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Smoothness in normed spaces

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Smoothness 101

- Renorming theory = Find an equivalent norm on $\mathcal{X}$ with the strongest possible form of a certain property.
- Let $\mathcal{U} \subseteq \mathcal{X}$ be open. $f\colon \mathcal{U} \to \mathcal{Y}$ is differentiable at $x \in \mathcal{U}$ if there is $f'(x) \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ such that

$$\lim_{h\to 0} \frac{f(x+h) - f(x) - \langle f'(x), h\rangle}{||h||} = 0.$$

- C^k-smoothness, rules of calculus, Implicit Function theorem, ...
- If $p \notin \mathbb{N}$, the ℓ_p norm is $C^{\lfloor p \rfloor}$-smooth, but not $C^{\lfloor p \rfloor + 1}$.
  And ℓ_p has no $C^{\lfloor p \rfloor + 1}$-smooth norm!
- Smooth references:
  - **Deville, Godefroy, Zizler,** ***Smoothness and Renormings in Banach Spaces*****.**
  - **Hájek, Johanis,** ***Smooth analysis in Banach spaces*****.**

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Smoothness and structure

- If a separable Banach space $\mathcal{X}$ has a C^1-smooth norm, then $\mathcal{X}$ is Asplund (*i.e.*, $\mathcal{X}^*$ is separable).
    - No closed, inf-dim subspace of ℓ_1 has a C^1-smooth norm.
- **Meshkov (1978).** If $\mathcal{X}$ and $\mathcal{X}^*$ admit a C^2-smooth norm, then $\mathcal{X}$ is isomorphic to a Hilbert space.
- **Fabian, Whitfield, Zizler (1983).** If $\mathcal{X}$ admits a C^2-smooth norm, either it contains c_0, or it is super-reflexive with type 2.
- **Deville (1989).** If $\mathcal{X}$ has a C^∞-smooth norm, either it contains c_0, or it is super-reflexive, with exact cotype $2k$, and it contains ℓ_{2k}.
    - Can the first case actually happen? The c_0 norm doesn't seem smooth...
- **Pechanec, Whitfield, Zizler (1981) – Fabian, Zizler (1997).** If $\mathcal{X}$ has a LFC norm, then it is c_0-saturated and Asplund.

All these proofs require $\mathcal{X}$ to be complete (variational principles).

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Normed spaces

- Let $\mathcal{X}$ be a normed space with a countable algebraic basis.
  - **Vanderwerff (1992).** $\mathcal{X}$ has a C^1-smooth norm.
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- **Guirao, Montesinos, Zizler**, *Open problems...*, Problem 149: Does the space of finitely supported vectors in $\ell_1(\Gamma)$ have a C^1-smooth norm (when Γ is uncountable)?
- **Dantas, Hájek, R. (JMAA'20).** Given a Banach space $\mathcal{X}$, is there a dense subspace of $\mathcal{X}$ that admits a C^k-smooth norm?
- **Benyamini, Lindenstrauss**, *Geometric Nonlinear Functional Analysis*
  - Does the existence of a smooth norm on some 'large' subset of a Banach space $\mathcal{X}$ imply that $\mathcal{X}$ is Asplund?
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Strong maxima
A.k.a. The secret of smoothness unveiled

- $c_0(\Gamma)$ has a C^∞-smooth norm.
- Normed spaces of countable dim have a C^∞-smooth norm.

Theorem (Dantas, Hájek, R., JMAA'20)

$\ell_\infty^F := span\{1_A \colon A \subseteq \mathbb{N}\}$ has a C^∞-smooth norm.

- Take a sequence $\varepsilon_j \searrow 0$ and define $T \colon \ell_\infty^F \to \ell_\infty$ by

$$(x(j))_{j=1}^\infty \mapsto ((1+\varepsilon_j)\cdot x(j))_{j=1}^\infty .$$

- Look at the picture. $\blacksquare$
- Take $X \subseteq \ell_\infty$ of countable dimension, $X = \mathrm{span}\{e_j\}_{j=1}^\infty$.
- Take $\{v_j\}_{j=1}^\infty \subseteq \ell_\infty^F$ 'very close' to e_js.
- X is isomorphic to $\mathrm{span}\{v_j\}_{j=1}^\infty$ (small perturbation lemma).
  - Well, I'm cheating a bit, $\{e_j\}_{j=1}^\infty$ has to be an M-basis.

