<A,B>:=Tr(B^*A),\ A,B\in\mathcal{L}_2(H),$$

where B^* is adjoint operator of B and Tr is the usual trace. The weak-$\mathcal{L}_1$ ideal $\mathcal{L}_{1,\infty}(H)$ is defined as

$$\mathcal{L}_{1,\infty}(H):=\{A\in K(H):s(A)\in\ell_{1,\infty}(\mathbb{Z}_+)\}.$$

The Lorentz ideal $\Lambda_\varphi(H)$ (see [23, Example 1.2.7, p. 25]) is

$$\Lambda_\varphi(H) := \{A \in K(H) : s(A) \in \Lambda_\varphi(\mathbb{Z}_+)\}.$$

If $\varphi(t) := \log(1+t)\,(t > 0)$, then the corresponding Lorentz ideal $\Lambda_{\log}(H)$ is defined by

$$\Lambda_{\log}(H) := \left\{ A \in K(H) : \|A\|_{\Lambda_{\log}(H)} = \sum_{n=0}^{\infty} \frac{s(n,A)}{n+1} < \infty \right\}.$$

This ideal contains all Schatten-von Neumann classes $\mathcal{L}_p(H)$ $(1 \le p < \infty)$. It corresponds to the double index $(\infty, 1)$ on the Lorentz scale and is known as the Macaev ideal (see [24]).
23 24

[23] S. Lord, F. Sukochev, D. Zanin, *Singular traces. Theory and applications*, De Gruyter Studies in Mathematics, **46**. De Gruyter, Berlin, 2013.

[24] I.C. Gohberg and M.G. Kreĭn, *Introduction to the theory of linear non-selfadjoint operators*, Transl. Math. Monogr., **18**, Amer. Math. Soc., Providence, R.I., 1969.

Abstarct operator $\mathcal{T}$

Theorem (Sukochev/T/Zanin, JFA, 2019)

Let $\mathcal{T} : \mathcal{L}_2(H) \to \mathcal{L}_2(H)$ be a self-adjoint contraction. Suppose that $\mathcal{T}$ admits a bounded linear extension from $\mathcal{L}_1(H)$ to $\mathcal{L}_{1,\infty}(H)$, that is

$$\|\mathcal{T}\|_{\mathcal{L}_1(H)\to\mathcal{L}_{1,\infty}(H)} \lesssim 1. \tag{6}$$

Then

$$s(\mathcal{T}(A)) \leq c_{abs} S_d(s(A)), \quad A \in \Lambda_{\log}(H),$$

where S_d is the discrete operator defined in (5).

Theorem (Sukochev/T/Zanin, JFA, 2019)

For all $V \in \mathcal{L}_1(H)$, we have

$$\|T(V)\|_{\mathcal{L}_{1,\infty}(H)} \leq 20\|V\|_{\mathcal{L}_1(H)}. \tag{7}$$

Theorem (Sukochev/T/Zanin, JFA, 2019)

Let $E(\mathbb{Z}_+) \subset \Lambda_{\log}(\mathbb{Z}_+)$ be a quasi-Banach symmetric space of sequences and $\mathcal{E}(H)$ be the corresponding non-commutative quasi-Banach ideal. The following are equivalent

(i) $S_d : E(\mathbb{Z}_+) \to F(\mathbb{Z}_+)$;

(ii) $T : \mathcal{E}(H) \to \mathcal{F}(H)$.

First restatement of the motivational question

Question

For which functions f we have that $f(A) - f(B) \in \mathcal{L}_1$ whenever are self-adjoint operators on H such that $A - B \in \mathcal{L}_1$, and, in addition,

$$\|f(A) - f(B)\|_{\mathcal{L}_1} \leq const(f)\|A - B\|_{\mathcal{L}_1}?$$

Such functions f are called **operator Lipshitz** *functions from $\mathcal{L}_1$ to $\mathcal{L}_1$.*

M.G. Krein's Conjecture

Considering the special case, when operators A, B are operators on one-dimensional Hilbert space $\mathbb{C}$, we see that if a function f is operator Lipschitz, then f is necessarily Lipschitz function, and therefore, the following conjecture arises naturally.
Conjecture (M.G. Krein 1964) Every Lipschitz functions is operator Lipschitz function from $\mathcal{L}_1$ to $\mathcal{L}_1$.
However, this conjecture fails.

