# Banach spaces consisting of Strongly NA Lipschitz mappings

**Óscar Roldán**

A joint work with
**Vladimir Kadets**

Workshop on Banach spaces and Banach lattices II
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## About the talk:

This talk is based in a recent joint work with Vladimir Kadets, initiated during a stay at V. N. Karazin Kharkiv National University, in Kharkiv, Ukraine:

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This talk is dedicated with love and support to all my Ukrainian friends and colleagues, and especially, to the kind and lovely persons who welcomed me so warmly in Kharkiv and who are now living a terrible nightmare.

#StopThisWar
#StopPutin
#StandWithUkraine

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# Notation and motivation

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- $\text{NA}(X, Y)$ is **NOT ALWAYS** dense in $\mathcal{L}(X, Y)$ (**Lindenstrauss, 1963**).

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**OBJECTIVE:** Study similar questions adapted to Lipschitz mappings.

# Lipschitz mappings

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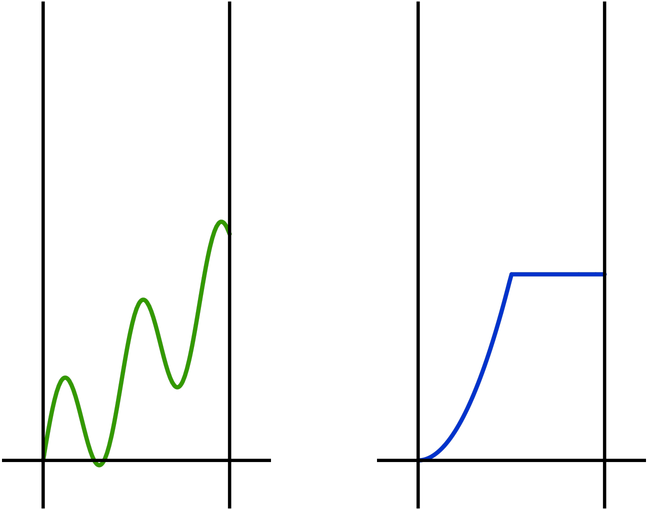
# Lipschitz mappings

Figure: Non vertical(∗)

(∗): Picture from: https://unsplash.com/photos/yON4XwM7OyA

# Lipschitz mappings

These **ARE** Lipschitz mappings:



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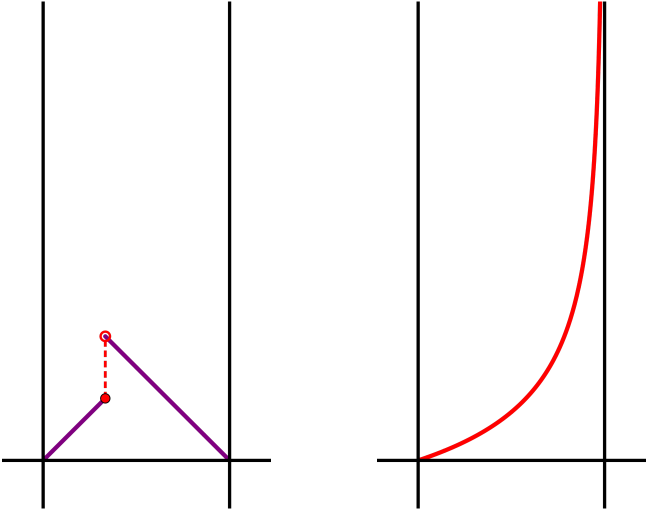

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There is a natural way to define norm-attainment. $f \in \mathrm{Lip}_0(M)$ **attains its norm strongly** if there exist $x \neq y \in M$ such that

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- This norm-attainment is called **strong** because there are other weaker, much less restrictive, norm-attainments considered that are also natural.

## The strong norm-attainment is restrictive

**Lemma 2.2 (Kadets-Martín-Soloviova, 2016)**

If $f \in \mathrm{Lip}_0(M)$ attains its norm on a pair $(x, y) \in M \times M$, $x \neq y$, and if $z \in M \setminus \{x, y\}$ is such an element that $\rho(x, y) = \rho(x, z) + \rho(z, y)$, then $f$ strongly attains its norm on the pairs $(x, z)$ and $(z, y)$, and

