# $C(K)$-spaces associated with almost disjoint families (Part II)

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Main parts: joint work with Piotr Koszmider (IMPAN, Warsaw)
and with Max Arnott (Lancaster)

## Overview

Today:

- An addition to yesterday's talk.
- Outline of the proof of the main theorem (joint work with Koszmider).
- Automatic continuity of homomorphisms from $\mathscr{B}(X)$.
- Uniqueness of algebra norm on all quotients of $\mathscr{B}(X)$ by its closed ideals (joint work with Arnott).

## $C(K)$-spaces for which all complemented subspaces are known

Yesterday I gave the following list:

- $c_0$ is prime (Pełczyński, *Studia Math.* 1960), that is, every complemented, $\infty$-dimensional subspace of $c_0$ is isomorphic to $c_0$.
- More generally, for $\Gamma$ of arbitrary cardinality, every complemented subspace of $c_0(\Gamma)$ is isomorphic to $c_0(\Delta)$ for some $\Delta \subseteq \Gamma$ (Granero, 1998).
- Every complemented, $\infty$-dimensional subspace of $C[0, \omega^\omega]$ is isomorphic to either $c_0$ or $C[0, \omega^\omega]$ (Benyamini, *Israel J. Math.* 1978).
- $\ell_\infty$ is prime (Lindenstrauss, *Israel J. Math.* 1967).

However, this list is **incomplete**. Additional results are:

- $C_0(K_{\mathcal{A}})$, where $\mathcal{A} \subseteq [\mathbb{N}]^\omega$ is an uncountable, almost disjoint family admitting few operators.

  Suppose that $C_0(K_{\mathcal{A}}) = X \oplus Y$ for some closed, $\infty$-dimensional subspaces $X$ and $Y$. Then $X \cong C_0(K_{\mathcal{A}})$ and $Y \cong c_0$, or vice versa (Koszmider, *PAMS* 2005).
- $\ell_\infty^c(\Gamma)$ for any uncountable set $\Gamma$.

  Every $\infty$-dimensional complemented subspace of $\ell_\infty^c(\Gamma)$ is isomorphic to either $\ell_\infty$ or $\ell_\infty^c(\Delta)$ for some uncountable $\Delta \subseteq \Gamma$ (Johnson–Kania–Schechtman, *PAMS* 2016). *(to be continued)*

## $C(K)$-spaces for which all complemented subspaces are known (continued)

Finally, all complemented subspaces are known for any $C(K)$-space of the form

$$C(K \sqcup L) \cong C(K) \oplus C(L),$$

where $K$ is scattered and $C(L)$ is a Grothendieck space, and all the complemented subspaces of $C(K)$ and $C(L)$ are classified (Johnson–Kania–Schechtman, *PAMS* 2016).

Specifically, this result applies in the following cases:

- $C(K) = c_0(\Gamma)$ for infinite $\Gamma$, or $C(K) = C[0, \omega^\omega]$, or $C_0(K_{\mathcal{A}})$ for $\mathcal{A} \subseteq [\mathbb{N}]^\omega$ an uncountable, almost disjoint family admitting few operators; and
- $C(L) = \ell_\infty$ or $C(L) = \ell_\infty^c(\Upsilon)$ for $\Upsilon$ uncountable.

# Proof of the main theorem (joint work with Piotr Koszmider)

Aim: Outline the main steps of the proof of following result.

**Theorem** (Koszmider–L, *Adv. Math.* 2021, working within ZFC).
*There is an uncountable, almost disjoint family* $\mathcal{A} \subseteq [\mathbb{N}]^\omega$ *such that*

$$\mathscr{B}(C_0(K_\mathcal{A})) = \mathbb{K}\,\mathrm{Id} + \mathscr{X}(C_0(K_\mathcal{A})),$$

*where*

$$\mathscr{X}(C_0(K_\mathcal{A})) = \{T \in \mathscr{B}(C_0(K_\mathcal{A})) : T \text{ has separable range}\}.$$

Recall: Since $K_\mathcal{A}$ is scattered, for $T \in \mathscr{B}(C_0(K_\mathcal{A}))$, $f \in C_0(K_\mathcal{A})$ and $s \in K_\mathcal{A}$, we have

$$T(f)(s) = \langle T(f), \delta_s \rangle = \langle f, T^*(\delta_s) \rangle = \sum_{t \in K_\mathcal{A}} T^*(\delta_s)(\{t\}) f(t),$$

where $\delta_s \in C_0(K_\mathcal{A})^*$ is the point evaluation at $s$.

We regard $\big(T^*(\delta_s)(\{t\})\big)_{s,t \in K_\mathcal{A}}$ as a "**matrix representation**" of $T$.

# Using $\mathbb{N} \times \mathbb{N}$-matrices to analyze operators on $C_0(K_{\mathcal{A}})$

Recall:

- $\big(T^*(\delta_s)(\{t\})\big)_{s,t\in K_{\mathcal{A}}}$ is a matrix representation of $T \in \mathscr{B}(C_0(K_{\mathcal{A}}))$.
- $K_{\mathcal{A}} = \{x_n : n \in \mathbb{N}\} \cup \{y_A : A \in \mathcal{A}\}$, where $x_n$ is isolated for every $n \in \mathbb{N}$ and $x_n \underset{A\ni n\to\infty}{\longrightarrow} y_A$ for every $A \in \mathcal{A}$.