Counterexamples

Theorem (Farforovskaya 1972)

There exists a Lipschitz function f and A, B with $A - B \in \mathcal{L}_1$ such that $f(A) - f(B)$ does not belong to $\mathcal{L}_1$

Theorem (Davies, 1988)

Given $d \in \mathbb{N}$, there exist $A = A^*, B = B^* \in \mathcal{L}_\infty(\mathbb{C}^{2d})$ such that

$$\||A| - |B|\|_{\mathcal{L}_1} \geq const \cdot \log(d) \cdot \|A - B\|_{\mathcal{L}_1}.$$

Lipschitz estimates in $\mathcal{L}_p(H)$, $1 < p < \infty$

Theorem (Solomyak 1967, Davies 1988)

If A, B are linear bounded operators on H and if $A - B \in \mathcal{L}_p$, $1 < p < \infty$, then $|A| - |B| \in \mathcal{L}_p$ and

$$\||A| - |B|\|_{\mathcal{L}_p} \leq c_p \|A - B\|_{\mathcal{L}_p}.$$

Here, c_p depends only on p.

Theorem (Potapov-Sukochev, Acta Math. 2011)

Let f be a Lipschitz function on $\mathbb{R}$. Then there is a constant $c_p > 0$ such that

$$\|f(A) - f(B)\|_{\mathcal{L}_p} \leq c_p \|A - B\|_{\mathcal{L}_p}$$

holds for all $A = A^*, B = B^* \in \mathcal{L}_p(H)$ if and only if $1 < p < \infty$.

Lipschitz estimates in $\mathcal{L}_1(H)$.

In 2009, Nazarov and Peller proved the following assertion

Theorem (Nazarov-Peller, C. R. Math. Acad. Sci. Paris, 2009)

If $A - B$ is rank one operator, then $f(A) - f(B) \in \mathcal{L}_{1,\infty}(H)$ for every Lipschitz function f.

They formulated the conjecture below.

Conjecture. Every Lipschitz functions is operator Lipschitz function from $\mathcal{L}_1(H)$ to $\mathcal{L}_{1,\infty}(H)$, that is $f(A) - f(B) \in \mathcal{L}_{1,\infty}(H)$ whenever $A - B \in \mathcal{L}_1(H)$.

Partial positive result

Theorem (Caspers-Potapov-Sukochev-Zanin, JOT, 2015)

If A, B are linear bounded operators on H and if $A - B \in \mathcal{L}_1(H)$, then $|A| - |B| \in \mathcal{L}_{1,\infty}(H)$ and

$$\||A| - |B|\|_{\mathcal{L}_{1,\infty}} \leq \left(34 + \frac{2560e}{\pi}\right) \|A - B\|_{\mathcal{L}_1}.$$

Complete resolution of Nazarov-Peller conjecture

Theorem (Caspers-Potapov-Sukochev-Zanin, Amer.J.Math. 2019)

If A, B are self-adjoint bounded operators on H and if $A - B \in \mathcal{L}_1(H)$, then for any Lipschitz function f we have $f(A) - f(B) \in \mathcal{L}_{1,\infty}(H)$ and

$$\|f(A) - f(B)\|_{\mathcal{L}_{1,\infty}} \leq c_{abs}\|f'\|_\infty\|A - B\|_{\mathcal{L}_1}.$$

Some new directions

Question

Take an ideal $\mathcal{E}(H)$. Describe the optimal ideal $\mathcal{F}(H)$ such that every Lipschitz function f is operator Lipschitz from $\mathcal{E}(H)$ into $\mathcal{F}(H)$.

