Free Banach lattices p. 1 (work in progress)

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Why, and how, we study free objects

- Why. *Freedom, Sancho, is one of the most precious gifts that heaven has bestowed upon men ...*

— Miguel de Cervantes Saavedra, Don Quixote

- How. The subject is vast, our knowledge is limited.

What is freeness?

Suppose $\mathcal{K}$ and $\mathcal{L}$ are categories, and $\Box : \mathcal{K} \to \mathcal{L}$ is a functor (often, a forgetful functor). $\mathcal{K}$ is called a rigged category.

Fix an object $E \in \mathcal{L}$. A free object over E is a pair (ϕ, X) so that:

1. $X \in \mathcal{K}$, $\phi : E \to \Box X$ is an $\mathcal{L}$-morphism;
2. $\forall\ Z \in \mathcal{K}$, and $\forall$ $\mathcal{L}$-morphism $T : E \to \Box Z$, $\exists$ unique $\mathcal{K}$-morphism $\widehat{T} : X \to Z$ s.t. $T = \Box\widehat{T} \circ \phi$.

$$\begin{tikzcd} \Box X \arrow[dr, dashed, "\Box\widehat{T}"] & \\ E \arrow[u, "\phi"] \arrow[r, "T"'] & \Box Z \end{tikzcd}$$

Free object is unique, if it exists.

Examples of freeness

1. $\mathcal{K}$ = category of all Hausdorff compacts. $\mathcal{L}$ = category of all topological spaces. $\square$ = id. Free object on $S \in \mathcal{L}$ is its Stone-Cech compactification βS.

$$\begin{tikzcd} \beta S \arrow[dr, dashed, "\beta T"] & \\ S \arrow[u] \arrow[r, "T"] & K \end{tikzcd}$$

2. $\mathcal{K}$ = category of all normed spaces; morphisms = contractive linear operators. $\mathcal{L}$ = category of sets. $\square X = \mathbf{B}_X$ (the unit ball of X). Free object on $A \in \mathcal{L}$ is $\ell_1^0(A)$. Used to investigate projective normed spaces.

$$\begin{tikzcd} \ell_1^0(A) \arrow[dr, dashed, "\widehat{T}"] & \\ A \arrow[u] \arrow[r, "T"] & Z \end{tikzcd}$$

3. $\mathcal{K}$ = category of all Banach spaces; morphisms = linear contractions. $\mathcal{L}$ = category of metric spaces with 0; morphisms = 0-preserving contractions. $\square X = \mathbf{B}_X$. Free object $\mathcal{F}(A)$ is the Lipschitz-free space.

$$\begin{tikzcd} \mathcal{F}(A) \arrow[dr, dashed, "\widehat{T}"] & \\ A \arrow[u] \arrow[r, "T"] & Z \end{tikzcd}$$

Freeness and Banach lattices: our setting

$\mathbf{Ban}_1$ = category of Banach spaces; morphisms = linear contractions.

$\mathbf{BL}_1^p$ = category of Banach lattices, p-convex with constant 1; morphisms = contractive lattice homomorphisms.

Recall: X is q-convex with constant C if
$\left\|\left(\sum_i |x_i|^q\right)^{1/q}\right\| \leqslant C\left(\sum_i \|x_i\|^q\right)^{1/q}$ holds for any $x_1, \ldots, x_N \in X$.
Example. L_p is q-convex (with const. 1) iff $p \leqslant q$.

For $p = 1$ recover $\mathbf{BL}_1$ = category of Banach lattices.

$\square : \mathbf{BL}_1^p \to \mathbf{Ban}_1$ = "forgetful functor."

$$\begin{tikzcd}
\mathrm{FBL}^{(p)}[E] \arrow[dr, dashed, "\widehat{T}"] & \\
E \arrow[u, hook, "\phi_E"] \arrow[r, "T"] & Z
\end{tikzcd}$$

Goal: show that the free object $\mathrm{FBL}^{(p)}[E]$ exists, describe its properties.

One can also consider other lattice categories – for instance, the category $\mathbf{BL}_1^{\uparrow p}$ of Banach lattices with upper p-estimate.