**Definition.** The ***reduced matrix*** of $T \in \mathscr{B}(C_0(K_{\mathcal{A}}))$ is

$$M_T^{\mathrm{red}} = \big(T^*(\delta_{x_k})(\{x_n\})\big)_{k,n\in\mathbb{N}}.$$

**Fact.** $M_T^{\mathrm{red}} \in \mathbb{M}$, where

$$\mathbb{M} = \Big\{M = (m_{k,n})_{k,n\in\mathbb{N}} : \|M\| := \sup_{k\in\mathbb{N}} \sum_{n\in\mathbb{N}} |m_{k,n}| < \infty\Big\}.$$

These are precisely the matrices representing operators $c_0 \to \ell_\infty$ via

$$(Mf)(k) = \sum_{n\in\mathbb{N}} m_{k,n} f(n) \qquad (k \in \mathbb{N})$$

for $f \in c_0$ and $M = (m_{k,n})_{k,n\in\mathbb{N}} \in \mathbb{M}$.

*Note:* The above formula makes sense for $f \in \ell_\infty$.

**Fact.** The matrices in $\mathbb{M}$ are the transposes of operators on $\ell_1$, or in other words the weak*-continuous operators on $\ell_\infty$.

## An illustrative calculation

Let $T \in \mathscr{B}(C_0(K_{\mathcal{A}}))$ for some almost disjoint family $\mathcal{A} \subseteq [\mathbb{N}]^{\omega}$, and take $A, B \in \mathcal{A}$. Then $1_{U(A)} \in C_0(K_{\mathcal{A}})$, where $U(A) = \{x_n : n \in A\} \cup \{y_A\}$, and

$$x_k \underset{B \ni k \to \infty}{\longrightarrow} y_B, \qquad \text{so} \qquad (T1_{U(A)})(x_k) \underset{B \ni k \to \infty}{\longrightarrow} (T1_{U(A)})(y_B).$$

Now

$$\begin{aligned}(T1_{U(A)})(x_k) &= \langle T1_{U(A)}, \delta_{x_k} \rangle = \langle 1_{U(A)}, T^*\delta_{x_k} \rangle = \sum_{t \in K_{\mathcal{A}}} (T^*\delta_{x_k})(\{t\}) 1_{U(A)}(t) \\ &= \sum_{t \in U(A)} (T^*\delta_{x_k})(\{t\}) = \sum_{n \in A} \underbrace{(T^*\delta_{x_k})(\{x_n\})}_{=(M_T^{\mathbf{red}})_{k,n}} + (T^*\delta_{x_k})(\{y_A\}).\end{aligned}$$

Recall: $T^*\delta_{x_k} \in C_0(K_{\mathcal{A}})^* \cong \ell_1(K_{\mathcal{A}})$, so

$$\mathcal{A}_T := \bigcup_{k \in \mathbb{N}} \{A \in \mathcal{A} : (T^*\delta_{x_k})(\{y_A\}) \neq 0\} \quad \text{is countable.}$$

For $A \in \mathcal{A} \setminus \mathcal{A}_T$,

$$(T1_{U(A)})(y_B) \underset{B \ni k \to \infty}{\longleftarrow} (T1_{U(A)})(x_k) = \sum_{n \in A} (M_T^{\mathbf{red}})_{k,n} = (M_T^{\mathbf{red}} 1_A)(k).$$

**Conclusion:** $\big((M_T^{\mathbf{red}} 1_A)(k)\big)_{k \in B}$ converges for all but countably many $A, B \in \mathcal{A}$.

*Let $\mathcal{A} \subseteq [\mathbb{N}]^\omega$ be an almost disjoint family such that, for every $M \in \mathbb{M}$, one of the following three conditions holds:*

(i) *There are uncountably many $A \in \mathcal{A}$ that* ***reject*** *$M$, in the sense that*

$$\big((M1_A)(k)\big)_{k\in A} \quad \textit{diverges.}$$

(ii) *Or there are uncountably many $A \in \mathcal{A}$ that* ***undermine*** *$M$, in the sense that there is $n \in \mathbb{N}$ such that*

$$(M1_{\{n\}})(k) \underset{A\ni k\to\infty}{\not\longrightarrow} 0.$$

(iii) *Or there are $\lambda \in \mathbb{K}$ and a countable subset $\mathcal{A}' \subseteq \mathcal{A}$ such that $\mathcal{A} \setminus \mathcal{A}'$* ***admits*** *$M - \lambda I$, in the sense that*

$$((M-\lambda I)1_A)(k) \underset{B\ni k\to\infty}{\longrightarrow} 0 \qquad (A, B \in \mathcal{A} \setminus \mathcal{A}'),$$

*where $I = (\delta_{k,n})_{k,n\in\mathbb{N}} \in \mathbb{M}$.*

*Then*

$$\mathscr{B}(C_0(K_{\mathcal{A}})) = \mathbb{K}\,\mathrm{Id} + \mathscr{X}(C_0(K_{\mathcal{A}})).$$

**Task:** Construct an uncountable almost disjoint family $\mathcal{A} \subseteq [\mathbb{N}]^\omega$ as above.

## The notion of acceptance

Recall: $\mathcal{A}$ ***admits*** $M \in \mathbb{M}$ if

$$(M1_A)(k) \underset{B \ni k \to \infty}{\longrightarrow} 0 \qquad (A, B \in \mathcal{A}).$$