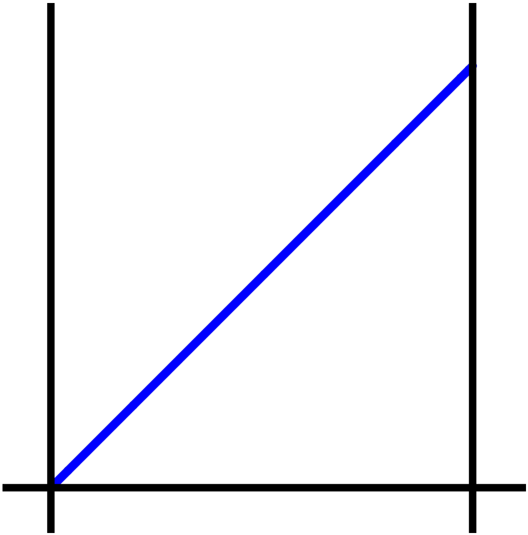
$$f(z) = \frac{\rho(z, y) f(x) + \rho(x, z) f(y)}{\rho(x, y)}.$$

In particular, if $M$ is a convex subset of a Banach space, then $f$ is affine on the closed segment $[x, y]$, that is, $f(\theta x + (1 - \theta) y) = \theta f(x) + (1 - \theta) f(y)$ for every $\theta \in [0, 1]$.

In other words: if $f$ attains its norm strongly at $(x, y)$, it must also attain its norm strongly at any pair of distinct points in between them!

Actually, if the maximum possible slope of $f$ is attained at the pair $(x, y)$, $f$ must be affine in between those points!

# The strong norm-attainment is restrictive



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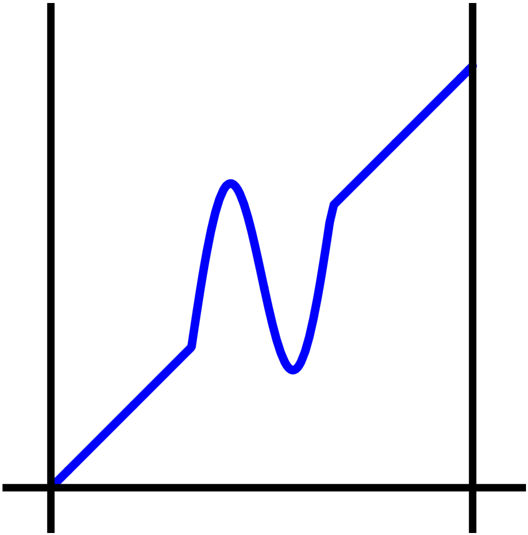

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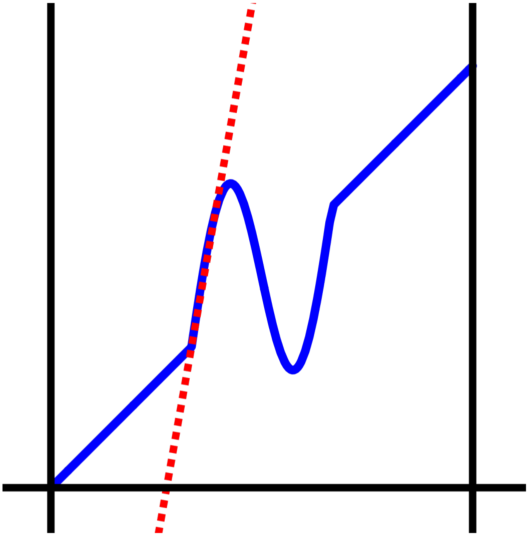

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In particular, we are interested in studying the existence and sizes of (closed) **linear spaces** of Lipschitz norms which attain their norms strongly

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Figure: Space (unrelated to our topic).

Picture from: https://pixabay.com/es/photos/estrellas-cielo-noche-1845140/

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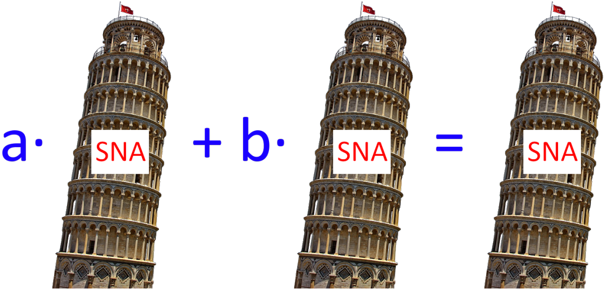


Figure: This is more like what we mean.