Free Banach lattices on Banach spaces

Definition

Suppose E is a Banach space, and $1 \leqslant p \leqslant \infty$. A free p-convex Banach lattice on E ($\mathrm{FBL}^{(p)}[E]$) is the unique Banach lattice X so that:

- X is p-convex with constant 1.
- There exists an isometry $\phi_E : E \to X$ so that $\phi_E(E)$ generates X as a Banach lattice.
- If Z is a Banach lattice, p-convex with constant 1, then any $T \in B(E, Z)$ extends to a lattice homomorphism $\widehat{T} : \mathrm{FBL}^{(p)}[E] \to Z$ so that $\|\widehat{T}\| = \|T\|$, and $\widehat{T} \circ \phi_E = T$.

$$\begin{tikzcd}
\mathrm{FBL}^{(p)}[E] \arrow[dr, dashed, "\widehat{T}"] & \\
E \arrow[u, hook, "\phi_E"] \arrow[r, "T"'] & Z
\end{tikzcd}$$

If Z is p-convex with constant C, we can construct an extension $\widehat{T}$ with $\|\widehat{T}\| \leqslant C\|T\|$ (renorming makes Z p-convex with constant 1).
Notation: $\mathrm{FBL}[E] := \mathrm{FBL}^{(1)}[E]$
(for $p = 1$, Z is an arbitrary lattice).

Explicit construction of $\mathrm{FBL}^{(p)}[E]$

Denote by $\mathbf{H}[E^*]$ the space of positively homogeneous functions on $\mathbf{B}_{E^*}$ ($f(tx^*) = tf(x^*)\ \forall t \geqslant 0$). $\mathbf{H}_p[E^*]$ consists of those $f \in \mathbf{H}[E^*]$ for which $\exists C > 0$ s.t. $\sum_{i=1}^N |f(x_i^*)|^p \leqslant C^p$ when $\|(x_i^*)_{i=1}^N\|_{p,\mathrm{weak}} \leqslant 1$. $\|f\|_p := \inf C$.

Here $\|(x_i^*)_{i=1}^N\|_{p,\mathrm{weak}} = \sup_{x \in \mathbf{B}(E)} \left(\sum_i |\langle x_i^*, x\rangle|^p\right)^{1/p}$.
$\|(x_i^*)_{i=1}^N\|_{\infty,\mathrm{weak}} = \max_i \|x_i^*\|$. $\|(x_i^*)_{i=1}^N\|_{1,\mathrm{weak}} = \max \|\sum_i \pm x_i^*\|$.
Let $\phi_E : E \to \mathbf{H}[E^*] : x \mapsto \delta_x$; $\delta_x(x^*) = \langle x^*, x\rangle$.

Theorem

$\phi_E : E \to \mathbf{H}_p[E^*]$ is an isometry. $\mathrm{FBL}^{(p)}[E]$ is the Banach lattice generated by $\phi_E(E)$ in $\mathbf{H}_p[E^*]$.

Note: All $f \in \mathrm{FBL}^{(p)}[E]$ are weak* continuous functions on B_{E^*}.

Explicit construction of $\mathrm{FBL}^{(p)}[E]$

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Note: All $f \in \mathrm{FBL}^{(p)}[E]$ are weak* continuous functions on B_{E^*}.

$\mathrm{FBL}^{(\infty)}[E]$ is the set of **all** weak* continuous functions in $\mathbf{H}_\infty[E^*]$:

Theorem

$\mathrm{FBL}^{(\infty)}[E]$ coincides with the space $\mathbf{CH}[E^*]$ of weak* continuous positively homogeneous functions on $\mathbf{B}_{E^*}$, with sup *norm $\|\cdot\|_\infty$.*

The case of $p \in [1, \infty)$ is different:

Theorem (Avilés, Rodríguez, Tradacete)

For $1 \leqslant p < \infty$ $\exists$ weak* continuous $0 \leqslant g \leqslant f$ s.t. $g, f \in \mathbf{H}_p[\ell_1^*] \backslash \{0\}$ s.t. $f \in |\mathrm{ran}\, \phi_{\ell_1}| \subset \mathrm{FBL}^{(p)}[\ell_1]$, $g \notin \mathrm{FBL}^{(p)}[\ell_1]$.

Free vector lattices

Let $\mathrm{FVL}[E]$ be the vector lattice generated by $\phi_E(E)$ in $\mathbf{H}[E^*]$.

