Generating bounded operators Prepublicaciones (2022)

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1. Introduction

The well-known Catalan numbers $(C_n)_{n\geq 0}$ given by the formula

$$C_n=rac{1}{n+1}\binom{2n}{n}, \quad n\geq 0,$$

appear in a wide range of problems. For instance, the Catalan number C_n counts the number of ways to triangulate a regular polygon with n + 2 sides; or, the number of ways that 2n people seat around a circular table are simultaneously shaking hands with another person at the table in such a way that none of the arms cross each other.

 $1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, \ldots$



Eugène Charles Catalan (1814 - 1894)

Triangulating Polygons



Leonhard Euler 1707 - 1783 In 1751, Euler asked the question:

In how many ways can we divide a convex polygon with n sides into triangles using non-intersecting diagonals? Letters between L. Euler and C. Goldbach (196) 1750's Euler wrote J.A. von Segner and Segner proved in 1758

$$C_{n+1} = C_0 C_n + C_1 C_{n-1} + \ldots C_1 C_{n-1} + C_0 C_n, \qquad n \ge 0,$$

We calculated C_n with $n \le 18$ but we made a mistake $C_{13} = 742,900$ which invalidated all larger values.

Kotelnikow (1766), Fuss (1795), Liouville (1836, *Journal of Mathématiques Pures and Appliquées*) Lamé (1838)

The first paper of E. C. Catalan (JMPA, 1838) III.

On sait que le $(n + 1)^{\epsilon}$ nombre figuré de l'ordre n + 1, a pour expression, $C_{2n,n}$: si donc, dans la table des nombres figurés, on prend ceux qui occupent la diagonale; savoir :

1, 2, 6, 20, 70, 252, 924...;

qu'on les divise respectivement par

1, 2, 3, 4, 5, 6, 7...;

on obtiendra une nouvelle suite de nombres,

1, 1, 2, 5, 14, 42, 132..., (A)

lesquels jouiront de cette propriété :

Un terme quelconque de la suite (A) est égal à la somme des produits que l'on obtient en écrivant au-dessous d'elle-même, et dans un ordre inverse, la série des termes précédents, et en multipliant les termes correspondants des deux séries.

Par exemple,

132 = 1.42 + 1.14 + 2.5 + 5.2 + 14.1 + 42.1.Tome III. - Octobere 1838 65

The last paper of E. C. Catalan (Rend.Con.MatPal, 1887) SUR LES NOMBRES DE SEGNER (*)

Par M. Eugène Catalan, à Liége

(Seduta del 19 dicembre 1886)

I. INTRODUCTION.

I. Divers Géomètres se sont occupés de ce problème : De combien de manières un polygone convexe, de n côlés, peut-il être décomposé en triangles, au moyen de diagonales ? (**)

Soit T_n le nombre des décompositions. On sait que

$$T_4 = 2, T_5 = 5, T_6 = 14, T_7 = 42, \dots$$

Les nombres T_n , considérés par Segner (***), satisfont aux relations

$$T_{n+1} = T_{2}T_{n} + T_{3}T_{n-1} + \ldots + T_{n-1}T_{3} + T_{n}T_{2}$$
 (****), (1)

The Ballot problem. Suppose A and B are candidates for office and there are 2n voters, n voting for A and n for B. In how many ways can the ballots be counted so that B is never ahead of A? The solution is a Catalan number C_n .

John Riordan introduced the term "Catalan number" in Math Reviews in 1948 and 1964.

Finally Riordan used "Catalan number" in *Combinatorial identies* (1968).

Martin Gardner used the term in his "Mathematical Games" column in Scientific American in 1976.

