Isomorphisms of Banach spaces of continuous functions

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Preliminaries

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- The *Banach-Mazur distance* $d_{BM}(E_1, E_2)$ of Banach spaces E_1, E_2 is the infimum of $\|T\| \left\|T^{-1}\right\|$ over the set of all isomorphisms $T : E_1 \to E_2$ (and $d_{BM}(E_1, E_2) = \infty$ if E_1 is not isomorphic to E_2).

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- If K is compact space and E is a Banach space, then $\mathcal{C}(K, E)$ stands for the Banach space of continuous E-valued functions on K endowed with the supremum norm, $\mathcal{C}(K) = \mathcal{C}(K, \mathbb{R})$. If E is a Banach lattice, then $\mathcal{C}(K, E)$ also has a natural Banach-lattice structure.

Motivation

Problem (Computing the distance between $\mathcal{C}(K, E)$ spaces)

Let K_1, K_2 be compact spaces and E_1, E_2 be Banach spaces. How to compute (or estimate) the value $d_{BM}(\mathcal{C}(K_1, E_1), \mathcal{C}(K_2, E_2))$?

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- These problems are interesting and difficult even in the case when $E = E_1 = E_2 = \mathbb{R}$.

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- Another possibility is to consider Banach lattices E and Banach-lattice isomorphisms (or isomorphisms preserving positive elements).

Possible values of distances between $\mathcal{C}(K)$ spaces

Theorem (Amir, 1965 and Cambern, 1966)

If there exists an isomorphism $T : \mathcal{C}(K_1) \to \mathcal{C}(K_2)$ *such that* $\|T\| \left\|T^{-1}\right\| < 2$, *then the spaces* K_1 *and* K_2 *are homeomorphic.*

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There exist non-homeomorphic compact spaces K_1, K_2 and an isomorphism $T : \mathcal{C}(K_1) \to \mathcal{C}(K_2)$ with $\|T\| \|T^{-1}\| = 2$.

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Conjecture (probably due to Pełczyński)

Let K_1, K_2 be compact spaces such that $\mathcal{C}(K_1)$ and $\mathcal{C}(K_2)$ are isomorphic. Is the Banach–Mazur distance between $\mathcal{C}(K_1)$ and $\mathcal{C}(K_2)$ an integer?

The case of Banach space-valued functions

- It is known that results similar to the Amir-Cambern theorem hold also for $\mathcal{C}(K, E)$ spaces some classes of Banach spaces E, e.g., uniformly convex spaces. That is, given a uniformly convex space E, there exists $\delta > 1$ such that there are no compact spaces K_1, K_2 with $1 < d_{BM}(\mathcal{C}(K_1, E), \mathcal{C}(K_2, E)) < \delta$ (Cambern, 1985). This property of Banach spaces is called the isomorphic Banach-Stone property (IBSP). The smallest such constant δ is called the Banach-Stone constant of E and is denoted by $BS(E)$.

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- The best known estimate of $BS(E)$ is the Schaffer constant of E, defined as

$$\lambda(E) = \inf\{\max\{\|e_1 + e_2\|, \|e_1 - e_2\|\}, e_1, e_2 \in S_E\},$$

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- For each Banach space E, $1 \leq \lambda(E) \leq 2$.
- $BS(l_p) = \lambda(l_p)$, $2 < p < \infty$.
- It is unknown whether $BS(E) = \lambda(E)$ for each Banach space E.

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- For a Banach lattice E, let $BS^+(E)$ denote the smallest constant $\delta > 1$ such that there are no compact spaces K_1, K_2 and a Banach-lattice isomorphism $T : \mathcal{C}(K_1, E) \to \mathcal{C}(K_2, E)$ with $1 < \|T\| \left\|T^{-1}\right\| < \delta$.

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The case of subspaces

- The Amir-Cambern theorem has been generalized to the context of subspaces of $\mathcal{C}(K, E)$ spaces satifsfying certain geometric condition, by subsequent results of Chu, Cohen (1985), Ludvík, Spurný (2011), Dostál, Spurný (2020), R., Spurný (2020) for scalar-valued functions, Al-Halees, Fleming (2015), R., Spurný (2020) for Banach space-valued functions, and R., Spurný (2021) for Banach lattice-valued functions.

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- In this context, from the fact that two subspaces $\mathcal{H}_1 \subseteq \mathcal{C}(K_1, E_1)$ and $\mathcal{H}_2 \subseteq \mathcal{C}(K_2, E_2)$ have small distance it is deduced that their Choquet boundaries $\operatorname{Ch}_{\mathcal{H}_1} K_1$ and $\operatorname{Ch}_{\mathcal{H}_2} K_2$ are homeomorphic.