$\mathrm{FVL}[E]$ is a realization of the free vector lattice on E: for any vector lattice Z, any linear map $T : E \to Z$ has a lattice homomorphic extension $\widehat{T}$.

$$\begin{tikzcd} \mathrm{FVL}[E] \arrow[dr, dashed, "\widehat{T}"] & \\ E \arrow[u, hook, "\phi_E"] \arrow[r, "T"'] & Z \end{tikzcd}$$

$$\mathrm{FBL}^{(p)}[E] = \overline{\mathrm{FVL}[E]}^{\mathbf{H}_p[E^*]}.$$

Theorem

For $1 \leqslant p \leqslant \infty$, $\mathrm{FVL}[E]$ is order dense in $\mathrm{FBL}^{(p)}[E]$ – that is, $\forall f \in \mathrm{FBL}^{(p)}[E]_+ \ \exists g \in \mathrm{FVL}[E]_+$ s.t. $f \geqslant g$.

Observations and examples

• $\mathrm{FBL}^{(\infty)}[E]$ coincides with the space $\mathbf{CH}[E^*]$ of weak* continuous positively homogeneous functions on $\mathbf{B}_{E^*}$, with sup norm $\|\cdot\|_\infty$.

Such functions extend to positively homogeneous functions on E^*, weak* continuous on bounded sets.

• For $1 \leqslant p \leqslant \infty$, $\mathrm{FBL}^{(p)}[E]$ is the sublattice of $\mathbf{H}[E^*]$ generated by $\{\delta_x : x \in E\}$, with norm $\|\cdot\|_p$.

• If $q > p$, then $\|\cdot\|_p \geqslant \|\cdot\|_q \Rightarrow \mathrm{FBL}^{(p)}[E] \subset \mathrm{FBL}^{(q)}[E]$, contractively.

• If $\dim E = n$, then $\mathrm{FBL}^{(\infty)}[E]$ is lattice isometric to $\mathbf{CH}[E^*] = C(\mathbf{S}_{E^*})$ $\simeq C(\mathbb{S}^{n-1})$ ($\mathbb{S}^{n-1}$ = sphere in $\mathbb{R}^n$). $\mathrm{FBL}^{(p)}[E]$ is lattice isomorphic to $C(\mathbb{S}^{n-1})$ (for $f \in \mathrm{FBL}^{(p)}[E]$, $\|f\|_\infty \leqslant \|f\|_p \leqslant n\|f\|_\infty$).

• If $E = \mathbb{R}$ ($n = 1$), then $\mathrm{FBL}^{(\infty)}[E] = C(\{1, -1\}) = \ell^2_\infty$.
$\phi_E(1) = \delta_1 = (1, -1)$. For a Banach lattice Z and $T : E \to Z$, write $T1 = z_+ - z_-$, where $z_+ = (T1)_+$ and $z_- = (T1)_-$ (disjoint). Then $\widehat{T}(a, b) = az_+ - bz_-$ ($\widehat{T}(1, 0) = z_+$, $\widehat{T}(0, 1) = z_-$).

Main questions

$\overline{T}$ is a lattice homomorphism extending $\phi_E \circ T$ (making the diagram commute).
If $G \xrightarrow{S} F \xrightarrow{T} E$, then $\overline{T \circ S} = \overline{T} \circ \overline{S}$.

$$\begin{tikzcd} \mathrm{FBL}^{(p)}[F] \arrow[r, dashed, "\overline{T}"] & \mathrm{FBL}^{(p)}[E] \\ F \arrow[u, hook, "\phi_F"] \arrow[r, "T"] & E \arrow[u, hook, "\phi_E"'] \end{tikzcd}$$

- Consider $T \in B(F, E)$ (E, F are Banach spaces). Which properties of T pass to $\overline{T} : \mathrm{FBL}^{(p)}[F] \to \mathrm{FBL}^{(p)}[E]$?
- Determine sublattice structure of $\mathrm{FBL}^{(p)}[E]$.
- Find connections between properties of E and $\mathrm{FBL}^{(p)}[E]$.
- When are $\mathrm{FBL}^{(p)}[E]$ and $\mathrm{FBL}^{(p)}[F]$ isometric? isomorphic?
If $T : E \to F$ is an isomorphism (isometry), then so is $\overline{T}$. If E and F are isomorphic (isometric), the same is true for $\mathrm{FBL}^{(p)}[E]$ and $\mathrm{FBL}^{(p)}[F]$.
Can $\mathrm{FBL}^{(p)}[E]$ and $\mathrm{FBL}^{(p)}[F]$ be "same", while E and F are "different?"
- How does $\operatorname{ran} \phi_E$ sit inside of $\mathrm{FBL}^{(p)}[E]$?
In particular: if $(e_i) \subset E$ is a "nice" sequence, what about $(|\phi_E(e_i)|)$?