The generating function of the Catalan sequence $c = (C_n)_{n \ge 0}$ is defined by

$$C(z) := \sum_{n=0}^{\infty} C_n z^n = \frac{1 - \sqrt{1 - 4z}}{2z}, \quad z \in D(0, \frac{1}{4}).$$
 (1)

This functions satisfies

$$zy^2 - y + 1 = 0.$$

The second solution is given by

$$rac{1}{zC(z)}=rac{1+\sqrt{1-4z}}{2z},\quad z\in D(0,rac{1}{4})ackslash\{0\}.$$

The main aim of this talk is to consider the quadratic equation

$$TY^2 - Y + I = 0, \qquad Y \in \mathcal{B}(X)$$

where

(1) $T \in \mathcal{B}(X)$ ([MR]). (2) T is the infinitesimal generator of a C second

(2) T is the infinitesimal generator of a C_0 -semigroup ([MM]).

$$Y = \frac{1 \pm \sqrt{1 - 4T}}{2T}$$

The problem of quadratic equation.

The study of quadratic equations in Banach space is much complicated than in the scalar case. There are infinite symmetric square roots of $I_2 \in \mathbb{R}^{2 \times 2}$ given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \begin{pmatrix} a & \pm\sqrt{1-a^2} \\ \pm\sqrt{1-a^2} & -a \end{pmatrix}$$

with $a \in [-1, 1]$, i.e., solutions of $Y^2 = I_2$.

As far as we are aware, no useful necessary and sufficient conditions for the existence of solution of quadratic equations in Banach spaces are known, even in the classical case of finite-dimensional spaces and square roots.

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2. New results about Catalan numbers

Theorem

Let A be a commutative algebra over \mathbb{R} or \mathbb{C} , $x \in A$ and y and z solutions of the quadratic equations

$$xy^2 - y + 1 = 0, \qquad -xz^2 - z + 1 = 0.$$

Then
$$\frac{y+z}{2}$$
 is a solution of the quartic equation
$$4x^2w^4 - w^2 + 1 = 0.$$

Remark.

$$\frac{y+z}{2} = 2x\left(\frac{y-z}{2}\right)\left(\frac{y+z}{2}\right).$$

Proposition

Let $c = (C_n)_{n \ge 0}$ be the Catalan sequence. Then

$$C_e(z) := \sum_{n=0}^{\infty} C_{2n} z^{2n} = \frac{\sqrt{1+4z} - \sqrt{1-4z}}{4z},$$
$$C_o(z) := \sum_{n=0}^{\infty} C_{2n+1} z^{2n+1} = \frac{2 - \sqrt{1+4z} - \sqrt{1-4z}}{4z},$$

for $|z| \leq rac{1}{4}$. In particular, $4z^2C_e^4(z)-C_e(z)^2+1=0$,

$$C_o(z)=\frac{C_e(z)-1}{2zC_e(z)}.$$

Catalan numbers have several integral representations, for example

$$C_n = rac{1}{2\pi} \int_0^4 t^n \sqrt{rac{4-t}{t}} dt = rac{2^{2n+1}}{\pi} eta(rac{3}{2}, n+rac{1}{2}),$$

Theorem Given $1 \neq z \in \mathbb{C}^+$, then

$$\int_0^\infty \frac{\sqrt{t}}{(t+1)(t+z)} dt = \frac{\pi}{z-1} \left(\sqrt{z} - 1 \right),$$

$$\int_0^\infty \frac{\sqrt{t}}{(t+1)(t+z)^{j+1}} dt = \frac{\pi}{2\sqrt{z}(z-1)^j} \sum_{k=j}^\infty C_k \left(\frac{z-1}{4z} \right)^k,$$

for $j \ge 1$ and where the last equality holds for $\Re(z) \ge \frac{1}{2}$.

3. The sequence of Catalan numbers

$$\lim_{z \to \frac{1}{4}} C(z) = \sum_{n=0}^{\infty} \frac{C_n}{4^n} = 2.$$

We consider the weight Banach algebra $\ell^1(\mathbb{N}^0, \frac{1}{4^n})$. This algebra is formed by sequence $a = (a_n)_{n \ge 0}$ such that

$$\|a\|_{1,\frac{1}{4^n}} := \sum_{n=0}^{\infty} \frac{|a_n|}{4^n} < \infty,$$

and the product is the usual convolution * defined by

$$(a * b)_n = \sum_{j=0}^n a_{n-j}b_j, \qquad a, b \in \ell^1(\mathbb{N}^*, \frac{1}{4^n}).$$

The base $\{\delta_j\}_{j\geq 0}$ is defined by $(\delta_j)_n := \delta_{j,n}$ is the delta Kronecker.