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Pisa tower picture from: https://www.kindpng.com/picc/m/109-1098941_leaning-tower-of-pisa-building-places-of-interest.png

# Index

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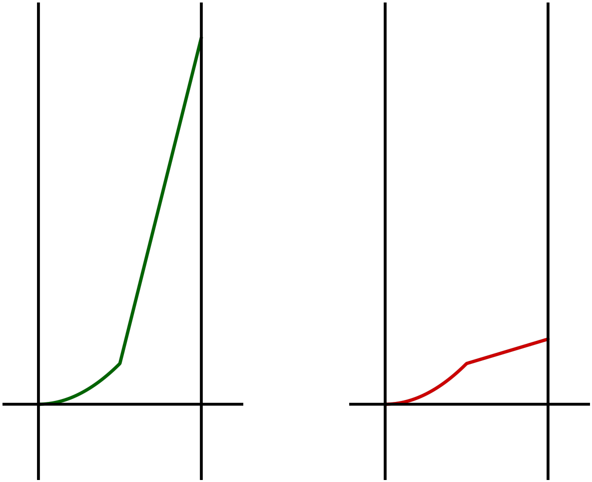

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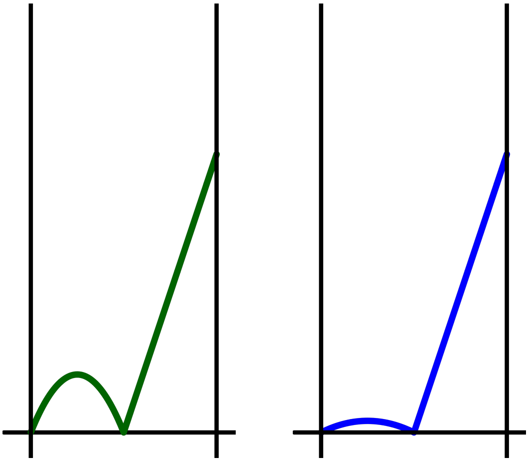

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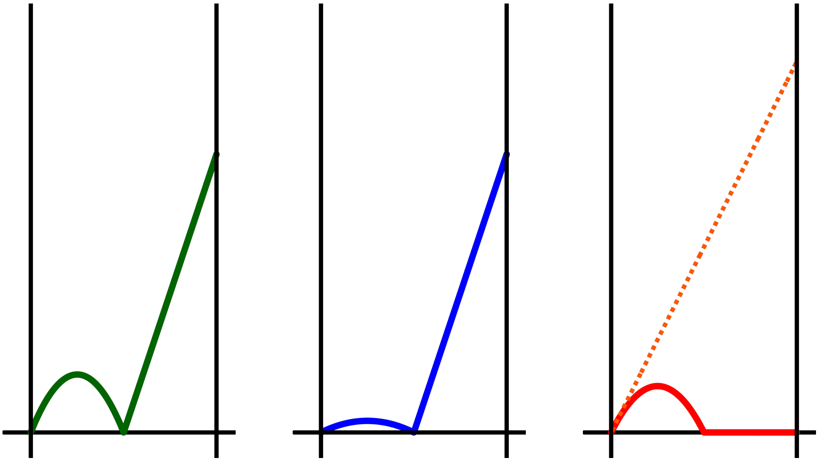
**First** and **second** mappings are SNA, but **their difference** isn't.



## Remarks

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For example, with $\rho$ given by $\|\cdot\|_\infty$, the complete space

$$\begin{array}{cccccc}
(1+1/1, & 0, & 0, & 0, & 0, & \ldots) \\
(1+1/2, & 1+1/2, & 0, & 0, & 0, & \ldots) \\
(1+1/3, & 1+1/3, & 1+1/3, & 0, & 0, & \ldots) \\
(1+1/4, & 1+1/4, & 1+1/4, & 1+1/4, & 0, & \ldots) \\
\ldots & & & & &
\end{array}$$

is uniformly discrete, but **NONE** of the points has a closest point.

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The **Lipschitz-free** space associated to $M$ is

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1) $\mathcal{F}(M)^*$ is isometrically isomorphic to $\operatorname{Lip}_0(M)$. This allows us to "linearize" Lipschitz mappings.

2) Link to classical functionals theory: $\operatorname{SNA}(M) \subset \operatorname{NA}(\mathcal{F}(M), \mathbb{R})$ in a natural way.

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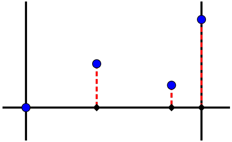

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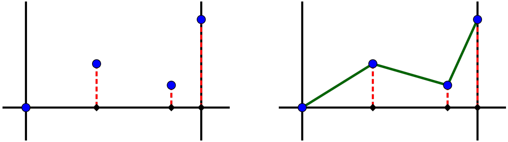

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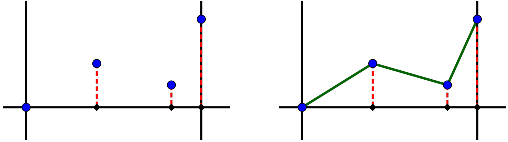

**McShane's extension theorem:** If $f \in \text{Lip}_0(M)$ and $M$ is a metric subspace of $M'$, there is a mapping $F \in \text{Lip}_0(M')$ such that $F = f$ on $M$ and $\|F\| = \|f\|$.