Properties of T which pass to $\overline{T}$

$$\begin{tikzcd} \mathrm{FBL}^{(p)}[F] \arrow[r, dashed, "\overline{T}"] & \mathrm{FBL}^{(p)}[E] \\ F \arrow[u, hook, "\phi_F"] \arrow[r, "T"] & E \arrow[u, hook, "\phi_E"'] \end{tikzcd}$$

Theorem

- *T is an isomorphism (isometry) iff $\overline{T}$ is the same.*
- *T is injective iff $\overline{T}$ is injective.*
- *T has dense range iff $\overline{T}$ has dense range.*
- *T is surjective iff $\overline{T}$ is surjective.*

What if T is an (isometric, or isomorphic) embedding? Under what conditions with $\overline{T}$ have the same property? Connected to extensions of operators into L_p.

Connections to extensions of operators

$$\begin{tikzcd} \mathrm{FBL}^{(p)}[F] \arrow[rrr, dashed, "\bar\iota"] & & & \mathrm{FBL}^{(p)}[E] \\ F \arrow[u, hook, "\phi_F"] \arrow[rrr, "\iota"] & & & E \arrow[u, hook, "\phi_E"] \end{tikzcd}$$

Consider an embedding $\iota : F \hookrightarrow E$. Then $\bar\iota : \mathrm{FBL}^{(p)}[F] \to \mathrm{FBL}^{(p)}[E]$ is contractive. Is it also a lattice isomorphic embedding? That is, is $\bar\iota$ bounded below?

Theorem

Suppose $C \geqslant 1$. The following [and more] are equivalent:

1. $\forall f \in \mathrm{FBL}^{(p)}[F]$, $\|\bar\iota f\| \geqslant C^{-1}\|f\|$.
2. $\forall n \in \mathbb{N}\ \forall \varepsilon > 0\ \forall u \in B(F, \ell_p^n)\ \exists$ *extension* $\widetilde{u} \in B(E, \ell_p^n)$, *with* $\|\widetilde{u}\| \leqslant C\|u\| + \varepsilon$.
3. $\forall u \in B(F, L_p(\mu))\ \exists$ *extension* $\widetilde{u} \in B(F, L_p(\mu))$, *with* $\|\widetilde{u}\| \leqslant C\|u\|$.

The POE-p

Theorem

Suppose $C \geqslant 1$. The following [and more] are equivalent:

1. $\forall f \in \mathrm{FBL}^{(p)}[F]$, $\|\bar{\iota} f\| \geqslant C^{-1}\|f\|$.
2. $\forall n \in \mathbb{N}\ \forall \varepsilon > 0\ \forall u \in B(F, \ell_p^n)\ \exists$ *extension* $\widetilde{u} \in B(E, \ell_p^n)$, *with* $\|\widetilde{u}\| \leqslant C\|u\| + \varepsilon$.
3. $\forall u \in B(F, L_p(\mu))\ \exists$ *extension* $\widetilde{u} \in B(F, L_p(\mu))$, *with* $\|\widetilde{u}\| \leqslant C\|u\|$.

A pair (F, E) with the above property is said to have C-POE-p (Property of Operator Extensions into L_p, with const. C).
(F, E) has the POE-p if it has C-POE-p, for some C.
F has the POE-p if (F, E) has the POE-p for any E containing F.
Equivalently, $\exists C$ s.t, for any E containing F, (F, E) has the C-POE-p.
Suffices to consider $E = \ell_\infty(\Gamma)$.

Which pairs (F, E), or spaces F, have the POE-p?

Which pairs (F, E), or spaces F, have the POE-p?

$$\begin{tikzcd} E \arrow[dr, dashed, "\widetilde{u}"] & \\ F \arrow[u, hook] \arrow[r, "u"] & \ell_p^N \end{tikzcd}$$

Trivial answers: **1.** Any pair has POE-∞.
2. F is complemented in $E \Rightarrow (F, E)$ has the POE-p (for $1 \leqslant p \leqslant \infty$).
Then $\bar{\iota} : \mathrm{FBL}^{(p)}[F] \to \mathrm{FBL}^{(p)}[E]$ is a lattice embedding.

Proposition

For $p \in [1, 2]$, F has the POE-p iff $B(F, L_p) = \Pi_2(F, L_p)$.

