Geometry of spaces of vector-valued Lipschitz functions

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Workshop on Banach spaces and Banach lattices II

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Spaces of Lipschitz functions

Given a **complete** metric space (M, d) with a distinguished point $0 \in M$ and a Banach space X, we consider the space

$$\mathrm{Lip}_0(M, X) := \{f : M \longrightarrow X : f \text{ is Lipschitz}, f(0) = 0\}$$

which is a Banach space when equipped with the norm

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When $X = \mathbb{R}$ we simply write $\mathrm{Lip}_0(M)$.

Geometric properties on $\mathrm{Lip}_0(M)$

A lot of properties have been analysed in $\mathrm{Lip}_0(M)$ as well as in its predual $\mathcal{F}(M)$ (approximation properties, the property of being an L_1 or L_∞ space, octahedrality, Daugavet property etc.).

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It is known that in the previous definition we can replace “finite dimensional subspace” with “finite subset”.

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3. *For every Lipschitz function* f *and* $\varepsilon > 0$*, the set of* ε*-points is infinite, where* $x \in M$ *is an* ε*-point if* $\inf\limits_{\delta>0} \|f_{|B(x,\delta)}\| \geq \|f\| - \varepsilon$.

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4. $\mathrm{Lip}_0(M)$ *has the Daugavet property (if $T : \mathrm{Lip}_0(M) \longrightarrow \mathrm{Lip}_0(M)$ rank-one linear continuous then $\|T + I\| = 1 + \|T\|$).*

Sketch of proof (3)⇒(1)

Pick $f_1, \ldots, f_n \in S_{\mathrm{Lip}_0(M)}$ and $\varepsilon > 0$. Let us find $g \in S_{\mathrm{Lip}_0(M)}$ with $\|f_i + g\| > 2 - \varepsilon$.

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No, McShane extension theorem is false in higher dimensions.

A new structure on $\mathrm{Lip}_0(M, X)$

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This operator $\hat{f}$ satisfies that $\|\hat{f}\| = \|f\|_L$ and that the following diagram is conmutative

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, hook, "\delta"'] & X \\
\mathcal{F}(M) \arrow[ur, "\hat{f}"'] &
\end{tikzcd}$$

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From here it follows that the mapping

$$\begin{array}{ccc} \mathrm{Lip}_0(M,X) & \longrightarrow & L(\mathcal{F}(M),X), \\ f & \longmapsto & \hat{f} \end{array}$$

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$$\mathrm{Lip}_0(M,X^*) = L(\mathcal{F}(M),X^*) = L(X,\mathcal{F}(M)^*) = L(X,\mathrm{Lip}_0(M)).$$

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Proposition

Let X be finitely representable in ℓ_1, Y be octahedral. If X has MAP then $L(X, Y)$ is octahedral.

Idea: $T_1, \ldots, T_n \in S_{L(X,Y)}$ and $\varepsilon > 0$, we need to find $T \in S_{L(X,Y)}$ s.t.

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Take $\Psi : \ell_1^K \longrightarrow S$ a δ-isometry and $i : S \longrightarrow Y$ the inclusion operator. Now define

$$T : X \xrightarrow{P} P(X) \xrightarrow{\Phi} \ell_1^n \xrightarrow{\Psi} S \xrightarrow{i} Y$$

Main result

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X finitely representable in ℓ_1 with MAP. If $\mathrm{Lip}_0(M)$ octahedral $\Rightarrow L(X, \mathrm{Lip}_0(M))$ octahedral.

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Theorem

Let M be a complete metric space and X be a Banach space. If $\mathrm{Lip}_0(M)$ has octahedral norm, then $L(X, \mathrm{Lip}_0(M))$ has octahedral norm.

Key Lemma

Lemma

Let $\mathrm{Lip}_0(M)$ *be octahedral. Given* $f_1, \ldots, f_n \in S_{\mathrm{Lip}_0(M)}$ *we can find a subspace* $Y \subseteq \mathrm{Lip}_0(M)$ *so that*

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1. *$\|f_i + f\| > 2 - \varepsilon$ holds for every i and every $f \in S_Y$.*
2. *Y is isometric to c_0.*

Let $V_i := \{x \in M : \|f_i\|_{B(x,\delta)} > 1 - \varepsilon \ \forall \delta > 0\}$, which is infinite.

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Let $\mathrm{Lip}_0(M)$ *be octahedral. Given* $f_1, \ldots, f_n \in S_{\mathrm{Lip}_0(M)}$ *we can find a subspace* $Y \subseteq \mathrm{Lip}_0(M)$ *so that*

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Sketch for the theorem

Theorem

Let M be a complete metric space and X be a Banach space. If $\mathrm{Lip}_0(M)$ has octahedral norm, then $L(X, \mathrm{Lip}_0(M))$ has octahedral norm.

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$$\|T + T_i\| \geq \|T(x_i) + T_i(x_i)\| \approx \|T(x_i)\| + \|T_i(x_i)\| \approx 2.$$

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Theorem (Lindenstrauss, 1964)

Given a Banach space X, TFAE:

1. *X^* is isometrically isomorphic to $L_1(\mu)$, for certain measure μ.*
2. *All $T : Z \longrightarrow X$ compact admits, for all $\varepsilon > 0$ and all $Z \subseteq Y$, a compact extension $\hat{T} : Y \longrightarrow X$ s.t. $\|\hat{T}\| \leq (1+\varepsilon)\|T\|$.*

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The norm is $\leq 1+\varepsilon$. Moreover, $T(X)$ is clearly included in Y. Finally $\|T(x_i)\| = \|\Phi(x_i)\| \approx 1$, and we are done.

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3. This is a task for another day ...

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