Free Banach lattices, p. 2: lattice homomorphisms, and non-linear theory of Banach spaces

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ICMAT, May 2022

Free Banach lattices on Banach spaces

Definition

Suppose E is a Banach space, and $1 \leqslant p \leqslant \infty$. A free p-convex Banach lattice on E ($\mathrm{FBL}^{(p)}[E]$) is the unique Banach lattice X so that:

- X is p-convex with constant 1.
- There exists an isometry $\phi_E : E \to X$ so that $\phi_E(E)$ generates X as a Banach lattice.
- If Z is a Banach lattice, p-convex with constant 1, then any $T \in B(E, Z)$ extends to a lattice homomorphism $\widehat{T} : \mathrm{FBL}^{(p)}[E] \to Z$ so that $\|\widehat{T}\| = \|T\|$, and $\widehat{T} \circ \phi_E = T$.

$$
\begin{tikzcd}
\mathrm{FBL}^{(p)}[E] \arrow[dr, dashed, "\widehat{T}"] & \\
E \arrow[u, hook, "\phi_E"] \arrow[r, "T"'] & Z
\end{tikzcd}
$$

If Z is p-convex with constant C, we can construct an extension $\widehat{T}$ with $\|\widehat{T}\| \leqslant C\|T\|$ (renorming makes Z p-convex with constant 1).

Notation: $\mathrm{FBL}[E] := \mathrm{FBL}^{(1)}[E]$
(for $p = 1$, Z is an arbitrary lattice).

Functional representation of $\mathrm{FBL}^{(p)}[E]$

Denote by $\mathbf{H}[E^*]$ the space of positively homogeneous functions on $\mathbf{B}_{E^*}$ ($f(tx^*) = tf(x^*)\ \forall t \geqslant 0$). $\mathbf{H}_p[E^*]$ consists of those $f \in \mathbf{H}[E^*]$ for which $\exists C > 0$ s.t. $\sum_{i=1}^N |f(x_i^*)|^p \leqslant C^p$ when $\|(x_i^*)_{i=1}^N\|_{p,\mathrm{weak}} \leqslant 1$. $\|f\|_p := \inf C$.

Here $\|(x_i^*)_{i=1}^N\|_{p,\mathrm{weak}} = \sup_{x\in\mathbf{B}(E)} \big(\sum_i |\langle x_i^*, x\rangle|^p\big)^{1/p}$.

$\|(x_i^*)_{i=1}^N\|_{\infty,\mathrm{weak}} = \max_i \|x_i^*\|$. $\|(x_i^*)_{i=1}^N\|_{1,\mathrm{weak}} = \max \|\sum_i \pm x_i^*\|$.

Let $\phi_E : E \to \mathbf{H}[E^*] : x \mapsto \delta_x$; $\delta_x(x^*) = \langle x^*, x\rangle$.

Theorem

$\phi_E : E \to \mathbf{H}_p[E^*]$ is an isometry. $\mathrm{FBL}^{(p)}[E]$ is the Banach lattice generated by $\phi_E(E)$ in $\mathbf{H}_p[E^*]$.

Note: All $f \in \mathrm{FBL}^{(p)}[E]$ are weak* continuous functions on $\mathbf{B}_{E^*}$.

Theorem

$\mathrm{FBL}^{(\infty)}[E]$ coincides with the space $\mathbf{CH}[E^*]$ of weak* continuous positively homogeneous functions on $\mathbf{B}_{E^*}$, with sup norm $\|\cdot\|_\infty$.

When do we have $\mathrm{FBL}^{(p)}[E] = \mathrm{FBL}^{(q)}[E]$?

Theorem

Suppose E is a Banach space, $p > q \geqslant 1$. TFAE:

1. $\mathrm{FBL}^{(q)}[E]$ *is p-convex.*
2. $\mathrm{FBL}^{(p)}[E]$ *is (canonically) lattice isomorphic to* $\mathrm{FBL}^{(q)}[E]$.
3. *Any $T : E \to L_q(\mu)$ has a factorization $E \xrightarrow{S} L_p(\mu) \xrightarrow{M_g} L_q(\mu)$; M_g is the multiplication by $g \in L_r$, $1/q = 1/p + 1/r$.*