Proof. **(1)** Suppose F has the POE-p. Take $F \hookrightarrow E = \ell_\infty(\Gamma)$. Any $u : F \to L_p$ has an extension $\widetilde{u} : \ell_\infty(\Gamma) \to L_p$. Then $\pi_2(u) \leqslant \pi_2(\widetilde{u}) \leqslant K\|\widetilde{u}\| \leqslant KC\|u\|$.

(2) If $B(F, L_p) = \Pi_2(F, L_p)$, then $\exists K$ s.t. $\pi_2(u) \leqslant K\|u\|$ for any $u : F \to L_p$. If $F \hookrightarrow E$, then $\exists$ extension $\widetilde{u} : E \to L_p$ s.t. $\|\widetilde{u}\| \leqslant \pi_2(\widetilde{u}) = \pi_2(u) \leqslant K\|u\|$. ∎

More about POE-p

Proposition. For $p \in [1,2]$, F has the POE-p iff $B(F, L_p) = \Pi_2(F, L_p)$.

Proposition

- *If $1 \leqslant p \leqslant q \leqslant 2$, and F has the POE-p, then it has POE-q.*
- *If $2 \leqslant p < \infty$, and F has the POE-p, then it has POE-2.*
- *ℓ_1 has POE-p iff $2 \leqslant p \leqslant \infty$.*

Proof. **(1)** $L_q \hookrightarrow L_p$; $B(F, L_p) = \Pi_2(F, L_p) \Rightarrow B(F, L_q) = \Pi_2(F, L_q)$.

(2) $L_2 \hookrightarrow L_p$, complementably. ∎

Question. Other connections between POE-p's?

Casazza & Nielsen 2003: E has Property M_p (M for Maurey Extension) if $\forall F \hookrightarrow E$, any $u : F \to \ell_p$ has an extension $\widetilde{u} : E \to \ell_p$.

If E is a Köthe function space on $(0,1)$ with Property M_p with $2 < p < \infty$, then E is isomorphic to a Hilbert space.

Complemented sublattices of $\mathrm{FBL}^{(p)}[E]$

Suppose $\iota : F \hookrightarrow E$, and $P : E \to F$ is a projection ($id_F = P\iota$). Then $\overline{P}^2 = \overline{P^2} = \overline{P}$, and $\overline{id_F} = \overline{P\iota} = \overline{P}\overline{\iota}$. IOW, $\overline{\iota} : \mathrm{FBL}^{(p)}[F] \to \mathrm{FBL}^{(p)}[E]$ is a lattice embedding, and $\overline{P} : \mathrm{FBL}^{(p)}[E] \to \mathrm{FBL}^{(p)}[E]$ is a lattice projection (idempotent lattice homomorphism).

Proposition

If F is a complemented subspace of E, then $\mathrm{FBL}^{(p)}[F]$ is a lattice-complemented sublattice of $\mathrm{FBL}^{(p)}[E]$.

Questions. (1) For $F \hookrightarrow E$, under what conditions is $\mathrm{FBL}^{(p)}[F]$ lattice isomorphic to a lattice complemented subspace of $\mathrm{FBL}^{(p)}[E]$? Is complementation of F in E necessary?

(2) What can we say about (complemented) sublattices of $\mathrm{FBL}^{(p)}[E]$?

Sublattice and subspace structure

Proposition

If $\dim E \geqslant n$*, then* $\mathrm{FBL}^{(p)}[E]$ *contains a lattice complemented sublattice, lattice isomorphic to* $C(\mathbb{S}^{n-1})$*.*

Proof. Pick a n-dimensional (complemented) $F \hookrightarrow E$. Known: $\mathrm{FBL}^{(p)}[F]$ is lattice isomorphic to $C(\mathbb{S}^{n-1})$. ■

Corollary

If $\dim E \geqslant 2$*, then* $\mathrm{FBL}^{(p)}[E]$ *contains isomorphically any separable Banach space.*

Proposition

If Γ *is uncountable, and* $1 \leqslant p < 2 < q < \infty$*, then* $\mathrm{FBL}[\ell_p(\Gamma)]$ *does not embed, as a Banach space, in* $\mathrm{FBL}[\ell_q(\Gamma)]$*.*

This is due to WCG considerations.

Lattice copies of E inside of $\mathrm{FBL}^{(p)}[E]$

Note: Canonical embedding $\phi_E : E \hookrightarrow \mathrm{FBL}^{(p)}[E]$ is not a lattice embedding, even if E is a Banach lattice.