**Problem:** This notion is not closed under finite unions, or subsets.

**Definition.** For $A \in [\mathbb{N}]^\omega$, $M = (m_{k,n})_{k,n\in\mathbb{N}} \in \mathbb{M}$ and $j \in \mathbb{N}$, define

$$M_j^A = (m'_{k,n})_{k,n\in\mathbb{N}} \in \mathbb{M}, \quad \text{where} \quad m'_{k,n} = \begin{cases} m_{k,n} & \text{if } k, n \in A \text{ and } n > j \\ 0 & \text{otherwise.} \end{cases}$$

The set $A$ ***accepts*** $M$ if

$$\|M_j^A\| \to 0 \quad \text{as} \quad j \to \infty.$$

*Note:* This is equivalent to saying that the operator $T\colon c_0 \to \ell_\infty$ induced by compressing the matrix $M$ to the set $A$ is compact because

$$T \text{ is compact} \quad \iff \quad TP_j \to T \quad \text{as} \quad j \to \infty.$$

**Definition.** An almost disjoint family $\mathcal{A} \subseteq [\mathbb{N}]^\omega$ ***accepts*** $M \in \mathbb{M}$ if the set $A \cup B$ accepts $M$ for every $A, B \in \mathcal{A}$.

## Outline proof of the main result

Begin with an uncountable almost disjoint family $\mathcal{C} \subseteq [\mathbb{N}]^\omega$ such that $\{1_C : C \in \mathcal{C}\}$ is a Borel subset of the Cantor cube $\{0,1\}^{\mathbb{N}}$, and define

$$\mathbb{M}' = \{M \in \mathbb{M} : \exists\, \lambda_M \in \mathbb{K}, \mathcal{C}_M \subset \mathcal{C} \text{ countable: } \mathcal{C} \setminus \mathcal{C}_M \text{ accepts } M - \lambda_M I\}.$$

If $\mathbb{M}' = \mathbb{M}$, set $\mathcal{A} = \mathcal{C}$ and go straight to Case 2 (next slide).

Otherwise enumerate $\mathbb{M} \setminus \mathbb{M}'$ as

$$\mathbb{M} \setminus \mathbb{M}' = \{M_\xi : \xi < \mathfrak{c}\}$$

with each matrix repeated continuum many times.

By transfinite recursion on $\xi < \mathfrak{c}$, choose $B_\xi, C_\xi \in \mathcal{C}$ such that

- there is $A_\xi \in [B_\xi \cup C_\xi]^\omega$ which rejects $M_\xi$,
- and $\{B_\eta, C_\eta\} \cap \{B_\xi, C_\xi\} = \emptyset$ for every $\eta < \xi < \mathfrak{c}$.

That this is possible relies on a classical **theorem** of Alexandroff and Hausdorff:

*Every uncountable Borel subset of $\{0,1\}^{\mathbb{N}}$ has cardinality $\mathfrak{c}$.*

Define $\mathcal{A} = \{A_\xi : \xi < \mathfrak{c}\}$.

**Check:** Every $M \in \mathbb{M}$ satisfies one of the conditions (i)–(iii) in the Reduction Lemma with respect to this choice of $\mathcal{A}$.

# Outline proof of the main result (continued)

**The Reduction Lemma.** *Let $\mathcal{A} \subseteq [\mathbb{N}]^\omega$ be an almost disjoint family such that, for every $M \in \mathbb{M}$, one of the following three conditions holds:*

(i) *There are uncountably many $A \in \mathcal{A}$ that reject $M$.*

(ii) *Or there are uncountably many $A \in \mathcal{A}$ that undermine $M$.*

(iii) *Or there are $\lambda \in \mathbb{K}$ and a countable subset $\mathcal{A}' \subseteq \mathcal{A}$ such that $\mathcal{A} \setminus \mathcal{A}'$ admits $M - \lambda I$.*

*Then*

$$\mathscr{B}(C_0(K_{\mathcal{A}})) = \mathbb{K}\,\mathrm{Id} + \mathscr{X}(C_0(K_{\mathcal{A}})).$$

Recall:

- $\mathbb{M} \setminus \mathbb{M}' = \{M_\xi : \xi < \mathfrak{c}\}$, with each matrix repeated continuum many times;
- $\mathcal{A} = \{A_\xi : \xi < \mathfrak{c}\}$, where $A_\xi$ rejects $M_\xi$.

**Case 1.** If $M \in \mathbb{M} \setminus \mathbb{M}'$, then $M = M_\xi$ for continuum many $\xi < \mathfrak{c}$, and $A_\xi \in \mathcal{A}$ rejects $M$ for each of these $\xi$, so condition (i) in the Reduction Lemma holds.

**Case 2.** If

$$\mathcal{A}' := \{A \in \mathcal{A} : A \text{ undermines } M\}$$

is uncountable, then condition (ii) in the Reduction Lemma is satisfied.

# Outline proof of the main result (continued)

**Case 3.** It remains to consider the case where

$$\mathcal{A}' := \{A \in \mathcal{A} : A \text{ undermines } M\}$$

is countable and $M \in \mathbb{M}'$; that is, there are $\mathcal{C}_M \subseteq \mathcal{C}$ countable and $\lambda_M \in \mathbb{K}$ such that $\mathcal{C} \setminus \mathcal{C}_M$ accepts $M - \lambda_M I$.

*Recall:* Every $A \in \mathcal{A}$ is contained in $B \cup C$ for some $B, C \in \mathcal{C}$.

Observe: $B \in \mathcal{C}_M$ or $C \in \mathcal{C}_M$ for only countably many $A \in \mathcal{A}$.