Corollary

1. *If $2 > s > p > q \geqslant 1$, and E has type s, then $\mathrm{FBL}^{(p)}[E]$ is (canonically) lattice isomorphic to $\mathrm{FBL}^{(q)}[E]$.*
2. *If $1 \leqslant p \leqslant 2 \leqslant r$, then $\mathrm{FBL}^{(p)}[L_r(\mu)]$ is (canonically) lattice isomorphic to $\mathrm{FBL}^{(2)}[L_r(\mu)]$.*

Convexity

Theorem

If $\dim E = \infty$, *and* $\mathrm{FBL}^{(p)}[E]$ *is* q*-convex, then* $q \leqslant \max\{2, p\}$.

If $\dim E = n < \infty$, then $\mathrm{FBL}^{(p)}[E] \simeq C(\mathbb{S}^{n-1})$ is q-convex for any q.

If $\dim E = \infty$, and $p \geqslant 2$, then $\mathrm{FBL}^{(p)}[E]$ is, by definition, p-convex (with constant 1). By the above theorem, this cannot be improved: $\mathrm{FBL}^{(p)}[E]$ cannot be q-convex for $q > p$.

Corollary

If $\dim E = \infty$, *and* $\mathrm{FBL}^{(p)}[E]$ *is lattice isomorphic to* $\mathrm{FBL}^{(q)}[E]$, *then either* $p = q$, *or* $p, q \in [1, 2]$.

Structure of lattice homomorphisms between free lattices

Theorem (N. Laustsen, P. Tradacete)

Suppose $T : \mathrm{FBL}^{(p)}[E] \to \mathrm{FBL}^{(p)}[F]$ *is a lattice homomorphism. Then* $\exists$ *a positively homogeneous* $\Phi_T : F^* \to E^*$, *weak** *to weak** *continuous on bounded sets, s.t.* $[Tf](y^*) = f(\Phi_T y^*)$ *for* $y^* \in F^*$, $f \in \mathrm{FBL}^{(p)}[F]$; $\|\Phi_T y^*\| \leqslant \|T\| \|y^*\|$. *Moreover:*

- *Any positively homogeneous* $\Phi : F^* \to E^*$, *weak** *to weak** *continuous on bounded sets, determines* $T : \mathrm{FBL}^{(\infty)}[E] \to \mathrm{FBL}^{(\infty)}[F]$ *s.t.* $\Phi_T = \Phi$, $\|T\| = \sup_{\|y^*\| \leqslant 1} \|\Phi_T y^*\|$.
- *If* $q \geqslant p$, *then, for any* $y_1^*, \ldots, y_N^* \in F^*$, *we have* $\|(\Phi_T y_i^*)\|_{q,\mathrm{weak}} \leqslant \|T\| \|(y_i^*)\|_{q,\mathrm{weak}}$.
- *If* T *is a lattice isomorphism, then* $\Phi_{T^{-1}} = \Phi_T^{-1}$.

Recall: $\|(y_i^*)_{i=1}^N\|_{q,\mathrm{weak}} = \sup_{y \in \mathbf{B}(F)} \left(\sum_i |\langle y_i^*, y \rangle|^q \right)^{1/q}$.

Structure of lattice homomorphisms: sketch of a proof

Theorem (N. Laustsen, P. Tradacete)

Suppose $T : \mathrm{FBL}^{(p)}[E] \to \mathrm{FBL}^{(p)}[F]$ *is a lattice homomorphism. Then* $\exists$ *a positively homogeneous* $\Phi_T : F^* \to E^*$, *weak** *to weak** *continuous on bounded sets, s.t.* $[Tf](y^*) = f(\Phi_T y^*)$ *for* $y^* \in F^*$, $f \in \mathrm{FBL}^{(p)}[F]$*;* $\|\Phi_T y^*\| \leqslant \|T\|\|y^*\|$. *If* $q \geqslant p$, *then, for any* $y_1^*, \ldots, y_N^* \in F^*$, *we have* $\|(\Phi_T y_i^*)\|_{q,\mathrm{weak}} \leqslant \|T\|\|(y_i^*)\|_{q,\mathrm{weak}}$.

(1) Construction of Φ_T. $\{\widehat{y^*} : y^* \in F^*\}$ = set of atoms ($\widehat{y^*}$ is the evaluation at y^*). T^* is interval preserving, hence $T^*\widehat{y^*}$ is also an atom, say $\widehat{x^*}$. Define $\Phi_T y^* = x^*$.