Theorem

Suppose the Banach lattice structure on E is determined by its 1-unconditional basis, and, moreover, E is p-convex with constant 1. Then $\exists$ an isometric lattice embedding $J : E \to \mathrm{FBL}^{(p)}[E]$, and a contractive lattice projection from $\mathrm{FBL}^{(p)}[E]$ onto JE.

Example: If $q \geqslant p$, then $\mathrm{FBL}^{(p)}[\ell_q]$ contains ℓ_q as a lattice-complemented sublattice (for $q = \infty$, take c_0 instead of ℓ_∞).

Lattices that cannot embed into $\mathrm{FBL}^{(p)}[E]$

Theorem

Suppose Z is a sublattice of $\mathrm{FBL}^{(p)}[E]$. Then lattice homomorphisms in Z^* separate points in Z.

$z^* \in Z^*$ is a lattice homomorphism iff (i) $z^* \geqslant 0$, and (ii) if $x, y \in Z$, $x \perp y$, then either $\langle z^*, x \rangle = 0$ or $\langle z^*, y \rangle = 0$. Example: $Z = C(K)$; $z^* \in C(K)^*$ is a lattice homomorphism iff $z^* = \delta_t$, for some $t \in K$.

Proof. $\mathrm{FBL}^{(p)}[E] \hookrightarrow \mathbf{H}_p[E^*]$ (as a sublattice). Known: $\phi \in \mathrm{FBL}^{(p)}[E]^*$ is a lattice homomorphism iff $\phi = \widehat{x^*}$ for some $x^* \in E^*$ (that is, $\langle \phi, f \rangle = f(x^*)$). These separate points in $\mathrm{FBL}^{(p)}[E]$. ■

Corollary

$\mathrm{FBL}^{(p)}[E]$ *cannot contain $L_q(0,1)$ $(1 \leqslant q < \infty)$ lattice isomorphically.*

Proof. If $Z = L_q(0,1)$ $(1 \leqslant q < \infty)$, then Z^* contains no lattice homomorphisms. ■

Dual of a free lattice: describing the atoms

Consider a Banach lattice Z. Then $z^* \in Z^*_+$ is an atom ($0 \leqslant x^* \leqslant z^* \Leftrightarrow x^* \in [0,1] \cdot z^*$) iff it is a lattice homomorphism $Z \to \mathbb{R}$.

Notation. For a Banach space E, and $e^* \in E^*$, $\widehat{e^*} : \mathrm{FBL}^{(p)}[E] \to \mathbb{R}$ is the lattice homomorphism extending $e^* : E \to \mathbb{R}$, with $\|e^*\| = \|\widehat{e^*}\|$.

Observation. If $T : X \to Y$ is a lattice homomorphism, then T^* is interval preserving, hence $T^* y^*$ is an atom whenever $y^* \in Y^*$ is.

Proposition (Folklore?)

$\phi \in \mathrm{FBL}^{(p)}[E]^*$ is an atom iff $\phi = \widehat{e^*}$ for some $e^* \in E^*$.

Proof. $\widehat{e^*}$ is a lattice homomorphic extension of $e^* : E \to \mathbb{R}$.

For a lattice homomorphism $\phi : \mathrm{FBL}^{(p)}[E] \to \mathbb{R}$, let $e^* = \phi|_E$. Then $\widehat{e^*}$ is a lattice homomorphism, coinciding with ϕ on E, hence also on the closed sublattice generated by E, which is the whole $\mathrm{FBL}^{(p)}[E]$. ∎

Dual of a free lattice: sums of atoms

Notation. For $e_1, \dots, e_n \in E$,
$\|(e_i)\|_{q,\mathrm{weak}} = \max\left\{\|\sum_i \lambda_i e_i\| : \sum_i |\lambda_i|^{q'} \leqslant 1\right\} = \|u : \ell_{q'}^n \to E\|$
$= \|u^* : E^* \to \ell_q^n\|$, where $1/q + 1/q' = 1$, and $u : \delta_i \mapsto e_i$.

Proposition

For $e_1^*, \dots, e_n^* \in E^*$, consider $\widehat{e_1^*}, \dots, \widehat{e_n^*} \in \mathrm{FBL}^{(p)}[E]^*$. Then, for $p \leqslant q$, $\|(e_i^*)\|_{q,\mathrm{weak}} = \|(\widehat{e_i^*})\|_{q,\mathrm{weak}}$.