Since "acceptance" passes to subsets of finite unions,

$$\exists\, \mathcal{A}'' \subseteq \mathcal{A} \text{ countable:} \quad \mathcal{A} \setminus \mathcal{A}'' \text{ accepts } M - \lambda_M I.$$

*Note:* no $A \in \mathcal{A} \setminus \mathcal{A}'$ undermines $M - \lambda_M I$ because, for every $n \in \mathbb{N}$,

$$(M - \lambda_M I)(1_{\{n\}})(k) = M(1_{\{n\}})(k) - \lambda_M 1_{\{n\}}(k) \underset{A \ni k \to \infty}{\longrightarrow} 0 - 0 = 0.$$

Hence

$$\left.\begin{array}{r} \mathcal{A} \setminus (\mathcal{A}' \cup \mathcal{A}'') \text{ accepts } M - \lambda_M I \\ \text{no } A \in \mathcal{A} \setminus (\mathcal{A}' \cup \mathcal{A}'') \text{ undermines } M - \lambda_M I \end{array}\right\} \Rightarrow \mathcal{A} \setminus (\mathcal{A}' \cup \mathcal{A}'') \text{ admits } M - \lambda_M I,$$

so condition (iii) in the Reduction Lemma is satisfied. $\square$

## Any questions?

Two questions from Koszmider–L:

- Is there within ZFC a *maximal* almost disjoint family $\mathcal{A} \subseteq [\mathbb{N}]^\omega$ such that

$$\mathscr{B}(C_0(K_\mathcal{A}))/\mathscr{X}(C_0(K_\mathcal{A})) \cong \mathbb{K}\,?$$

- Which unital Banach algebras of density at most $\mathfrak{c}$ are isomorphic to

$$\mathscr{B}(C_0(K_\mathcal{A}))/\mathscr{X}(C_0(K_\mathcal{A}))$$

  for some uncountable almost disjoint family $\mathcal{A} \subseteq [\mathbb{N}]^\omega$?

  We propose this as a counterpart for Mrówka spaces of the

  **Calkin algebra question:** Which unital Banach algebras are isomorphic to the quotient $\mathscr{B}(X)/\mathscr{K}(X)$ for some Banach space $X$?

*Mini-break*

**Terminology.** "Homomorphism" means algebra homomorphism.

**Theorem.** (Koszmider–L). *Let $\mathcal{A} \subseteq [\mathbb{N}]^{\omega}$ be an uncountable, almost disjoint family such that*

$$\mathscr{B}(C_0(K_{\mathcal{A}})) = \mathbb{K}\,\mathrm{Id} + \mathscr{X}(C_0(K_{\mathcal{A}})).$$

*Then every homomorphism from $\mathscr{B}(C_0(K_{\mathcal{A}}))$ into a Banach algebra is continuous.*

In fact, this result is an immediate consequence of a more general result:

**Theorem.** (Koszmider–L). *Let $\mathcal{A} \subseteq [\mathbb{N}]^{\omega}$ be an almost disjoint family. Then every homomorphism from $\mathscr{X}(C_0(K_{\mathcal{A}}))$ into a Banach algebra is continuous.*

**Question.** Is there an uncountable, almost disjoint family $\mathcal{A} \subseteq [\mathbb{N}]^{\omega}$ such that $\mathscr{B}(C_0(K_{\mathcal{A}}))$ admits a **dis**continuous homomorphism into a Banach algebra?

The automatic continuity of homomorphisms (or lack thereof) is one of the oldest lines of research in the theory of Banach algebras.

The seminal automatic-continuity result for $\mathscr{B}(X)$ is due to B.E. Johnson (*JLMS*, 1967):

**Theorem.** *Let $X$ be a Banach space which is isomorphic to its square $X \oplus X$. Then every homomorphism from $\mathscr{B}(X)$ into a Banach algebra is continuous.*

**Remark.** The hypothesis $X \cong X \oplus X$ can be weakened somewhat.

The conclusion that every homomorphism from $\mathscr{B}(X)$ into a Banach algebra is continuous remains true for many Banach spaces $X$ for which the hypothesis $X \cong X \oplus X$ fails (as well as any possible weakenings of it), notably:

- $X = J_p$ (the $p^{\text{th}}$ quasi-reflexive James space) for $1 < p < \infty$ (Willis, *Studia Math.* 1995).
- $X = C[0, \omega_1]$ (Ogden, *JLMS* 1996; simplified proof in Kania–Koszmider–L, *TLMS* 2014).

**Question.** Let $X$ be the Argyros–Haydon space (or the Gowers–Maurey space). Is every homomorphism from $\mathscr{B}(X)$ into a Banach algebra continuous?

Within ZFC, not many Banach spaces $X$ are known for which $\mathscr{B}(X)$ admits a discontinuous homomorphism into a Banach algebra.

The most important example is Read's Banach space $X_R$ such that $\mathscr{B}(X_R)$ admits a discontinuous "point derivation". (Read, *JLMS* 1989).