This is a generalization of the case when $\mathcal{E}(H) = \mathcal{L}_1(H)$ and $\mathcal{F}(H) = \mathcal{L}_{1,\infty}(H)$.

Theorem (Caspers-Potapov-Sukochev-Zanin, JOT, 2015)

If A, B are compact operators on H such that $A - B \in \mathcal{L}_1(H)$, then $|A| - |B| \in (\mathcal{L}_{1,\infty})^0$.

Here, the ideal $(\mathcal{L}_{1,\infty})^0$ in $B(H)$ is defined by setting

$$(\mathcal{L}_{1,\infty})^0 = \{A \in \mathcal{L}_{1,\infty}(H) : ks(k, A) \to 0 \quad \text{as} \quad k \to \infty\}.$$

This ideal coincides with the closure of the ideal of all finite rank operators in $\mathcal{L}_{1,\infty}$ and is commonly called the separable part of $\mathcal{L}_{1,\infty}$.

Theorem (Sukochev/T/Zanin, JFA, 2019)

Let $E(\mathbb{Z}_+) \subset \Lambda_{\log}(\mathbb{Z}_+)$ be a quasi-Banach symmetric space of sequences and $\mathcal{E}(H)$ be the corresponding non-commutative quasi-Banach ideal. The following are equivalent

(i) $S_d : E(\mathbb{Z}_+) \to F(\mathbb{Z}_+)$;

(ii) $T : \mathcal{E}(H) \to \mathcal{F}(H)$.

Moreover, if any one of above conditions holds, then for all self-adjoint operators $A, B \in B(H)$ such that $A - B \in \mathcal{E}(H)$ and for every Lipschitz function f defined on $\mathbb{R}$, we have

$$\|f(A) - f(B)\|_{\mathcal{F}(H)} \le c_{\mathcal{F},\mathcal{E}} \|f'\|_{L_\infty(\mathbb{R})} \|A - B\|_{\mathcal{E}(H)}.$$

Theorem (Sukochev/T/Zanin/, Unpublished)

Let φ be an increasing concave function on $[0,\infty)$ such that $\phi(0+)=0$ and let ψ be the function as above. Suppose $M_\varphi(\mathbb{Z}_+) \subset \Lambda_{\log}(\mathbb{Z}_+)$. The following are equivalent:

(i) $T : M_\varphi(H) \to M_\psi(H)$;

(ii) $S_d : M_\varphi(\mathbb{Z}_+) \to M_\psi(\mathbb{Z}_+)$.

If any one of above conditions holds, then for all self-adjoint operators $A, B \in B(H)$ such that $A - B \in M_\varphi(H)$ and for every Lipschitz function $f : \mathbb{R} \to \mathbb{R}$, we have

$$\|f(A) - f(B)\|_{M_\psi(H)} \leq c_{\varphi,\psi} \|f'\|_{L_\infty(\mathbb{R})} \|A - B\|_{M_\varphi(H)}.$$

Another application of these results to boundedness of the higher dimensional Hilbert operator can be found from the recently accepted paper [25]

[25] F. Sukochev, K. Tulenov, and D. Zanin. A weak type (1, 1) estimate for the Hilbert operator in higher-dimensional setting. Studia Mathematica.,(2022), 1–16. DOI:10.4064/sm201201-4-11.

Open questions

Question

What is the least ideal $\mathcal{F}(H)$ such that

$$\|f(A) - f(B)\|_{\mathcal{F}(H)} \leq c_{\mathcal{F},\mathcal{E}}\|A - B\|_{\mathcal{E}(H)}?$$

Here, A, B are self-adjoint bounded operators such that $A - B \in \mathcal{E}(H)$.

In Particular,

Question

What is the least ideal $\mathcal{F}(H)$ such that

$$\|f(A) - f(B)\|_{\mathcal{F}(H)} \leq c_{\mathcal{F},\mathcal{E}}\|A - B\|_{\mathcal{L}_1(H)}?$$

Here, A, B are self-adjoint bounded operators such that $A - B \in \mathcal{L}_1(H)$.

Thank you for your attention !