Weak compactness in variable exponent Lebesgue spaces

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Variable Lebesgue spaces

Let (Ω, μ) be a finite separable non-atomic measure space.

Definition

Given $p(\cdot) : \Omega \to [1, \infty)$, the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is the Banach function space consisting of all $f \in L_0(\Omega)$ with

$$\int_\Omega \left| \frac{f(t)}{r} \right|^{p(t)} dt < \infty, \text{ for some } r > 0,$$

with the Luxemburg norm

$$\|f\|_{p(\cdot)} := \inf \left\{ r > 0 \colon \int_\Omega \left| \frac{f(t)}{r} \right|^{p(t)} dt \leq 1 \right\}.$$

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$p^+ := ess\sup_{t\in\Omega} p(t)$ and $p^- := ess\inf_{t\in\Omega} p(t)$.

Also, the essential range of $p(\cdot)$ is
$R_{p(\cdot)} := \{q \in [1, \infty) \colon \forall\varepsilon > 0, \mu\left(p^{-1}(q-\varepsilon, q+\varepsilon)\right) > 0\}$.

Musielak-Orlicz spaces

Definition

A Musielak-Orlicz function is a measurable function $\Phi : \Omega \times [0, \infty) \to [0, \infty)$ *such that* $\Phi(t, \cdot)$ *is an Orlicz function for every* $t \in \Omega$.

The Musielak-Orlicz space is the space of all $f \in L_0(\Omega)$ *with*

$$\int_\Omega \Phi\left(t, \left|\frac{f(t)}{r}\right|\right) dt < \infty, \text{ for some } r > 0,$$

with the Luxemburg norm

$$\|f\|_\Phi = \inf\left\{r > 0 \colon \int_\Omega \Phi\left(t, \left|\frac{f(t)}{r}\right|\right) dt \le 1\right\}.$$

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• If $\Phi(t, x) = \varphi(x)$, then $L^\Phi(\Omega) = L^\varphi(\Omega)$ is an Orlicz space.
• If $\Phi(t, x) = x^{p(t)}$, then $L^\Phi(\Omega) = L^{p(\cdot)}(\Omega)$ is a variable Lebesgue space.

Properties of $L^{p(\cdot)}(\Omega)$

- $L^{p(\cdot)}(\Omega)$ is separable if and only if $p^+ < \infty$.

- If $p^+ < \infty$, then $\left(L^{p(\cdot)}(\Omega)\right)^* = L^{p^*(\cdot)}(\Omega)$, where $\frac{1}{p(t)} + \frac{1}{p^*(t)} = 1$, μ-a.e.

- $L^{p(\cdot)}(\Omega)$ is reflexive if and only if $1 < p^- \le p^+ < \infty$.

- If $1 < p^- \le p^+ < \infty$, every bounded subset $S \subset L^{p(\cdot)}(\Omega)$ is relatively weakly compact.

- Variable Lebesgue spaces are not rearrengement invariant, nor symmetric.

- ℓ_r is lattice embedded in $L^{p(\cdot)}(\Omega) \Leftrightarrow r \in R_{p(\cdot)}$.

- Which subsets $S \subset L^{p(\cdot)}(\Omega)$ are weakly compact for variable exponents with $p^- = 1$?

Weak compactness in $L_1(\Omega)$

Theorem (Dunford-Pettis)

Let $S \subset L_1(\Omega)$ be bounded. Then, the following are equivalent:

- *S is relatively weakly compact.*
- *S is equi-integrable.*
- *S does* **not** *contain a basic sequence equivalent to the canonical basis of ℓ_1.*

Recall that a bounded subset $S \subset L_1(\Omega)$ is equi-integrable if

$$\lim_{\mu(E)\to 0} \sup_{f\in S} \int_E |f(t)|dt = 0.$$

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Theorem (De La Vallée Poussin)

A subset $S \in L_1(\Omega)$ is equi-integrable $\Leftrightarrow$ S is norm bounded in some Orlicz space $L^{\varphi}(\Omega)$ for a N-function φ, i.e. with

$$\lim_{x\to\infty} \frac{\varphi(x)}{x} = \infty.$$

Equi-integrability in $L^{p(\cdot)}(\Omega)$

$S \subset L^{p(\cdot)}(\Omega)$ is equi-integrable $\Leftrightarrow$

$$\lim_{x\to\infty} \sup_{f\in S} \int_{\{|f|>x\}} |f(t)|^{p(t)} d\mu = 0.$$

Proposition

A bounded subset $S \subset L^{p(\cdot)}(\Omega)$ is equi-integrable $\Leftrightarrow$ S is norm bounded in the Musielak-Orlicz space $L^{\Phi}(\Omega)$, where $\Phi(t,x) = (\varphi(x))^{p(t)}$ for some N-function φ.