In fact, it has a stronger property:

Let $\ell_2^{\sim}$ be the unitization of the Hilbert space $\ell_2$ endowed with the trivial product; that is, $\ell_2^{\sim} = \ell_2 \oplus \mathbb{K}1$ as a vector space,

$$\|x + s1\| = \|x\|_2 + |s|$$

and

$$(x + s1)(y + t1) = tx + sy + st1 \qquad (x, y \in \ell_2,\ s, t \in \mathbb{K}).$$

**Theorem** (L–Skillicorn, *Studia Math.* 2017). *There is a continuous, surjective homomorphism* $\psi\colon \mathscr{B}(X_R) \to \ell_2^{\sim}$ *with* $\ker\psi = \mathscr{W}(X_R)$ *such that the extension*

$$\{0\} \longrightarrow \mathscr{W}(X_R) \longrightarrow \mathscr{B}(X_R) \xrightarrow{\ \psi\ } \ell_2^{\sim} \longrightarrow \{0\}$$

*splits in the category of Banach algebras.*

## Discontinuous homomorphisms from $\mathscr{B}(X)$ (assuming CH)

**Theorem** (Dales, *Am. J. Math.* 1979; Esterle, *PLMS* 1978; assuming CH). *Let $K$ be an infinite compact Hausdorff space. Then $C(K)$ admits a discontinuous homomorphism into a Banach algebra.*

**Corollary** (Assuming CH). *Let $X$ be a Banach space such that $\mathscr{B}(X)$ admits a continuous, surjective homomorphism onto $C(K)$ for some infinite compact Hausdorff space $K$. Then $\mathscr{B}(X)$ admits a discontinuous homomorphism into a Banach algebra.*

It is highly non-trivial that Banach spaces exist which satisfy the hypothesis of this corollary. Examples include:

- (Mankiewicz, *Israel J. Math.* 1989; Dales–Loy–Willis, *JFA* 1994) There is a Banach space $X$ such that $\mathscr{B}(X)$ admits a continuous, surjective homomorphism onto $\ell_\infty \cong C(\beta\mathbb{N})$.
- (Motakis, Puglisi and Zisimopoulou, *Indiana Univ. Math. J.* 2016) For every countably infinite, compact metric space $K$, there is a Banach space $X$ for which there is a continuous, surjective homomorphism $\varphi\colon \mathscr{B}(X) \to C(K)$. Moreover, $\ker\varphi = \mathscr{K}(X)$.
- (Motakis, preprint 2021). Same, but without requiring that the compact metric space $K$ be countably infinite.

- Koszmider's "first" $C(K)$-space with few operators.

  Recall: $K$ is an infinite, compact Hausdorff space such that

$$\mathscr{B}(C(K)) = \{M_f : f \in C(K)\} + \mathscr{W}(C(K)),$$

  where $M_f$ is the multiplication operator $M_f(g) = fg$.

  For this space, there is a continuous, surjective homomorphism $\varphi\colon \mathscr{B}(C(K)) \to C(K)$ with $\ker\varphi = \mathscr{W}(C(K))$.

  In fact, the extension

$$\{0\} \longrightarrow \mathscr{W}(C(K)) \longrightarrow \mathscr{B}(C(K)) \xrightarrow{\varphi} C(K) \longrightarrow \{0\}$$

  splits in the category of Banach algebras provided $K$ has no isolated points.

**Question.** Within ZFC, is there a compact Hausdorff space $K$ such that $\mathscr{B}(C(K))$ admits a discontinuous homomorphism into a Banach algebra?

Any questions?

# Uniqueness of algebra norm (joint work with Max Arnott)

**Aim:** For a Banach space $X$, decide whether every quotient of $\mathscr{B}(X)$ by a closed ideal has a unique algebra norm.

**Main result** (Arnott–L, in preparation). *Let $\mathcal{A} \subseteq [\mathbb{N}]^\omega$ be an uncountable, almost disjoint family such that*

$$\mathscr{B}(C_0(K_\mathcal{A})) = \mathbb{K}\,\mathrm{Id} + \mathscr{X}(C_0(K_\mathcal{A})).$$

*Then every quotient of $\mathscr{B}(C_0(K_\mathcal{A}))$ by a closed ideal has a unique algebra norm.*

**Definition.** A submultiplicative norm on an algebra $\mathscr{B}$ is an ***algebra norm***; that is,

$$\|ab\| \leqslant \|a\|\,\|b\| \quad \text{for every} \quad a, b \in \mathscr{B}.$$

We say that $\mathscr{B}$ has a ***unique algebra norm*** (up to equivalence) if every pair of algebra norms $\|\cdot\|$ and $|\!|\!|\cdot|\!|\!|$ on $\mathscr{B}$ are equivalent, that is, there are constants $C, c > 0$ such that

$$c\|a\| \leqslant |\!|\!|a|\!|\!| \leqslant C\|a\| \quad \text{for every} \quad a \in \mathscr{B}.$$

**Remark.** When $\mathscr{B}$ is unital, one usually requires that an algebra norm must satisfy $\|1\| = 1$. However, for simplicity, we ignore that convention. This will not cause problems because one can always pass to an equivalent algebra norm which satisfies $\|1\| = 1$.

Recall: $\mathscr{B}$ has a unique algebra norm if, for every pair of algebra norms $\|\cdot\|$ and $|||\cdot|||$, there are constants $C, c > 0$ such that

$$c\|a\| \leqslant |||a||| \leqslant C\|a\| \quad \text{for every} \quad a \in \mathscr{B}. \tag{†}$$

**Remarks.**

- It is more common to study whether $\mathscr{B}$ has a unique **complete** algebra norm (up to equivalence). To do so, it suffices to establish either of the two inqualities in (†) by the Banach Isomorphism Theorem.
- We shall study the two inequalities in (†) separately.