This Banach algebra has identity element, δ_0 , its spectrum set is $\overline{D(0, \frac{1}{4})}$ and its Gelfand transform is given by the Z-transform

$$Z(a)(z) := \sum_{n=0}^{\infty} a_n z^n, \qquad z \in \overline{D(0, \frac{1}{4})}.$$

It is straightforward to check that $Z(\delta_n)(z) = z^n$ for $n \ge 0$.

Proposition

Take $c = (C_n)_{n \ge 0}$. Then (i) $\|c\|_{1,\frac{1}{4^n}} = 2$. (ii) C(z) = Z(c)(z) for $z \in D(0,\frac{1}{4})$. (iii) $\delta_1 * c^{*2} - c + \delta_0 = 0$. The resolvent set $\rho(a) := \{\lambda \in \mathbb{C} : (\lambda \delta_0 - a)^{-1} \in \ell^1(\mathbb{N}^0, \frac{1}{4^n})\},\$ and the spectrum of *a* is given by $\sigma(a) := \mathbb{C} \setminus \rho(a).$

Proposition

The spectrum of the Catalan sequence $c = ((C_n))_{n\geq 0}$ is given by $\sigma(c) = C(\overline{D(0, \frac{1}{4})})$ and its boundary by

$$\partial(\sigma(c)) = \{2e^{-i heta}\left(1-\sqrt{2|\sin(rac{ heta}{2})|}e^{rac{i(\pi- heta)}{4}}
ight) \ : \ heta\in(-\pi,\pi)\}.$$



Given $\lambda \in \mathbb{C}$, we consider the geometric progression $p_{\lambda} = (\frac{1}{\lambda^n})_{n \geq 0}$. Note that $p_{\lambda} \in \ell^1(\mathbb{N}^0, \frac{1}{4^n})$ if and only if $|\lambda| > \frac{1}{4}$. Moreover

$$egin{aligned} &(\lambda-\delta_1)^{-1}=rac{1}{\lambda} oldsymbol{p}_\lambda, & |\lambda|>rac{1}{4}, \ &\Omega:=\{\lambda\in\mathbb{C}\;:\; \left|rac{\lambda-1}{\lambda^2}
ight|>rac{1}{4}\}. \end{aligned}$$

Theorem

The inverse of the Catalan sequence c is given $c^{-1} = \delta_0 - \delta_1 * c$ and

$$(\lambda-c)^{-1}=rac{\delta_0}{\lambda}+rac{1}{\lambda(\lambda-1)}p_{rac{\lambda-1}{\lambda^2}}+rac{1}{\lambda^2}c-rac{1}{\lambda^2}c*p_{rac{\lambda-1}{\lambda^2}},\qquad\lambda\in\Omegaackslash\{0\}.$$



The set $\partial(\Omega)$ in blue and $\partial(\sigma(c))$ in red.

4. Inverse spectral mapping theorem.

$$TY^2 - Y + I = 0, \qquad Y \in \mathcal{B}(X). \tag{2}$$

Lemma

Given $T \in \mathcal{B}(X)$ and Y a solution of (2). Then Y has left-inverse and $Y_l^{-1} = I - TY$.

Theorem

Given $T \in \mathcal{B}(X)$ and Y a solution of (2). Then TFAE

(i) $0 \in \rho(Y)$. (ii) $T = Y^{-1} - Y^{-2}$. (iii) T and Y commute. (iv) $TY^2 = YTY$.

Corollary

Let X be a Banach space with $dim(X) < \infty$, $T \in \mathcal{B}(X)$ and Y a solution of (2). Then Y is invertible, T and Y commute and

$$T = Y^{-1} - Y^{-2}$$
.