(2) $(q, \mathbf{weak})$ norms. $\|(y_i^*)_{i=1}^N\|_{q,\mathrm{weak}} = \|(\widehat{y_i^*})_{i=1}^N\|_{q,\mathrm{weak}}$.
$\|(\widehat{T^* y_i^*})_{i=1}^N\|_{q,\mathrm{weak}} \leqslant \|T\|\|(\widehat{y_i^*})_{i=1}^N\|_{q,\mathrm{weak}}$. ■

Remark. (1) $\forall y^* \|\Phi_T y^*\| \leqslant \|T\|\|y^*\|$. (2) If T is a lattice quotient, then T^* is bounded below, hence $\exists c > 0$ s.t. $\forall y^* \|\Phi_T y^*\| \geqslant c\|y^*\|$.

Non-linear maps between Banach spaces

Many classes of non-linear maps between Banach spaces have been considered.

Continuous maps are too common to be useful: topological homeomorphisms are not an efficient tool for classifying Banach spaces.

Theorem (Kadec; Torunczyk)

If Banach spaces E and F have the same density character, then they are homeomorphic.

Need more restrictive classes of maps: Lipschitz, uniformly continuous, coarse, ...

Lipschitz equivalence of Banach spaces

Theorem

Suppose E is separable and reflexive, and Lipschitz isomorphic to F. If F has the RNP, then E linearly isomorphic to a complemented subspace of F. Under some extra conditions E and F are linearly isomorphic.

Theorem

If E is Lipschitz isomorphic to L_p or ℓ_p $(1 < p < \infty)$, then the two are linearly isomorphic.

Theorem

If a separable E is Lipschitz isomorphic to a subset of F, then $E \hookrightarrow F^{**}$.

Theorem

There exist an uncountable Γ and a Banach space E so that $c_0(\Gamma)$ is Lipschitz isomorphic to, but does not embed into, E.

Rigidity of isometries

Theorem (Mazur-Ulam)

Any surjective isometry between Banach spaces is affine. Consequently, if Banach spaces E and F are isometric, then they are also linearly isometric.

Non-surjective isometries need not be affine. But:

Theorem (T. Figiel)

Suppose X, Y are separable Banach spaces, $T : X \to Y$ is an isometry s.t. $T0 = 0$, and $\overline{\mathrm{span}}[TX] = Y$. Then $\exists$ linear contraction $S : Y \to X$ s.t. $ST = I_X$.

Theorem

Suppose X, Y are separable Banach spaces, and $T : X \to Y$ is an isometry. Then Y contains an isometric copy of X.

When do we have $\mathrm{FBL}^{(p)}[E] = \mathrm{FBL}^{(p)}[F]$?

$$\begin{tikzcd} \mathrm{FBL}^{(p)}[F] \arrow[r, dashed, "\overline{T}"] & \mathrm{FBL}^{(p)}[E] \\ F \arrow[u, hook, "\phi_F"] \arrow[r, "T"] & E \arrow[u, hook, "\phi_E"'] \end{tikzcd}$$

If T is isomorphic (isometric), then so is $\overline{T}$. Can $\mathrm{FBL}^{(p)}[E]$ and $\mathrm{FBL}^{(p)}[F]$ be "similar," while E and F are "different?"

The answer for $p = \infty$ differs from what we get for $p \in [1, \infty)$.

The isometric and isomorphic settings are different.

Theorem (N. Laustsen, P. Tradacete)

Suppose $T : \mathrm{FBL}^{(p)}[E] \to \mathrm{FBL}^{(p)}[F]$ *is a lattice homomorphism. Then* $\exists$ *a positively homogeneous* $\Phi_T : F^* \to E^*$, *weak** *to weak** *continuous on bounded sets, s.t.* $[Tf](y^*) = f(\Phi_T y^*)$ *for* $y^* \in F^*$, $f \in \mathrm{FBL}^{(p)}[F]$; $\|\Phi_T y^*\| \leqslant \|T\| \|y^*\|$. *If* $q \geqslant p$, *then, for any* $y_1^*, \ldots, y_N^* \in F^*$, *we have* $\|(\Phi_T y_i^*)\|_{q,\mathrm{weak}} \leqslant \|T\| \|(y_i^*)\|_{q,\mathrm{weak}}$.