**Definition.** An algebra norm $\|\cdot\|$ on an algebra $\mathscr{B}$ is ***maximal*** if, for every algebra norm $|||\cdot|||$ on $\mathscr{B}$, there is a constant $C > 0$ such that

$$|||a||| \leqslant C\|a\| \quad \text{for every} \quad a \in \mathscr{B}.$$

**Proposition.** *The following conditions are equivalent for a Banach algebra* $\mathscr{B}$:

(a) *Its norm* $\|\cdot\|$ *is maximal.*

(b) *Every homomorphism from* $\mathscr{B}$ *into a Banach algebra is continuous.*

(c) *For every closed ideal* $\mathscr{I}$ *of* $\mathscr{B}$*, the quotient norm on* $\mathscr{B}/\mathscr{I}$ *is maximal.*

(d) *For every closed ideal* $\mathscr{I}$ *of* $\mathscr{B}$*, every homomorphism from* $\mathscr{B}/\mathscr{I}$ *into a Banach algebra is continuous.*

**Proposition.** *The following conditions are equivalent for a Banach algebra $\mathscr{B}$:*

(a) *Its norm $\|\cdot\|$ is maximal; that is, for every algebra norm $|\!|\!|\cdot|\!|\!|$ on $\mathscr{B}$, there is a constant $C > 0$ such that $|\!|\!|a|\!|\!| \leqslant C\|a\|$ for every $a \in \mathscr{B}$.*

(b) *Every homomorphism from $\mathscr{B}$ into a Banach algebra is continuous.*

(c) *For every closed ideal $\mathscr{I}$ of $\mathscr{B}$, the quotient norm on $\mathscr{B}/\mathscr{I}$ is maximal.*

(d) *For every closed ideal $\mathscr{I}$ of $\mathscr{B}$, every homomorphism from $\mathscr{B}/\mathscr{I}$ into a Banach algebra is continuous.*

*Proof.* (a)$\Rightarrow$(b). Given a homomorphism $\varphi\colon \mathscr{B} \to \mathscr{C}$,

$$|\!|\!|a|\!|\!| = \max\{\|a\|, \|\varphi(a)\|\} \quad \text{for} \quad a \in \mathscr{B}$$

defines an algebra norm on $\mathscr{B}$. By hypothesis, there is $C > 0$ such that

$$\|\varphi(a)\| \leqslant |\!|\!|a|\!|\!| \leqslant C\|a\| \quad \text{for every} \quad a \in \mathscr{B},$$

so $\varphi$ is continuous.

(b)$\Rightarrow$(a). Given an algebra norm $|\!|\!|\cdot|\!|\!|$ on $\mathscr{B}$, let $\mathscr{C}$ be the completion of $\mathscr{B}$ with respect to $|\!|\!|\cdot|\!|\!|$, and consider the inclusion map $a \mapsto a$, $\mathscr{B} \to \mathscr{C}$.

(c)$\Leftrightarrow$(d). Apply (a)$\Leftrightarrow$(b) with $\mathscr{B}/\mathscr{I}$ instead of $\mathscr{B}$.

(b)$\Rightarrow$(d). Apply the Open Mapping Theorem. (d)$\Rightarrow$(b). Clear! $\square$

**Definition.** An algebra norm $\|\cdot\|$ on an algebra $\mathscr{B}$ is ***minimal*** if, for every algebra norm $|\!|\!|\cdot|\!|\!|$ on $\mathscr{B}$, there is a constant $C > 0$ such that

$$\|a\| \leqslant C|\!|\!|a|\!|\!| \quad \text{for every} \quad a \in \mathscr{B}.$$

**Proposition.** *The following conditions are equivalent for a Banach algebra* $\mathscr{B}$:

(a) *Its norm* $\|\cdot\|$ *is minimal.*

(b) *Every injective homomorphism from* $\mathscr{B}$ *into a Banach algebra is bounded below.*

*Proof.* (a)⇒(b). Given an injective homomorphism $\varphi\colon \mathscr{B} \to \mathscr{C}$,

$$|\!|\!|a|\!|\!| = \|\varphi(a)\| \quad \text{for} \quad a \in \mathscr{B}$$

defines an algebra norm on $\mathscr{B}$. By hypothesis, there is $C > 0$ such that

$$\|a\| \leqslant C|\!|\!|a|\!|\!| = C\|\varphi(a)\| \quad \text{for every} \quad a \in \mathscr{B},$$

so $\varphi$ is bounded below by $1/C$.

(b)⇒(a). Given an algebra norm $|\!|\!|\cdot|\!|\!|$ on $\mathscr{B}$, let $\mathscr{C}$ be the completion of $\mathscr{B}$ with respect to $|\!|\!|\cdot|\!|\!|$, and consider the inclusion map $a \mapsto a$, $\mathscr{B} \to \mathscr{C}$. □

## Characterization of uniqueness of algebra norm for all quotients

**Corollary.** *The following conditions are equivalent for a Banach algebra $\mathscr{B}$:*

(a) *Every quotient algebra of $\mathscr{B}$ by a closed ideal has a unique algebra norm.*

(b) *Every homomorphism from $\mathscr{B}$ into a Banach algebra is continuous and has closed range.*

*Proof.* (a)$\Rightarrow$(b). Let $\varphi\colon \mathscr{B} \to \mathscr{C}$ be a homomorphism.
It is continuous because the norm on $\mathscr{B}$ is maximal.
In particular, the ideal $\ker\varphi$ is closed, so the quotient norm on $\mathscr{B}/\ker\varphi$ is minimal.
Let $\widetilde{\varphi}\colon \mathscr{B}/\ker\varphi \to \mathscr{C}$ be the induced homomorphism, which satisfies:

- $\widetilde{\varphi}[\mathscr{B}/\ker\varphi] = \varphi[\mathscr{B}]$.
- $\widetilde{\varphi}$ is injective, hence bounded below.
- $\widetilde{\varphi}$ is continuous, hence it has closed range, and therefore so does $\varphi$.

