$C(K)$-spaces in the twisted realm

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Workshop on Banach Spaces and Banach Lattices II

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Outline

Spaces of continuous functions

- K compact Hausdorff space.

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Definition

A $\mathcal{C}$-space is a Banach space which is isomorphic to a space $C(K)$ for some compact Hausdorff space K.

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Definition

A twisted sum Z is *trivial* if $i(Y)$ is complemented in Z.

$$0 \longrightarrow Y \xrightarrow{\ i\ } Y \oplus X \xrightarrow{\ q\ } X \longrightarrow 0$$

The principal actor

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☞ If $|\mathcal{A}|$ is uncountable, there is a non-trivial exact sequence

$$0 \longrightarrow c_0 \longrightarrow C_0(K_{\mathcal{A}}) \longrightarrow c_0(\mathcal{A}) \longrightarrow 0$$

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Notation: $\mathrm{ult}(\mathfrak{B}) = \{\text{ultrafilters on } \mathfrak{B}\}$ is the *Stone* space associated to $\mathfrak{B}$.

$$\widehat{B} = \{p \in \mathrm{ult}(\mathfrak{B}) : B \in p\}, \; B \in \mathfrak{B}$$

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Theorem (Plebanek, S-A., 2022)

There are two almost disjoint families $\mathcal{A}$, $\mathcal{B}$ so that

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and Z is not a $\mathcal{C}$-space.

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Theorem (Plebanek, S-A., 2022)

There are two almost disjoint families $\mathcal{A}$, $\mathcal{B}$ so that

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and Z is not a $\mathcal{C}$-space.

We will show: how to construct Z so that it is not a $\mathcal{C}$-space.

Norming and free subsets

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Proposition

X is a $\mathcal{C}$-space if and only if there is a weak* compact set $K \subseteq B_{X^*}$ which is norming and free.

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Key Lemma 1

Let $\mathfrak{B}$ be a subalgebra and $Z \in \mathfrak{B}$. If $M \subseteq M_1(\mathrm{ult}(\mathfrak{B}))$ lies on a free set, for every $\varepsilon > 0$ there is a simple $\mathfrak{B}$-measurable function

$$g = \sum_{i=1}^{n} a_i \cdot 1_{B_i} \in C(\mathrm{ult}(\mathfrak{B}))$$

so that $\big| |\mu(Z)| - \langle \mu, g \rangle \big| < \varepsilon$ for every $\mu \in M$.

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Proof:

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- There is $h \in C(\mathrm{ult}(\mathfrak{B}))$ so that $|\mu(Z)| = \langle \mu, h \rangle$ for $\mu \in M$.
- Pick some simple $\mathfrak{B}$-measurable g so that $\|h - g\| < \varepsilon$. □

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- ...or rather, in the subalgebra $\mathfrak{B}$ that it generates.

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- $\mu_n(S_k) = 0$ for every $n, k \in \mathbb{N}$.
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Definition

A subalgebra $\mathfrak{B}$ *kills* a c-admissible sequence $(\mu_n)_{n=1}^{\infty}$ if there is no c-norming free set in $M_1(\mathrm{ult}(\mathfrak{B}))$ containing $(\mu_n)_{n=1}^{\infty}$.

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☞ *Aim:* given a subalgebra $\mathfrak{B}$ with $|\mathfrak{B}| < \mathfrak{c}$ and a c-admissible sequence $(\mu_n)_{n=1}^\infty$, find $Z \subset \Delta$ so that $\mathfrak{B}[Z]$ **kills** $(\mu_n)_{n=1}^\infty$.

Notation: $\mathfrak{B}[Z]$ = smallest subalgebra containing $\mathfrak{B}$ and Z.

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2. If $M \subseteq M_1(\text{ult}(\mathfrak{B}))$ is infinite, there exists $M' \subseteq M$ so that M' and $M \setminus M'$ are *not* $\mathfrak{B}$-separated.

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$$\mu_n(Z) \approx \begin{cases} c/2 & \text{if } n \in J_3 \\ -c/2 & \text{if } n \in J_2 \setminus J_3 \\ 0 & \text{if } n \in J_1 \setminus J_2 \end{cases}$$

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- If (μ_n) lies inside a free set in $M(\mathrm{ult}(\mathfrak{B}[Z]))$, then

$$|\mu_n(Z)| \approx \langle \mu_n, h \rangle \quad \forall n \in \mathbb{N}$$

for some $h = r \cdot 1_Z + g$, where g is simple $\mathfrak{B}$-measurable.

Dealing with how to deal with $\mathfrak{B}$-separation

- Therefore

$$\langle \mu_n, g\rangle \approx \begin{cases} (1-r)c/2 & \text{if } n \in J_3 \\ (1+r)c/2 & \text{if } n \in J_2 \setminus J_3 \\ 0 & \text{if } n \in J_1 \setminus J_2 \end{cases}$$

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  – If $r \approx 0$ then the sets $\{\mu_n : n \in J_2\}$ and $\{\mu_n : n \in J_1 \setminus J_2\}$ are $\mathfrak{B}$-separated.

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  - If $r \approx 0$ then the sets $\{\mu_n : n \in J_2\}$ and $\{\mu_n : n \in J_1 \setminus J_2\}$ are $\mathfrak{B}$-separated.
  - If $|r| \gg 0$ then the sets $\{\mu_n : n \in J_3\}$ and $\{\mu_n : n \in J_2 \setminus J_3\}$ are $\mathfrak{B}$-separated.

...Is that all?

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Key Lemma 2

Suppose we have

- A subalgebra $\mathfrak{B} \subseteq \mathcal{P}(\Delta)$ containing finite sets and $|\mathfrak{B}| < \mathfrak{c}$.
- A collection of pairs of non-$\mathfrak{B}$-separated subsets of $M_1(\text{ult}(\mathfrak{B}))$ $\{(M_\alpha, M'_\alpha) : \alpha < \xi\}$, $\xi < \mathfrak{c}$.

Then for every almost disjoint family $\mathcal{Z}$ of size $\mathfrak{c}$ there is $Z \in \mathcal{Z}$ so that M_α and M'_α are not $\mathfrak{B}[Z]$-separated for every $\alpha < \kappa$.

Open problems

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1. *An Amir-Cambern theorem?*

   Our counterexample Z is 2-complemented in a $\mathcal{C}$-space.

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   Our counterexample Z is 2-complemented in a $\mathcal{C}$-space. What about 1-complemented?

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   Our counterexample Z is 2-complemented in a $\mathcal{C}$-space. What about 1-complemented?

2. *A stronger counterexample*

   Is there a $C(K)$ so that $C(K) = Z \oplus Z'$ where both Z and Z' are not $\mathcal{C}$-spaces?

Thank you very much
for your attention!