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- Each countable compact space K is homeomorphic to $[0, \omega^\alpha m]$ for some countable ordinal α and $m \in \mathbb{N}$ (Mazurkiewicz, Sierpinski, 1920), and hence $\mathcal{C}(K)$ is isometric to $\mathcal{C}([0, \omega^\alpha m])$.

Banach–Mazur distances between spaces of continous functions on countable ordinal intervals

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- $d_{BM}(\mathcal{C}([0, \omega^\alpha m]), c_0) = 2\alpha + 1, \alpha \in \mathbb{N}$ (Candido, 2018).

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- If $\alpha \in \mathbb{N}$, then

$$2\alpha - 1 \leq d_{BM}(\mathcal{C}([0, \omega^\alpha]), \mathcal{C}([0, \omega])) \leq \alpha + \sqrt{(\alpha - 1)(\alpha + 3)}, \text{ and}$$

$$2\alpha + 1 \leq d_{BM}(\mathcal{C}([0, \omega^\alpha m]), \mathcal{C}([0, \omega])) \leq \alpha + 1 + \sqrt{(\alpha(\alpha + 4))}$$

if $m > 1$.(Candido, Galego, 2013).

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  if $m > 1$.(Candido, Galego, 2013).
- If $\alpha, \beta \in \mathbb{N}$, $\alpha \geq \beta$, then $d_{BM}(\mathcal{C}([0, \omega^\alpha m]), \mathcal{C}([0, \omega^\beta n])) \geq \frac{2\alpha - \beta}{\beta}$ and $d_{BM}(\mathcal{C}([0, \omega^\alpha m]), \mathcal{C}([0, \omega^\beta n])) \geq \frac{2\alpha + 2 - \beta}{\beta}$ if $m > n$ (R., 2020).

A more general case of scattered compacta

Theorem (R., Somaglia, in preparation)

Let K_1, K_2 be infinite compact spaces. If $n, k \in \mathbb{N}$, $n > k$, and α is an ordinal such that

$$\omega^\alpha k < ht(K_2) \leq \omega^\alpha (k+1) \text{ and } ht(K_1) > \omega^\alpha n,$$

then

$$d_{BM}(\mathcal{C}(K_1), \mathcal{C}(K_2)) \geq \max\{3, \frac{2n-k}{k}\}.$$

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- For example, the property of being Fréchet-Urysohn is not preserved (Okunev, 2005).

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- For example, the property of being Fréchet-Urysohn is not preserved (Okunev, 2005).
- A long-standing open problem is whether the property of being Corson is preserved (partial results were given by (Argyros, Mercourakis, Negrepontis, 1988), (Marciszewski, Plebanek, 2010) and (Plebanek, 2015)).

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- The assumption that c_0 does not embed in E_1, E_2 cannot be plainly removed: For each countable compact space K,

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Isomorphic classification of $\mathcal{C}(K, E)$ spaces

Problem (Isomorphic classification of $\mathcal{C}(K, E)$ spaces)

Let K_1, K_2 be compact spaces and E be a Banach space. Describe when are the Banach spaces $\mathcal{C}(K_1, E), \mathcal{C}(K_2, E)$ isomorphic and when they are not.

- If E is a Banach space not containing an isomorphic copy of c_0 and there exists an isomorphism $T : \mathcal{C}(K_1, E) \to \mathcal{C}(K_2, E)$, then either K_1 and K_2 are both finite or they have the same cardinality (Galego, Rincón-Villamizar, 2015). Moreover, K_1 is scattered iff K_2 is scattered (Candido, 2016).
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- Concerning the property of being scattered, a more detailed information can be given.

The Szlenk index

- X Banach space, $K \subseteq X^*$ w* closed, $\varepsilon > 0$, then

$$s_\varepsilon(K) = \{x^* \in K, \text{ for each w}^* - \text{open neighbourhood } U \text{ of } x^* : \operatorname{diam}(U \cap K) > \varepsilon\}.$$

- $s_\varepsilon^0(K) = K$,
  $s_\varepsilon^{\alpha+1}(K) = s_\varepsilon(s_\varepsilon^\alpha(K))$,
  and $s_\varepsilon^\alpha(K) = \bigcap_{\beta<\alpha} s_\varepsilon^\beta(K)$, if α is limit.
- $Sz_\varepsilon(K) = \min\{\alpha : s_\varepsilon^\alpha(K) = \emptyset\}$, if this minimum exists, and $Sz_\varepsilon(K) = \infty$ otherwise.
- The Szlenk index of K is $Sz(K) = \sup_{\varepsilon>0} Sz_\varepsilon(K)$.
- The Szlenk index of a Banach space X is $Sz(X) = Sz(B_{X^*})$.