Theorem Given $T \in \mathcal{B}(X)$ and Y a solution of (2) such that $0 \in \rho(Y)$. (i) Given $\lambda \in \mathbb{C}$ such that $\frac{\lambda - 1}{\lambda^2} \in \rho(T)$ then $\lambda \in \rho(Y)$ and

$$(\lambda - Y)^{-1} = \frac{1}{\lambda} + \frac{1}{\lambda^3} (\frac{\lambda - 1}{\lambda^2} - T)^{-1} + \frac{Y}{\lambda^2} - \frac{(\lambda - 1)Y}{\lambda^4} (\frac{\lambda - 1}{\lambda^2} - T)^{-1}.$$

(ii) Given
$$\lambda \in \rho(Y)$$
 such that $\frac{\lambda}{\lambda - 1} \in \rho(Y)$ then $\frac{\lambda - 1}{\lambda^2} \in \rho(T)$
and

$$\left(\frac{\lambda-1}{\lambda^2}-T\right)^{-1} = \frac{\lambda^4}{\lambda-1}\left(\frac{\lambda}{\lambda-1}-Y\right)^{-1}\left((\lambda-Y)^{-1}-\frac{\lambda+Y}{\lambda^2}\right)$$

٠

5. Catalan generating functions

In this section, $T \in \mathcal{B}(X)$, such that

$$\sup_{n\geq 0} \|4^n T^n\| := M < \infty, \tag{3}$$

i.e., 4T is a power-bounded operator. Then $\sigma(T) \subset D(0, \frac{1}{4})$ and we define the following bounded operator,

$$C(T) := \sum_{n \ge 0} C_n T^n.$$
(4)

Theorem

Given $T \in \mathcal{B}(X)$ such that 4T is power-bounded and $c = (C_n)_{n \ge 0}$ the Catalan sequence. Then

 (i) The operator C(T) defined by (4) is well-defined, T and C(T) commute, and C(T) is a solution of the quadratic equation (2). (ii) The following integral representation holds

$$C(T)x = \frac{1}{\pi} \int_{\frac{1}{4}}^{\infty} \frac{\sqrt{\lambda - \frac{1}{4}}}{\lambda} (\lambda - T)^{-1} x d\lambda, \qquad x \in X.$$

(iii) The following integral representation holds

$$TC(T)=\frac{l}{2}-\sqrt{\frac{1}{4}-T}.$$

(iv) The spectral mapping theorem hols for C(T), i.e, $\sigma(C(T)) = C(\sigma(T))$ and

$$\sigma(\mathcal{C}(\mathcal{T})) \subset \mathcal{C}(\overline{D(0,\frac{1}{4})}) \subset \sigma(c).$$

(v) Given $\lambda \in \mathbb{C}$ such that $\frac{\lambda - 1}{\lambda^2} \in \rho(T)$ then $\lambda \in \rho(Y)$ and

$$(\lambda - C(T))^{-1} =$$

$$\frac{1}{\lambda} + \frac{1}{\lambda^3} \left(\frac{\lambda - 1}{\lambda^2} - T\right)^{-1} + \frac{C(T)}{\lambda^2} - \frac{(\lambda - 1)C(T)}{\lambda^4} \left(\frac{\lambda - 1}{\lambda^2} - T\right)^{-1}.$$

Remark. In the case that $\sigma(T) \subset D(0, \frac{1}{4})$, the generating function C(z) given in (1) is an holomorphic function in a neighborhood of $\sigma(T)$. Then the Dunford functional calculus, defined by the integral Cauchy-formula,

$$f(T)x = \int_{\Gamma} f(z)(z-T)^{-1}xdz, \qquad x \in X,$$

(Γ is a path around the spectrum set $\sigma(T)$) allows to defined C(T), ([Y, Section VIII.7]) which, of course, coincides with the expression gives in (4).

6. Examples, applications and final comments

6.1 Matrices on \mathbb{C}^2 We consider \mathbb{C}^2 and $T = \lambda I_2$ with $0 \neq \lambda \in \mathbb{C}$. Then the solution of (2) is given by

$$Y=egin{pmatrix}rac{1\pm\sqrt{1-4\lambda(1+\lambda bc)}}{2\lambda}&b\ c&rac{1\mp\sqrt{1-4\lambda(1+\lambda bc)}}{2\lambda}\end{pmatrix},$$

for |c|+|b|> 0, the allowed signs are (-,+) and (+,-); and

$$Y = egin{pmatrix} rac{1\pm\sqrt{1-4\lambda}}{2\lambda} & 0 \ 0 & rac{1\pm\sqrt{1-4\lambda}}{2\lambda} \end{pmatrix},$$

for c = b = 0. In both cases, note that $\sigma(Y) = \{C(\lambda), \frac{1}{\lambda C(\lambda)}\}$ and $\sigma(T) = \{\lambda\}$. For $|\lambda| \leq \frac{1}{4}$.