Case of $p = \infty$: structure of lattice homomorphisms

Theorem

$\mathrm{FBL}^{(\infty)}[E]$ *coincides with the space* $\mathbf{CH}[E^*]$ *of weak** *continuous positively homogeneous functions on* $\mathbf{B}_{E^*}$*, with* sup *norm* $\|\cdot\|_\infty$.

Corollary (Lattice homomorphism vs compositions)

(1) *If* $\Phi : F^* \to E^*$ *is a positively homogeneous and weak** *continuous on bounded sets, then* $\exists$ *a lattice homomorphism*
$T : \mathrm{FBL}^{(\infty)}[E] \to \mathrm{FBL}^{(\infty)}[F]$ *s.t.* $\Phi = \Phi_T$ *– that is,* $Tf = f \circ \Phi$.
(2) *If, moreover,* Φ *is surjective, and*
$\forall y^* \in F^* \ C_1\|y^*\| \geqslant \|\Phi y^*\| \geqslant \|y^*\|/C_2$,
then T *is a lattice isomorphism, with* $\|T\| \leqslant C_1$ *and* $\|T^{-1}\| \leqslant C_2$.

Comparing $\mathrm{FBL}^{(\infty)}[E]$ with $\mathrm{FBL}^{(\infty)}[F]$, and FDDs

A Banach space E has Finite Dimensional Decomposition (FDD) if $\exists$ a sequence of finite rank projections $P_n \in B(E)$ s.t. (i) $P_i P_j = P_{\min\{i,j\}}$, and (ii) $\forall x \in E$, $\lim_N P_N x = x$ (then $\sup_N \|P_N\| < \infty$).

Example: a Schauder basis (e_i) gives rise to an FDD ($P_i =$ canonical basis projection).

An FDD (P_i) is monotone if $\sup_N \|P_N\| = 1$.
Example: canonical basis in ℓ_p.

Theorem

If E is a Banach space with a monotone FDD, then $\mathrm{FBL}^{(\infty)}[E]$ is lattice isometric to $\mathrm{FBL}^{(\infty)}[c_0]$.

$\mathrm{FBL}^{(\infty)}[E] = \mathrm{FBL}^{(\infty)}[c_0]$, construction in particular case

Suppose E has strictly monotone normalized basis (e_i) – that is, $\| \sum_{i=1}^{N} \alpha_i e_i \| \leqslant \| \sum_{i=1}^{N+1} \alpha_i e_i \|$, with equality iff $\alpha_{N+1} = 0$.

Goal: construct positively homogeneous norm-preserving surjection $\Phi : E^* \to \ell_1 = c_0^*$ s.t. Φ and Φ^{-1} are weak* continuous on bounded sets.

Let $e_i^* \in E^*$ be the biorthogonal functionals, and let f_i^* be the canonical basis of ℓ_1. Let P_n be the n-th basis projection in E, then $P_n^* : E^* \to X_n = \mathrm{span}[e_i^* : 1 \leqslant i \leqslant n]$ canonically.

Construct maps $\Phi_n : X_n \to Y_n = \mathrm{span}[f_i^* : 1 \leqslant i \leqslant n] \subset \ell_1$.
Let $\Phi_1 \alpha_1 e_1^* := \alpha_1 f_1^*$.
Define Φ_n recursively: for $e^* = x^* + \alpha e_n^*$ ($x^* \in X_{n-1}$), let $\Phi_n e^* := \Phi_{n-1} x^* + t f_n^*$, where $t = \mathrm{sign}\alpha \cdot \big(\|e^*\| - \|x^*\| \big)$.

The $\Phi = \mathrm{weak}^* - \lim \Phi_n P_n^*$ has the desired properties. ∎

Question. Can we avoid the FDD assumption?

Properties of $\mathrm{FBL}^{(\infty)}[c_0]$

Theorem

If E is a Banach space with a monotone FDD, then $\mathrm{FBL}^{(\infty)}[E]$ is lattice isometric to $\mathrm{FBL}^{(\infty)}[c_0] =: \mathfrak{U}$ (for "universal").