Proof. Consider $T : E \to \ell_q^n : e \mapsto (e_i^*(e))_{i=1}^n$. Then $\|T\| = \|(e_i^*)\|_{q,\mathrm{weak}}$; $\widehat{T} : \mathrm{FBL}^{(p)}[E] \to \ell_q^n : f \mapsto (\langle \widehat{e_i^*}, f\rangle)_{i=1}^n$ is the lattice homomorphic extension, of the same norm. ∎

Dual of a free lattice: atoms fill the universe

An element of $\mathrm{FBL}^{(p)}[E]^*$ is finitely presented if if it can be written as $f_{\overline{e^*},\overline{\alpha}} = \sum_{i=1}^n \alpha_i \widehat{e_i^*}$ for some $\overline{e^*} = (e_1^*, \ldots, e_n^*) \subset E^*$ (with $e_i^* \notin [0, \infty) \cdot e_j^*$ for $i \neq j$) and $\overline{\alpha} = (\alpha_1, \ldots, \alpha_n) \subset \{-1, 1\}$.

We know that, for $p = 1$, $\|f_{\overline{e^*},\overline{\alpha}}\| = \|\overline{e^*}\|_{1,\mathrm{weak}}$.

Proposition

$f \in \mathrm{FBL}^{(p)}[E]^*$ satisfies $\|f\| \leqslant 1$ iff it is a weak* limit of a net $(f_{\overline{e^*}_j \overline{\alpha}_j})$ of finitely presented functionals of norm not exceeding 1.
Moreover, $f \in \mathrm{FBL}^{(p)}[E](\ell_p^m)^*$ satisfies $\|f\| \leqslant 1$ iff it is a weak* limit of a net (f_j), where $f_j = (f_{\overline{e^*}_{js},\overline{\alpha}_{js}})_{s=1}^m$, and $\sup_j \|f_j\| \leqslant 1$.

Convexity and summing properties of operators

Suppose E, F are Banach spaces, and $T \in B(F, E)$.

T is $(r, 1)$-summing if here exists a constant C so that the inequality $(\sum \|f_i\|^r)^{1/r} \leqslant \|(f_i)\|_{1,\mathrm{weak}}$; $\pi_{r,1}(T) = \inf C$.

If E is a Banach lattice, T is (r, ∞)-convex if there exists a constant C so that the inequality $\| \vee_i |Tf_i|\| \leqslant C(\sum_i \|f_i\|^r)^{1/r}$; the (r, ∞)-convex norm $\mathrm{cvx}_{r,\infty}(T)$ is the smallest such C.

Proposition

For $u \in B(F, E)$ and $r \in (1, \infty)$, the lattice homomomorphism $\overline{u} : \mathrm{FBL}[F] \to \mathrm{FBL}[E]$ is (r, ∞)-convex iff u^* is $(r', 1)$-summing, with $1/r + 1/r' = 1$; then $\mathrm{cvx}_{r,\infty}(\overline{u}) = \pi_{r',1}(u^*)$.

Idea of the proof: dualize.

Upper estimates for $\mathrm{FBL}[E]$

Definition

Suppose $1 \leqslant q \leqslant \infty$. A Banach lattice X:

- is *q-convex with constant C* if $\left\|\left(\sum_{i=1}^N |x_i|^q\right)^{1/q}\right\| \leqslant C\left(\sum_i \|x_i\|^q\right)^{1/q}$ holds for any $x_1, \ldots, x_N \in X$.
- has *upper q-estimate with constant C* if $\mathrm{cvx}_{q,\infty}(id_E) \leqslant C$: $\left\| \bigvee_{i=1}^N |x_i| \right\| \leqslant C\left(\sum_i \|x_i\|^q\right)^{1/q}$ holds for any $x_1, \ldots, x_N \in X$. Suffices to check this inequality when $x_1, \ldots, x_N$ are disjoint.

Theorem

For a Banach space E, $p \geqslant 1$, and $C \geqslant 1$, TFAE:

1. $\mathrm{FBL}[E]$ *has upper p-estimate with constant C.*
2. id_{E^*} *is $(p', 1)$-summing with constant C $(1/p + 1/p' = 1)$.*

Only $p \in [1, 2]$ are feasible.