(b)$\Rightarrow$(a). Let $\mathscr{I}$ be a closed ideal of $\mathscr{B}$. The quotient norm on $\mathscr{B}/\mathscr{I}$ is maximal because homomorphisms from $\mathscr{B}$ are continuous.
To see that it is minimal, let $\varphi\colon \mathscr{B}/\mathscr{I} \to \mathscr{C}$ be an injective homomorphism.
By hypothesis $\varphi \circ \pi$ is continuous and has closed range, where $\pi\colon \mathscr{B} \to \mathscr{B}/\mathscr{I}$ is the quotient homomorphism.
Hence $\varphi$ is continuous and has closed range, so $\varphi$ is bounded below. $\square$

Any questions?

# Key known results for $\mathscr{B}(X)$

**Theorem** (Yood, *Pacific Math. J.* 1954).
*The operator norm on $\mathscr{B}(X)$ is minimal for every Banach space $X$.*

**Corollary.** *Let $X$ be a Banach space. Then:*

$\mathscr{B}(X)$ has a unique algebra norm $\iff$

every homomorphism from $\mathscr{B}(X)$ into a Banach algebra is continuous.

For all the Banach spaces $X$ we consider, the latter condition will be satisfied, usually by **Johnson's Theorem:** *Suppose that $X \cong X \oplus X$. Then every homomorphism from $\mathscr{B}(X)$ into a Banach algebra is continuous.*

**Theorem** (Meyer, *BLMS* 1992). *Let $X = c_0$ or $X = \ell_p$ for some $p \in [1, \infty)$. Then $\mathscr{B}(X)/\mathscr{K}(X)$ has a unique algebra norm.*

Since $\mathscr{K}(X)$ is the **only** non-trivial closed ideal of $\mathscr{K}(X)$ for $X = c_0$ and $X = \ell_p$ for some $p \in [1, \infty)$, it follows that every quotient algebra of $\mathscr{B}(X)$ by a closed ideal has a unique algebra norm.

**Generalization** (Ware, PhD thesis 2014). *Let $\Gamma$ be an uncountable index set, and let $X = c_0(\Gamma)$ or $X = \ell_p(\Gamma)$ for some $p \in [1, \infty)$. Then every quotient algebra of $\mathscr{B}(X)$ by a closed ideal has a unique algebra norm.*

# Daws' classification of the closed operator ideals on $c_0(\Gamma)$ and $\ell_p(\Gamma)$ for $1 \leqslant p < \infty$

Recall:

**Generalization** (Ware, PhD thesis 2014). *Let $\Gamma$ be an uncountable index set, and let $X = c_0(\Gamma)$ or $X = \ell_p(\Gamma)$ for some $p \in [1,\infty)$. Then every quotient algebra of $\mathscr{B}(X)$ by a closed ideal has a unique algebra norm.*

Key ingredient: Daws' classification of the closed ideals of $\mathscr{B}(X)$ (*Math. Proc. Cambr. Phil. Soc.* 2006):

$$\{0\} \subsetneq \mathscr{K}(X) \subsetneq \mathscr{K}_{\aleph_1}(X) \subsetneq \cdots \subsetneq \mathscr{K}_\kappa(X) \subsetneq \mathscr{K}_{\kappa^+}(X) \subsetneq \cdots$$
$$\cdots \subsetneq \mathscr{K}_\Gamma(X) \subsetneq \mathscr{K}_{\Gamma^+}(X) = \mathscr{B}(X),$$

where $\mathscr{K}_\kappa(X)$ is the ideal of "$\kappa$-compact operators" for a cardinal $\kappa \geqslant \aleph_1$.

Daws' result generalizes results of Gramsch (*J. Reine Angew. Math.* 1967) and Luft (*Czech. Math. J.* 1968) for $X = \ell_2(\Gamma)$.

See also Johnson–Kania–Schechtman (*PAMS* 2016) and Horváth–Kania (*Q. J. Math.* 2021).

## Key known results for $\mathscr{B}(X)$ (continued)

Ware's main focus: Uniquenenss of algebra norm for the Calkin algebra.

**Theorem** (Ware, "thesis highlights"). *$\mathscr{B}(X)/\mathscr{K}(X)$ has a unique algebra norm in each of the following cases:*

- $X = \ell_{p_1} \oplus \cdots \oplus \ell_{p_n}$ *for any* $n \in \mathbb{N}$ *and* $1 \leqslant p_1 < \cdots < p_n < \infty$.
  (One can also add $c_0$.)
- $X = \left(\bigoplus_{n=1}^{\infty} \ell_p^n\right)_{c_0}$ *and* $X = \left(\bigoplus_{n=1}^{\infty} \ell_p^n\right)_{\ell_q}$ *for* $1 \leqslant p \leqslant \infty$ *and* $1 \leqslant q < \infty$.
- $X = J_p$ *for* $1 < p < \infty$, *the* $p^{th}$ *quasi-reflexive James space.*
- $X = T$, *the Tsirelson space.*

Banach spaces $X$ such that the quotient norm on $\mathscr{B}(X)/\mathscr{K}(X)$ is **not** minimal were first found by Astala–Tylli (*JFA* 1987) and Tylli (*Israel J. Math.* 1995).