Corollary

(1) *If E has FDD, then $\mathrm{FBL}^{(\infty)}[E]$ is lattice isomorphic to $\mathfrak{U}$.*
(2) *If E is a separable Banach space, then $\exists$ linear isometry $J : \mathrm{FBL}^{(\infty)}[E] \to \mathfrak{U}$ and a contractive lattice homomorphism $P : \mathfrak{U} \to \mathrm{FBL}^{(\infty)}[E]$ so that $PJ = id_{\mathrm{FBL}^{(\infty)}[E]}$.*
(3) *If F has the BAP, J can be a lattice isomorphism (not isometric).*

Proof of (2). $X = \mathrm{FBL}^{(\infty)}[E]$ is an $\mathcal{L}_{\infty,1+}$ space, hence it has a monotone basis, hence $\mathrm{FBL}^{(\infty)}[X]$ is lattice isometric to $\mathfrak{U}$.
$id : X \to X$ extends to a lattice homomorphism $\widehat{id} : \mathrm{FBL}^{(\infty)}[X] \to X$.
$\widehat{id} \circ \phi_X$ is the desired factorization via $\mathrm{FBL}^{(\infty)}[X] = \mathfrak{U}$. ∎

Properties of $\mathfrak{U}$

Question. What can we say about $\mathfrak{U}$?

Proposition ($\mathfrak{U}$ is not homogeneous (Gurarii))

If $1 \leqslant p \leqslant \infty$ *and* $\dim E \geqslant 2$, *there exist norm one* $\phi, \psi \in \mathrm{FBL}^{(p)}[E]_+$ *s.t.* $\|T\phi - \psi\| \geqslant 1/3$ *for any lattice isometry* T.

Proposition

$\mathfrak{U}$ *is Banach space isomorphic to* $C[0,1]$.

$\mathrm{FBL}^{(\infty)}[E]$ vs. $\mathrm{FBL}^{(\infty)}[F]$, non-separable case

Theorem (Keller)

If E and F are separable Banach spaces, then $(\mathbf{B}_{E^*}, w^*)$ is homeomorphic to $(\mathbf{B}_{F^*}, w^*)$. In fact, both are homeomorphic to $[0,1]^\omega$.

No analogue exists in the non-separable setting.

Proposition

For Γ with $|\Gamma| \geqslant \mathfrak{c}$, and $p \in (1,\infty)$, let $E = \ell_1(\Gamma)$ and $F = \ell_p(\Gamma)$. Then:

1. $(\mathbf{B}_{E^*}, w^*)$ *is not homeomorphic to* $(\mathbf{B}_{F^*}, w^*)$.
2. $\mathrm{FBL}^{(\infty)}[E]$ *is not lattice isomorphic to* $\mathrm{FBL}^{(\infty)}[F]$.

Lattice homomorphisms of $\mathrm{FBL}^{(p)}$ for $1 \leqslant p < \infty$

Theorem (N. Laustsen, P. Tradacete)

Suppose $T : \mathrm{FBL}^{(p)}[E] \to \mathrm{FBL}^{(p)}[F]$ *is a lattice homomorphism. Then* $\exists$ *a positively homogeneous* $\Phi_T : F^* \to E^*$, *weak** *to weak** *continuous on bounded sets, s.t.* $[Tf](y^*) = f(\Phi_T y^*)$ *for* $y^* \in F^*$, $f \in \mathrm{FBL}^{(p)}[F]$; $\|\Phi_T y^*\| \leqslant \|T\| \|y^*\|$. *If* $q \geqslant p$, *then, for any* $y_1^*, \ldots, y_N^* \in F^*$, *we have* $\|(\Phi_T y_i^*)\|_{q,\mathrm{weak}} \leqslant \|T\| \|(y_i^*)\|_{q,\mathrm{weak}}$.

Although any lattice homomorphism is a composition operator, a composition need not generate a lattice homomorphism.

Proposition

$\forall p \in [1, \infty) \, \exists$ *positively homogeneous* $\Phi : \ell_1^* \to \ell_1^*$, *weak** *continuous on bounded sets, s.t.* $\forall q \in [p, \infty]$, $\forall y_1^*, \ldots, y_N^* \in \ell_1^*$ *we have* $\|(\Phi y_i^*)\|_{q,\mathrm{weak}} \leqslant \|(y_i^*)\|_{q,\mathrm{weak}}$, *but s.t.* $f \circ \Phi \notin \mathrm{FBL}^{(p)}[\ell_1]$ *for some* $f \in \mathrm{FBL}^{(p)}[\ell_1]$.