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Time flies

Fundamental biorthogonal systems

Theorem (Dantas, Hájek, R., JMAA'20)

Let $\mathcal{X}$ be a Banach space with long unconditional basis and let $\mathcal{Y}$ be the linear span of such basis. Then, $\mathcal{Y}$ has a C^∞-smooth norm.

Theorem (Dantas, Hájek, R., `arXiv:2201.03379`)

Let $\mathcal{X}$ be a Banach space with a fundamental biorthogonal system $\{e_\alpha; \varphi_\alpha\}_{\alpha\in\Gamma}$. Consider $\mathcal{Y} := \operatorname{span}\{e_\alpha\}_{\alpha\in\Gamma}$. Then:

(i) *$\mathcal{Y}$ admits a polyhedral and LFC norm.*

(ii) *$\mathcal{Y}$ admits a C^∞-smooth and LFC norm.*

(iii) *$\mathcal{Y}$ admits a C^1-smooth LUR norm.*

Moreover, such norms are dense.

The norm $\|\cdot\|$ is *LFC* on $\mathcal{X}$ if for each $x \in S_{\mathcal{X}}$ there exist an open nhood $\mathcal{U}$ of x, functionals $\varphi_1, \dots, \varphi_k \in \mathcal{X}^*$, and $G\colon \mathbb{R}^k \to \mathbb{R}$ such that

$$\|y\| = G(\langle\varphi_1, y\rangle, \dots, \langle\varphi_k, y\rangle) \qquad \text{for every } y \in \mathcal{U}.$$

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Let $\mathcal{X}$ be a Banach space with a fundamental biorthogonal system $\{e_\alpha; \varphi_\alpha\}_{\alpha\in\Gamma}$. Consider $\mathcal{Y} := \operatorname{span}\{e_\alpha\}_{\alpha\in\Gamma}$. Then:

(i) *$\mathcal{Y}$ admits a polyhedral and LFC norm.*

(ii) *$\mathcal{Y}$ admits a C^∞-smooth and LFC norm.*

(iii) *$\mathcal{Y}$ admits a C^1-smooth LUR norm.*

Moreover, such norms are dense.

The norm $\|\cdot\|$ is *LFC* on $\mathcal{X}$ if for each $x \in \mathcal{S}_\mathcal{X}$ there exist an open nhood $\mathcal{U}$ of x, functionals $\varphi_1, \dots, \varphi_k \in \mathcal{X}^*$, and $G\colon \mathbb{R}^k \to \mathbb{R}$ such that

$$\|y\| = G(\langle\varphi_1, y\rangle, \dots, \langle\varphi_k, y\rangle) \qquad \text{for every } y \in \mathcal{U}.$$

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  - See Slide 8 for more about this.
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Can we aim for more?

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$\{e_\alpha; \varphi_\alpha\}_{\alpha\in\Gamma} \subseteq \mathcal{X} \times \mathcal{X}^*$ is a *fundamental biorthogonal system* for $\mathcal{X}$ if

- $\langle \varphi_\beta, e_\alpha \rangle = \delta_{\alpha,\beta}$,
- $\operatorname{span}\{e_\alpha\}_{\alpha\in\Gamma}$ is dense in $\mathcal{X}$.

Which Banach spaces admit a fundamental biorthogonal system?

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- $\ell_\infty(\Gamma)$, $\ell_\infty^c(\Lambda)$ when $|\Lambda| \leqslant \mathfrak{c}$,
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Other dense subspaces

Theorem (Dantas, Hájek, R., *in preparation*)

Let $1 \leqslant p < \infty$ and $r \in (0,p)$. The dense subspace $\ell_r(\Gamma)$ of $\ell_p(\Gamma)$ has a C^∞-smooth norm.

- ℓ_p has a dense subspace of dimension continuum with a C^∞-smooth norm.
- If $p > 1$ such subspace is an operator range.
- **Rosenthal (1970).** Every non-separable operator range in $\ell_1(\Gamma)$ contains $\ell_1(\omega_1)$.

- Let X be a separable Banach space. Does it have a dense subspace of dimension continuum with a C^∞-smooth norm?
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