Relations between the linear and the lattice structure of Banach lattices

joint work with A. Avilés, A. Rueda Zoca and P. Tradacete

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In particular, $\|x\| = \||x|\|$.

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- ℓ_p and $L_p(\mu)$ are Banach lattices for every $1 \leq p \leq \infty$.
- Any Banach space with a 1-unconditional basis has a natural structure of Banach lattice.

L. Kantorovič and his school first recognized the importance of studying vector lattices in connection with Banach's theory of normed vector spaces; they investigated normed vector lattices as well as order-related linear operators between such vector lattices. (Cf. Kantorovič-Vulikh-Pinsker [1950] and Vulikh [1967].) However, in the years following that early period, functional analysis and vector lattice theory began drifting more and more apart; it is my impression that "linear order theory" could not quite keep pace with the rapid development of general functional analysis and thus developed into a theory largely existing for its own sake, even though it had interesting and beautiful applications here and there.

H.H. Schaefer, 1974

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If, in addition, X and Y are Banach lattices and T is also a lattice homomorphism then X is said to be lattice embeddable into Y.

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A Banach lattice is Dedekind σ-complete whenever every nonempty countable subset bounded above has a supremum.

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EMBEDDINGS OF $C(\Delta)$ AND $L^1[0, 1]$ IN BANACH LATTICES

BY

HEINRICH P. LOTZ AND HASKELL P. ROSENTHAL†

ABSTRACT

It is proved that if E is a separable Banach lattice with E' weakly sequentially complete, F is a Banach space and $T: E \to F$ is a bounded linear operator with $T'F'$ non-separable, then there is a subspace G of E, isomorphic to $C(\Delta)$, such that $T_{|G}$ is an isomorphism, where $C(\Delta)$ denotes the space of continuous real valued functions on the Cantor discontinuum. This generalizes an earlier result of the second-named author. A number of conditions are proved equivalent for a Banach lattice E to contain a subspace order isomorphic to $C(\Delta)$. Among them are the following: L^1 is lattice isomorphic to a sublattice of E'; $C(\Delta)'$ is lattice isomorphic to a sublattice of E'; E contains an order bounded sequence with no weak Cauchy subsequence; E has a separable closed sublattice F such that F' does not have a weak order unit.

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Theorem (H.P. Lotz and H.P. Rosenthal, 1978)

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5. *there exists a lattice embedding* $T\colon C[0,1] \longrightarrow X$ *(A. Avilés, G.M.C., A. Rueda Zoca, P. Tradacete, 2021);*

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For any compact spaces K and L and any lattice homomorphism $T : C(K) \longrightarrow C(L)$ there exist continuous functions $u : L \longrightarrow [0,\infty)$ and $h : L \setminus u^{-1}(\{0\}) \longrightarrow K$ such that $T(f)(t) = u(t)f(h(t))$ whenever $t \in L \setminus u^{-1}(\{0\}$ and $T(f)(t) = 0$ otherwise.

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Definition (B. de Pagter, A.W. Wickstead, 2015)

A Banach lattice P is said to be projective if, whenever X is a Banach lattice, J is a closed ideal in X and $Q\colon X \longrightarrow X/J$ is the quotient map, for every lattice homomorphism $T\colon P \longrightarrow X/J$ and every $\varepsilon > 0$ there exists a lattice homomorphism $\widehat{T}\colon P \longrightarrow X$ such that $T = Q \circ \widehat{T}$ and $\|\widehat{T}\| \leq (1+\varepsilon)\|T\|$.

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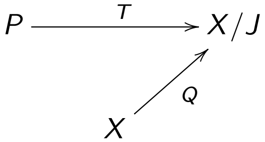

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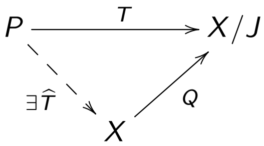

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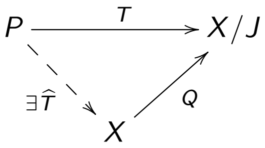

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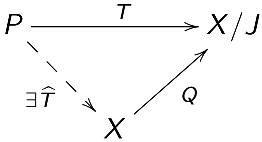

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Corollary

Let X be a Banach lattice and J be a closed ideal in X. If $C[0,1]$ is lattice embeddable into X/J then $C[0,1]$ is lattice embeddable into X.

Theorem (A. Avilés, G.M.C., A. Rueda Zoca, P. Tradacete, 2021)

The Banach lattice $C[0,1]$ *is linearly embeddable in a Banach lattice if and only if it is lattice embeddable.*

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- Instead of working directly with $C[0,1]$ we construct a sublattice $Z \subseteq X$ with a quotient lattice isomorphic to $C(\Delta)$.

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For a separable Banach lattice X the following conditions are equivalent:

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An infinite-dimensional sublattice of $C[0,1]$ is either lattice isomorphic to c_0 or it contains a sublattice isomorphic to $C[0,1]$.

Open Problems

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Thank you for your attention.