Lattice quotients $\mathrm{FBL}^{(p)}[E] \to \mathrm{FBL}^{(p)}[F]$, $1 \leqslant p < \infty$

Theorem

Fix $u, v \in [2, \infty]$, $p \in [1, \infty]$, and $u < \min\{v, p'\}$, $1/p + 1/p' = 1$. Suppose E^* has cotype u, and F^* does not have cotype less than v. Then $\mathrm{FBL}^{(p)}[F]$ is not lattice isomorphic to a lattice quotient of $\mathrm{FBL}^{(p)}[E]$.

Cotype u: $\exists K$ s.t. $\forall x_1^*, \ldots, x_N^* \in E^*$, $\mathbb{E}\| \sum_i \pm x_i^* \| \geqslant K(\sum_i \|x_i^*\|^u)^{1/u}$.

L_r has cotype $\max\{r, 2\}$.

Corollary

Suppose $\beta \in [1, 2]$, and $\alpha \in (\beta, \infty]$. Then $\mathrm{FBL}[L_\beta]$ is not a lattice quotient of $\mathrm{FBL}[L_\alpha]$.

Known: L_β is not a quotient of L_α.

Proof. Apply theorem with $E = L_\alpha$, $F = L_\beta$, $p = 1$. E^* has cotype $\max\{2, \alpha'\}$, F^* has cotype $\beta' > \max\{2, \alpha'\}$. ∎

Lattice quotients $\mathrm{FBL}^{(p)}[E] \to \mathrm{FBL}^{(p)}[F]$, $1 \leqslant p < \infty$

Theorem

Fix $u, v \in [2, \infty]$, $p \in [1, \infty]$, and $u < \min\{v, p'\}$, $1/p + 1/p' = 1$. Suppose E^* has cotype u, and F^* does not have cotype less than v. Then $\mathrm{FBL}^{(p)}[F]$ is not lattice isomorphic to a lattice quotient of $\mathrm{FBL}^{(p)}[E]$.

Cotype u: $\exists K$ s.t. $\forall x_1^*, \dots, x_N^* \in E^*$, $\mathbb{E}\| \sum_i \pm x_i^* \| \geqslant K(\sum_i \|x_i^*\|^u)^{1/u}$.

Proof. Find $q \in (u, \min\{v, p'\})$. $\exists C$ s.t. $\forall n$ $\exists y_1^*, \dots, y_n^* \in F^*$ s.t.
$\forall (\alpha_i)$ $\vee_i |\alpha_i| \leqslant \| \sum_i \alpha_i y_i^* \| \leqslant C(\sum_i |\alpha_i|^q)^{1/q}$
$\Rightarrow \min_i \|y_i^*\| \geqslant 1$, $\|(y_i^*)\|_{q', \mathrm{weak}} \leqslant C \Rightarrow \|(\Phi_T y_i^*)\|_{q', \mathrm{weak}} \leqslant C\|T\|$.

Suppose $T : \mathrm{FBL}^{(p)}[E] \to \mathrm{FBL}^{(p)}[F]$ is a surjective lattice homomorphism. T^* is bounded below by $c > 0$, hence
$\forall y^* \in F^*$ $\|\Phi_T y^*\| \geqslant c\|y^*\|$.
By cotype u, $\max_\pm \| \sum_i \pm \Phi_T y_i^* \| \geqslant Kcn^{1/u}$, so
$\|(\Phi_T y_i^*)\|_{q', \mathrm{weak}} \geqslant \max_\pm \| \sum_i \pm n^{-1/q} \Phi_T y_i^* \| \geqslant Kcn^{1/u - 1/q} > \|T\|C$ for large n. Contradiction! ∎

Lattice quotients $\mathrm{FBL}^{(p)}[E] \to \mathrm{FBL}^{(p)}[F]$, $1 \leqslant p < \infty$

Theorem

Suppose $\infty \geqslant u > \max\{v, p\} \geqslant v \geqslant 1$, $E = (\sum_j E_j)_u$ ($E_1, E_2, \ldots$ *are finite dimensional*), *and* F^* *contains a copy of* $\ell_{v'}$, *with* $1/v + 1/v' = 1$. *Then* $\mathrm{FBL}^{(p)}[F]$ *is not a lattice quotient of* $\mathrm{FBL}^{(p)}[E]$.