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• The inclusion $L^{q(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$ is L-weakly compact $\Leftrightarrow$ there exists an N-function φ with $L^{q(\cdot)}(\Omega) \subset L^{\Phi}(\Omega)$, where Φ is the Musielak-Orlicz function $\Phi(t,x) = (\varphi(x))^{p(t)}$.

• If $ess\inf(p(x) - q(x)) = \delta > 0$, then the inclusion $L^{p(\cdot)}(\Omega) \subset L^{q(\cdot)}(\Omega)$ is L-weakly compact.

Weak compactness in Orlicz spaces

Theorem

A subset $S \subset L^{\varphi}(\Omega)$ for a N-function φ satisfying the Δ_2-condition is relatively weakly compact $\Leftrightarrow$ for every $g \in L^{\varphi^*}(\Omega)$,

$$\lim_{\mu(E)\to 0} \sup_{f\in S} \int_E |f(t)\cdot g(t)|dt = 0.$$

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Theorem (Ândo)

A subset $S \subset L^{\varphi}(\Omega)$ for a N-function φ satisfying the Δ_2-condition is relatively weakly compact $\Leftrightarrow$ it is norm bounded and

$$\lim_{\lambda\to 0} \sup_{f\in S} \frac{1}{\lambda} \int_\Omega \varphi(|\lambda\cdot f(t)|)\, dt = 0.$$

Weak compactness in non reflexive $L^{p(\cdot)}(\Omega)$-spaces

Denote $\Omega_1 := p^{-1}(\{1\})$.

Theorem

Let $L^{p(\cdot)}(\Omega)$ with $p^+ < \infty$ and $\mu(\Omega_1) = 0$. A subset $S \subset L^{p(\cdot)}(\Omega)$ is relatively weakly compact $\Leftrightarrow$ it is norm bounded and

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Proposition

Under the same hypothesis, a sequence (f_n) in $L^{p(\cdot)}(\Omega)$ converges weakly to $f \in L^{p(\cdot)}(\Omega) \Leftrightarrow$

(*i*) $\lim_n \int_A f_n \, dt = \int_A f \, dt$ *for each measurable $A \subset \Omega$, and*

(*ii*) $\lim_{\lambda \to 0} \sup_n \frac{1}{\lambda} \int_\Omega |\lambda(f_n - f)|^{p(t)} \, dt = 0.$

Consequences

Proposition

The inclusion $L^{q(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$ *is weakly compact* $\Leftrightarrow$

$$\lim_{\lambda \to 0} \sup_{\|f\|_{q(\cdot)} \leq 1} \frac{1}{\lambda} \int_{\Omega \setminus \Omega_1} |\lambda f(t)|^{p(t)} d\mu = 0$$

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Theorem

$L^{p(\cdot)}(\Omega)$ *is weakly Banach-Saks* $\Leftrightarrow p^+ < \infty$.

X is weakly Banach-Saks if every sequence (x_n) weakly convergent to $x \in X$ has a Cèsaro convergent subsequence (x_{n_k}), i.e. such that

$$\lim_{k \to \infty} \left\| \frac{x_{n_1} + \ldots + x_{n_k}}{k} - x \right\|_X = 0.$$

The case of σ-finite measure space

For σ-finite separable non-atomic measures (Ω, μ), again:

Theorem

Let $L^{p(\cdot)}(\Omega)$ with $p^+ < \infty$ and $\mu(\Omega_1) = 0$. A subset $S \subset L^{p(\cdot)}(\Omega)$ is relatively weakly compact $\Leftrightarrow$ it is norm bounded and

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Theorem

Let (Ω, μ) be a non-atomic separable σ-finite measure space. Then, $L^{p(\cdot)}(\Omega)$ is isomodular lattice isomorphic to some space $L^{\widehat{p}(\cdot)}(0,1)$, where $R_{p(\cdot)} = R_{\widehat{p}(\cdot)}$.

Some bibliography

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THANKS YOU VERY MUCH