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**Question:** is it possible to get $\ell_1^n$ spaces, $n > 1$, in $\mathrm{SNA}(M)$, where $M$ is finite?

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**Remark:** Contrast with classical norm-attainment theory!

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**Lemma 2 (Theorem 14.5, Khan-Mim-Ostrovskii, 2020)**

If $|M| = 2n$, then $\mathcal{F}(M)$ contains a 1-complemented subspace isometric to $\ell_1^n$.

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If $n \in \mathbb{N}$, then $\ell_1^n$ is isometric to a subspace of $\ell_\infty^{2^{n-1}}$.

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Compare this to the classical theory again, where there are Banach spaces $X$ such that $\mathrm{NA}(X)$ does not contain 2-dimensional linear subspaces.

# Index

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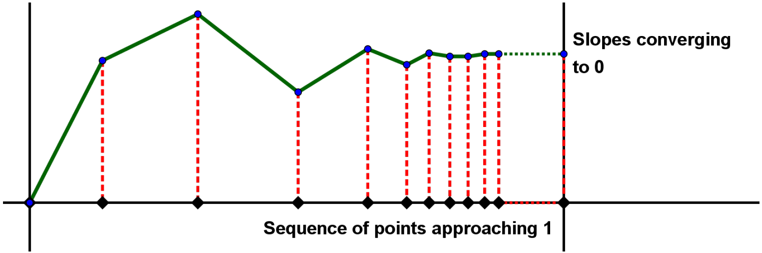

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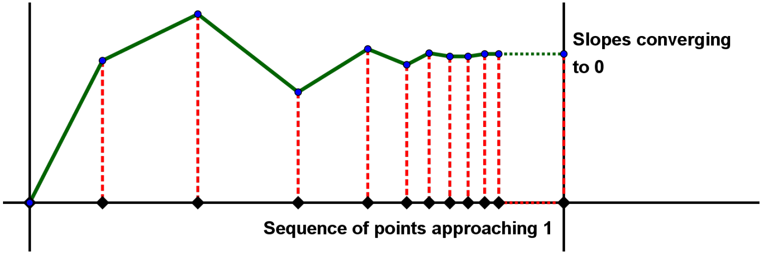


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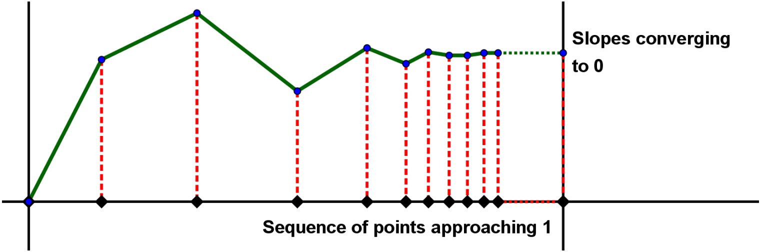


With retractions and McShane, this construction can be extended:

**Theorem**

If $M$ is a metric space containing $[0,1]$ isometrically, then $\mathrm{SNA}(M)$ contains $c_0$ isometrically.

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However, surprisingly enough, **ANY** Banach space can be formed in some SNA($M$) if you choose the right $M$.

**Theorem**

If $Y$ is a Banach space, then it embeds isometrically in SNA($B_{Y^*}$).

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So if $Y^*$ is not separable, $M$ **CANNOT** be separable either. Hence, some $\mathrm{SNA}(M)$ with $M$ infinite don't contain $\ell_1$, despite containing all the $\ell_1^n$ spaces.

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**Theorem**

Let $M$ be a $\sigma$-precompact pointed metric space, then all Banach subspaces in SNA($M$) are separable and isomorphic to polyhedral spaces.

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So all the subspaces of SNA([0, 1]) are separable and isomorphically polyhedral, and the same happens on all the $\mathbb{R}^n$ spaces, for instance.

# Index

# Sample of references

For the interested reader.

Figure: Interested reader

Manipulated pictures. Originals: Pisa tower, Book, Sunglasses.

# NON-EXHAUSTIVE sample of references


- G. Godefroy, On norm attaining Lipschitz maps between Banach spaces, *Pure Appl. Funct. Anal.* **1** (2016), no. 1, 39–46.
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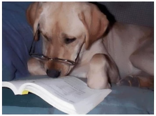

# Thank you for your attention!

**Vladimir Kadets**
v.kateds@karazin.ua
School of Mathematics and Computer Sciences
V.N. Karazin Kharkiv National University

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**Óscar Roldán**
Oscar.Roldan@uv.es
Departamento de Análisis Matemático
Universitat de València