More general results (involving games in Banach spaces) are known.

Questions. **(1)** Do there exist non-isomorphic E and F s.t. $\mathrm{FBL}^{(p)}[E]$ and $\mathrm{FBL}^{(p)}[F]$ are lattice isomorphic ($1 \leqslant p < \infty$)?

(2) If $\mathrm{FBL}^{(p)}[E]$ and $\mathrm{FBL}^{(p)}[F]$ are lattice isomorphic, which properties do E and F share?

Known for FBL ($p = 1$). If E contains ℓ_1 complementably (or contains ℓ_1^n uniformly complementably), then so does F.

Lattice isometries $\mathrm{FBL}^{(p)}[E] \to \mathrm{FBL}^{(p)}[F]$, $1 \leqslant p < \infty$

A Banach space Z is called strictly convex if $\forall$ norm one $z_1, z_2 \in Z$ we have $\|z_1 + z_2\| < 2$ unless $z_1 \neq z_2$.

Proposition (Preservation of strict convexity – warm-up)

Suppose $1 \leqslant p < \infty$, and E, F are Banach spaces s.t. $\mathrm{FBL}^{(p)}[E]$ is lattice isometric to $\mathrm{FBL}^{(p)}[F]$. If F^* is not strictly convex, neither is E^*.

Proof for $p = 1$. Suppose $T : \mathrm{FBL}^{(p)}[E] \to \mathrm{FBL}^{(p)}[F]$ is a lattice isometry. Then $\forall x^*, y^* \in F^*$ we have $\|(x^*, y^*)\|_{1,\mathrm{weak}} = \max_\pm \|x^* \pm y^*\| = \max_\pm \|\Phi_T x^* \pm \Phi_T y^*\| = \|(\Phi_T x^*, \Phi_T y^*)\|_{1,\mathrm{weak}}$.

Find distinct norm one $z_1, z_2 \in F^*$ so that $\|z_1 + z_2\| = 2$. Let $x^* = \frac{z_1+z_2}{2}$, $y^* = \frac{z_1-z_2}{2}$. Then $x^* \neq y^*$ and $\|(x^*, y^*)\|_{1,\mathrm{weak}} = 1$.

Let $w_1 = \Phi_T x^* + \Phi_T y^*$, $w_2 = \Phi_T x^* - \Phi_T y^*$ (in E^*). Then $\|w_1 + w_2\| = 2\|\Phi_T x^*\| = 2\|x^*\| = 1$, and $w_2 \neq 0$.
On the other hand, $\|w_1\|, \|w_2\| \leqslant \|(\Phi_T x^*, \Phi_T y^*)\|_{1,\mathrm{weak}} = 1$, so E^* is not strictly convex either. ∎

Lattice isometries $\mathrm{FBL}^{(p)}[E] \to \mathrm{FBL}^{(p)}[F]$, $1 \leqslant p < \infty$

A Banach space Z is called smooth if $\forall z \in Z \backslash \{0\}$ the norming functional $z^* \in Z^*$ is unique. Meaning: the unit sphere of Z has no "corners."

Theorem

Suppose $1 \leqslant p < \infty$, and E, F are Banach spaces s.t. either (i) E^*, F^* are smooth, or (ii) E, F are reflexive, and either E^* or F^* is smooth. Then $\mathrm{FBL}^{(p)}[E]$ is lattice isometric to $\mathrm{FBL}^{(p)}[F]$ iff E is isometric to F.

Idea. We have access to $\|(e_i^*)\|_{p,\mathrm{weak}}$, have to use these quantities to reach conclusion.

For $x \in Z$, denote by $\mathcal{F}(x)$ the set of support functionals for x – that is, of $x^* \in Z^*$ for which $\|x^*\| = \|x\|$ and $\|x\|^2 = \langle x^*, x\rangle$.