Complemented copies of ℓ_1^n

Proposition

For a Banach space E, TFAE:

1. *E contains copies of ℓ_1^n, uniformly complementably.*
2. $\mathrm{FBL}[E]$ *contains ℓ_1^n as sublattices, uniformly.*
3. $\mathrm{FBL}[E]$ *contains ℓ_1^n as sublattices, uniformly lattice complementably.*
4. $\mathrm{FBL}[E]$ *contains ℓ_1^n as subspaces, uniformly complementably.*

Proof of (1) $\Leftrightarrow$ (2). E contains ℓ_1^n, uniformly complementably
$\Leftrightarrow$ E^* contains copies of ℓ_∞^n $\Leftrightarrow$ E^* has trivial cotype
$\Leftrightarrow$ id_{E^*} is not $(r', 1)$-summing for any $r' < \infty$
$\Leftrightarrow$ $\mathrm{FBL}[E]$ has no non-trivial upper r-estimates for $r > 1$
$\Leftrightarrow$ $\mathrm{FBL}[E]$ contains ℓ_1^n as sublattices, uniformly [Krivine]. ■

Copies of ℓ_1

Proposition

For a Banach space E, TFAE:

1. *E contains a complemented copy of ℓ_1.*
2. $\mathrm{FBL}[E]$ *contains ℓ_1 as a sublattice.*
3. $\mathrm{FBL}[E]$ *contains ℓ_1 as a lattice complemented sublattice.*
4. $\mathrm{FBL}[E]$ *contains ℓ_1 as a complemented subspace.*

$\mathrm{FBL}[E]$ when E is a Banach lattice

Theorem

Suppose $1 < p < 2$, E is a Banach lattice. Then E has an upper p-estimate iff $\mathrm{FBL}[E]$ does.

Corollary

Suppose $1 < p < 2$, E, F are Banach lattices, and F is linearly isomorphic to a complemented subspace of E. If E has the upper p-estimate, then so does F.

Similar result is known for p-convexity.

Sketch of proof. Suppose $\iota : F \to E$ isomorphic embedding, and ιF is complemented in E. Then $\bar{\iota} : \mathrm{FBL}[F] \to \mathrm{FBL}[E]$ is a lattice isomorphic embedding. Upper estimates pass to sublattices. ■

Properties of $(e_k) \subset E$ versus $(|\phi_E(e_k)|) \subset \mathrm{FBL}^{(p)}[E]$

Proposition

Suppose (e_k) is a basis (an unconditional basis) in E. Then $(|\phi_E(e_k)|)$ is a basic sequence (unconditional basic sequence) in $\mathrm{FBL}^{(p)}[E]$.

Theorem (Avilés, Tradacete, Villanueva; 2019)

Suppose $(e_k) \subset \ell_q$ is the canonical basis. Then $(|\phi_E(e_k)|) \subset \mathrm{FBL}[E]$ is equivalent to the canonical ℓ_r-basis, where $r = 1$ if $1 \leqslant q \leqslant 2$, and $1/r = 1/q + 1/2$ for $2 < q \leqslant \infty$.

Proposition

Suppose $(e_k) \subset E$ is equivalent to the canonical basis of c_0. Then $(|\phi_E(e_k)|) \subset \mathrm{FBL}^{(p)}[E]$ is equivalent to the canonical ℓ_2 basis for $1 \leqslant p < \infty$, and to the canonical c_0-basis for $p = \infty$.

$(e_k) \subset E$ versus $(|\phi_E(e_k)|) \subset \mathrm{FBL}^{(p)}[E]$, part II

Proposition

Suppose (e_k) is a semi-normalized basis of E. Then, for $1 \leqslant p < \infty$, $(|\phi_E(e_k)|) \subset \mathrm{FBL}^{(p)}[E]$ dominates the canonical ℓ_2.

Theorem

Suppose (e_k) is an unconditional basis in E, and $p \in [1, \infty)$. Then $(|\phi_E(e_k)|) \subset \mathrm{FBL}^{(p)}[E]$ is equivalent to the canonical ℓ_2-basis iff (e_k) is the canonical basis for c_0.

Proposition

If $(e_k) \subset C(K)$ is equivalent to the canonical ℓ_2-basis, then, for $1 \leqslant p \leqslant \infty$, $(|\phi_E(e_k)|) \subset \mathrm{FBL}^{(p)}[C(K)]$ is equivalent to the canonical ℓ_2-basis.

Question. Under what conditions on the basis $(e_i) \subset E$ will $(|\phi_E(e_k)|)$ be equivalent to a prescribed sequence (such as ℓ_q-basis)?

Thank you for your attention! Questions welcome!