The Szlenk index and isomorphisms of Banach spaces

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- Thus if $\mathcal{C}(K_1)$ is isomorphic to $\mathcal{C}(K_2)$, then $Sz(\mathcal{C}(K_1)) = Sz(\mathcal{C}(K_2))$, and if E is a Banach space and $\mathcal{C}(K_1, E)$ is isomorphic to $\mathcal{C}(K_2, E)$, then $Sz(\mathcal{C}(K_1, E)) = Sz(\mathcal{C}(K_2, E))$.

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- $Sz(\mathcal{C}(K))$ (and for certain Banach spaces E also $Sz(\mathcal{C}(K, E))$) can be computed using $ht(K)$.

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- This was known in the case when K is an ordinal interval by subsequent results of Samuel (1983), Hájek and Lancien (2007) and Brooker (2013).

- Thus, if $\mathcal{C}(K_1)$ is isomorphic to $\mathcal{C}(K_2)$, then $Sz(\mathcal{C}(K_1)) = Sz(\mathcal{C}(K_2))$, that is, $\Gamma(ht(K_1)) = \Gamma(ht(K_2))$.

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- In the case of nonmetrizable compact spaces, such a classification is no longer true even for the case of ordinal intervals. For example, $\Gamma(ht(\omega_1)) = \Gamma(ht(\omega_1 2))$, but the space $\mathcal{C}(\omega_1)$ is not isomorphic to $\mathcal{C}(\omega_1 2)$ (Semadeni, 1960).

The Szlenk index and isomorphisms of C(K) spaces

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- What about $\mathcal{C}(K, E)$ spaces?

- $Sz(\mathcal{C}(K,E)) = \max\{Sz(\mathcal{C}(K)), Sz(E)\}$. (Causey, 2017)

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- $Sz(\mathcal{C}(K, E)) = \max\{Sz(\mathcal{C}(K)), Sz(E)\}$. (Causey, 2017)
- Thus if E_1, E_2 are Banach spaces with $Sz(E_1) \leq Sz(\mathcal{C}(K_1))$, $Sz(E_2) \leq Sz(\mathcal{C}(K_2))$ and $\Gamma(ht(K_1)) \neq \Gamma(ht(K_2))$, then the spaces $\mathcal{C}(K_1, E_1)$, $\mathcal{C}(K_2, E_2)$ are not isomorphic, because otherwise we would have

$$\begin{aligned}\Gamma(ht(K_1)) = & Sz(\mathcal{C}(K_1)) = Sz(\mathcal{C}(K_1, E_1)) = \\ & = Sz(\mathcal{C}(K_2, E_2)) = Sz(\mathcal{C}(K_2)) = \Gamma(ht(K_1)).\end{aligned}$$

The Szlenk index and isomorphisms of C(K, E) spaces

- If E is a Banach space not containing a copy of c_0 and $\mathcal{C}(K, E)$ is isomorphic to $\mathcal{C}_0(\Lambda, E)$, where Λ is a discrete space, then $ht(K)$ is finite. (Candido, 2018)

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- The assumption that c_0 does not embed in E_1, E_2 cannot be plainly removed, which follows for example from the fact that

$$\mathcal{C}(\omega, \mathcal{C}(\omega^\omega)) \simeq \mathcal{C}(\omega \times \omega^\omega) \simeq \mathcal{C}(\omega^\omega, \mathcal{C}(\omega)),$$

but $\Gamma(ht(\omega)) = \Gamma(2) = \omega \neq \omega^2 = \Gamma(\omega + 1) = \Gamma(ht(\omega^\omega))$.

An open problem

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Let K_1, K_2 be compact Hausdorff spaces and E be a Banach space not containing an isomorphic copy of c_0. Is it true that $\mathcal{C}(K_1, E)$ is isomorphic to $\mathcal{C}(K_2, E)$ if and only if $\mathcal{C}(K_1)$ is isomorphic to $\mathcal{C}(K_2)$?

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- This together with the classical isomorphic classification due to Bessaga, Pelczynski, and Milutin yields that the answer is positive in the case when K_1, K_2 are compact metric spaces.

A question

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Is there a natural class of nonmetrizable compact spaces K such that for each compact $L \subseteq K$, $\mathcal{C}(L)$ is isomorphic to a subspace of $\mathcal{C}(K)$ (e.g. scattered Eberlein spaces)?
(For the class of metrizable compact spaces it is true by a theorem of Borsuk, 1933)

Acknowledgement

Thank you.