$$C(T) = \begin{pmatrix} C(\lambda) & 0 \\ 0 & C(\lambda) \end{pmatrix}$$

Now we consider $T = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}$ with $\lambda \in \mathbb{C} \setminus \{0\}$. The solutions of (2) are given by $Y = \begin{pmatrix} a & \frac{a-1}{2\lambda a} \\ \frac{a-1}{2\lambda a} & a \end{pmatrix}$

where *a* is a solution of the quartic equation $4\lambda^2 a^4 - a^2 + 1 = 0$. In the case that $|\lambda| \leq \frac{1}{4}$, we get that

$$C(T) = \begin{pmatrix} C_e(\lambda) & C_o(\lambda) \\ C_o(\lambda) & C_e(\lambda) \end{pmatrix}$$

where functions C_e and C_o are defined in Proposition 2.1

6. Examples, applications and final comments

6.2 Catalan operators on ℓ^p We consider the space of sequences $\ell^p(\mathbb{N}^0,\frac{1}{4^n})$ where

$$\|a\|_{p,\frac{1}{4^n}} := \left(\sum_{n=0}^{\infty} \frac{|a_n|^p}{4^{np}}\right)^{\frac{1}{p}} < \infty, \qquad \|a\|_{\infty,\frac{1}{4^n}} := \sup_{n \ge 0} \frac{|a_n|}{4^n} < \infty.$$

for $1 \leq p \leq \infty$. Note that $\ell^1(\mathbb{N}^0, \frac{1}{4^n}) \hookrightarrow \ell^p(\mathbb{N}^0, \frac{1}{4^n}) \hookrightarrow \ell^\infty(\mathbb{N}^0, \frac{1}{4^n})$.

Now we consider the convolution operator C(f) := c * f for $f \in \ell^{p}(\mathbb{N}^{0}, \frac{1}{4^{n}})$ with $1 \leq p \leq \infty$. Since $C(f) = \sum_{n \geq 0} c_{n}(\delta_{1})^{n}(f)$, then

$$\sigma(C) = C(\sigma(\delta_1)) = C(D(0, \frac{1}{4}))$$

i.e., it is independent on p and equal to $\sigma(c)$ in $\ell^1(\mathbb{N}^0, \frac{1}{4^n})$.

Now we consider $\ell^p(\mathbb{Z})$ for $1 \leq p \leq \infty$ and $a = \delta_1 - \delta_0$ defines

$$a*(f)(n):=f(n-1)-f(n), \qquad f\in\ell^p(\mathbb{Z})$$

for $n \in \mathbb{Z}$. Note that ||a|| = 2, and

$$(\lambda\delta_0+a)^{-1}=\sum_{j\geq 0}rac{\delta_j}{(1+\lambda)^{j+1}},\quad 1<|1+\lambda|,$$

see [GLM, Theorem 3.3 (4)]. Now we need to consider $\frac{a}{8}$ and the associated Catalan generating operator $C(\frac{a}{8})$. Then

$$C(\frac{a}{8}) = \frac{8}{\pi} \int_{\frac{1}{4}}^{\infty} \frac{\sqrt{\lambda - \frac{1}{4}}}{\lambda} (8\lambda\delta_0 + a)^{-1} x d\lambda$$

= $\frac{4}{\pi} \sum_{j \ge 0} \frac{\delta_j}{2^{j+1}} \int_0^{\infty} \frac{\sqrt{t}}{(t+1)(t+\frac{3}{2})^{j+1}} dt$
= $(2\sqrt{6} - 4)\delta_0 + \sum_{j=1}^{\infty} \left(\frac{\sqrt{6}}{3} \sum_{k=j}^{\infty} \frac{C_k}{12^k}\right) \delta_j.$

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