Lemma

Suppose $x, y \in \mathbf{S}_Z$, and $1 \leqslant p < \infty$. Let $\kappa = \sup_{x^* \in \mathcal{F}(x)} |\langle x^*, y\rangle|$. Then, for $t \to 0$, $\|(x, ty)\|_{p,weak} = 1 + \frac{\kappa^p}{p}|t|^p + o(|t|^p)$.

Lattice isometries when E^*, F^* are smooth

Theorem

Suppose $1 \leqslant p < \infty$, and E, F are Banach spaces so that E^*, F^* are smooth. Then $T : \mathrm{FBL}^{(p)}[E] \to \mathrm{FBL}^{(p)}[F]$ is a lattice isometry iff $T = \overline{U}$, for some isometry $U : E \to F$.

Proof. If Z is smooth, then $\forall x \in Z\ \exists f_x \in Z^*$ s.t. $\mathcal{F}(x) = \{f_x\}$. Define semi-inner product $[y, x] := \langle f_x, y \rangle$. This has the same properties as the inner product, except symmetry and additivity in second variable.

Let $[\cdot, \cdot]_E$ and $[\cdot, \cdot]_F$ be the semi-inner products on E^* and F^* respectively.

For $x^*, y^* \in \mathbf{S}_{F^*}$, $\|(x^*, ty^*)\|_{p,\mathrm{weak}} = 1 + \frac{|t|^p}{p} |[y^*, x^*]_F| + o(|t|^p)$, hence $|[y^*, x^*]_F| = |[\Phi_T y^*, \Phi_T x^*]_E|$.

[Ilisevic & Turnsec]: $\exists \sigma : F^* \to \{-1, 1\}$ and $V : F^* \to E^*$ s.t. $\forall x^* \in F^*$ $\Phi_T x^* = \sigma(x^*) V x^*$. Show that $\sigma = \mathrm{const}$ and V is weak* continuous, so $V = U^*$. ∎

Lattice isometries and the lack of smoothness

Definition

An element x of a Banach space X is called an ℓ_1-point if the equality $\max_{\pm} \|x \pm y\| = \|x\| + \|y\|$ holds for any $y \in X$.

Proposition

Suppose $1 \leqslant p < \infty$. *x is an ℓ_1-point iff* $\|(x, y)\|_{p,\mathrm{weak}} = (\|x\|^p + \|y\|^p)^{1/p}$ *for any* $y \in X$.

Proposition

Suppose $T : \mathrm{FBL}^{(p)}[F] \to \mathrm{FBL}^{(p)}[E]$ *is a lattice isometry. Then* Φ_T *is a bijection between ℓ_1-points of E^* and F^*.*

Application: free lattices on AM-spaces

Proposition

Suppose E and F are AM-spaces. For $1 \leqslant p < \infty$, TFAE:

1. $\mathrm{FBL}^{(p)}[E]$ *is lattice isometric to* $\mathrm{FBL}^{(p)}[F]$.
2. E *and* F *are isometric.*
3. E *and* F *are lattice isometric.*

Idea of proof. Realize E as a set of functions of $\overline{\Omega_E}$ (the weak* closure of the state space). Identify ℓ_1 points with point masses. Do the same for F. Any lattice isometry $\mathrm{FBL}^{(p)}[E] \to \mathrm{FBL}^{(p)}[F]$ will generate a homeomorphism between Ω_E and Ω_F. ∎

Weak compact generation

Definition

A Banach space E is weakly compactly generated (WCG) if $\exists$ a weakly compact set $K \subseteq E$ whose linear span is dense. A Banach lattice X is weakly compactly generated as a lattice (LWCG) if there is a weakly compact set $K \subseteq X$ so that the sublattice generated by K is dense in X.

Proposition

Suppose E is either an AM-space, or an order continuous p-convex Banach lattice. Then $\mathrm{FBL}^{(p)}[E]$ is LWCG iff E is WCG.

Theorem (A. Avilés, J. Rodríguez, and P. Tradacete)

For $1 < q \leqslant 2$, and uncountable Γ, $\mathrm{FBL}[\ell_q(\Gamma)]$ is LWCG, but not WCG.

Sketch of a proof. $\ell_q(\Gamma)$ is order continuous. However, $\mathrm{FBL}[\ell_q(\Gamma)]$ contains a complemented copy of $\ell_1(\Gamma)$, hence it is not WCG. ■

Thank you for your attention! Questions welcome!