**Theorem** (Johnson–Phillips–Schechtman, in preparation). *Let* $L_p = L_p[0,1]$ *for some* $1 < p < \infty$. *Then:*

- *The Calkin algebra* $\mathscr{B}(L_p)/\mathscr{K}(L_p)$ *has a unique algebra norm.*
  *More generally, this is true for every complemented subspace of* $L_p$.
- *Suppose that* $p \neq 2$. *Then* $\mathscr{B}(L_p)$ *contains a closed ideal* $\mathscr{I}$ *for which the quotient norm on* $\mathscr{B}(L_p)/\mathscr{I}$ *is not minimal.*

There is a simple way of proving that a maximal norm on a Banach algebra $\mathscr{B}$ is also minimal (and therefore $\mathscr{B}$ has a unique algebra norm). It can be found in the works of Meyer and Johnson–Phillips–Schechtman.

**Lemma.** *Let $\|\cdot\|$ be a maximal algebra norm on an algebra $\mathscr{B}$. Suppose that there is a constant $K \geqslant 1$ such that, for every $b \in \mathscr{B}$ with $\|b\| = 1$, there are $a, c \in \mathscr{B}$ with $\|a\|\,\|c\| \leqslant K$ such that $abc$ is a non-zero idempotent.*
*Then $\|\cdot\|$ is minimal, and therefore $\mathscr{B}$ has a unique algebra norm.*

*Proof.* Given an algebra norm $|\!|\!|\cdot|\!|\!|$ on $\mathscr{B}$, choose $C > 0$ such that $|\!|\!|\cdot|\!|\!| \leqslant C\|\cdot\|$.
Given $b \in \mathscr{B} \setminus \{0\}$, choose $a, c \in \mathscr{B}$ with $\|a\|\,\|c\| \leqslant K$ such that $abc/\|b\|$ is a non-zero idempotent. Then

$$\left|\!\left|\!\left|\frac{abc}{\|b\|}\right|\!\right|\!\right| \geqslant 1,$$

so

$$\|b\| \leqslant |\!|\!|abc|\!|\!| \leqslant |\!|\!|a|\!|\!|\,|\!|\!|b|\!|\!|\,|\!|\!|c|\!|\!| \leqslant C^2\|a\|\,|\!|\!|b|\!|\!|\,\|c\| \leqslant C^2K|\!|\!|b|\!|\!|. \qquad \square$$

Any questions?

## New results I: Uniqueness of algebra norm on quotients of $\mathscr{B}(C_0(K_{\mathcal{A}}))$

**Theorem** (Arnott–L, in preparation). *Let $\mathcal{A} \subseteq [\mathbb{N}]^\omega$ be an uncountable, almost disjoint family such that*

$$\mathscr{B}(C_0(K_{\mathcal{A}})) = \mathbb{K}\,\mathrm{Id} + \mathscr{X}(C_0(K_{\mathcal{A}})).$$

*Then every quotient of $\mathscr{B}(C_0(K_{\mathcal{A}}))$ by a closed ideal has a unique algebra norm.*

Recall: Kania–Kochanek's closed ideal classification:

$$\{0\} \subset \mathscr{K}(C_0(K_{\mathcal{A}})) \subset \mathscr{X}(C_0(K_{\mathcal{A}})) \subset \mathscr{B}(C_0(K_{\mathcal{A}})).$$

**Maximality** of the quotient norms follows from the automatic continuity of homormorphisms from $\mathscr{B}(C_0(K_{\mathcal{A}}))$.

**To do:** Show that the quotient norm on $\mathscr{B}(C_0(K_{\mathcal{A}}))/\mathscr{K}(C_0(K_{\mathcal{A}}))$ is minimal.

One can view this as a quantitative version of the key step in the proof of K–K: For $T \in \mathscr{B}(C_0(K_{\mathcal{A}})) \setminus \mathscr{K}(C_0(K_{\mathcal{A}}))$, there are $U, V \in \mathscr{B}(C_0(K_{\mathcal{A}}))$ such that

$$\mathrm{Id}_{c_0} = VTU.$$

We do the same but, assuming $\|T + \mathscr{K}(C_0(K_{\mathcal{A}}))\| = 1$, we need a uniform bound on $\|U\|\,\|V\|$. This will follow from two more general results.

# The two main steps

**Step 1.** A "blueprint" for proving minimality of the quotient norm of the Calkin algebra of $C(K)$ for $K$ scattered.

**Theorem.** (Arnott–L). *Let $T \in \mathscr{B}(C_0(K))$ for a scattered, locally compact Hausdorff space $K$, and suppose that $B_{C_0(K)}$ contains a disjoint sequence $(f_n)_{n\in\mathbb{N}}$ such that*

$$\delta := \inf\{\|Tf_n\| : n \in \mathbb{N}\} > 0. \tag{‡}$$

*Then, for every $C > 1$, there is an operator $U \in \mathscr{B}(C_0(K))$ with $\|U\| \leqslant C/\delta$ such that $TU$ is a projection whose image is $C$-isomorphic to $c_0$.*

Key ingredients: Quantitative versions of theorems of Rosenthal and Dowling–Randrianantoanina–Turett.

**Step 2.** We show that (‡) is satisfied (for $\delta < 1/4$) whenever $K = K_{\mathcal{A}}$ for an an uncountable, almost disjoint family $\mathcal{A} \subseteq [\mathbb{N}]^\omega$.

This is a somewhat involved recursive construction.

Work in progress:
Can we verify (‡) for other scattered spaces, such as $K = [0, \omega^\omega)$?

It would imply that the Calkin algebra of $C_0[0, \omega^\omega)$ has a unique algebra norm.

## New results II: The Banach space $X = \left(\bigoplus_{n=1}^{\infty} \ell_2^n\right)_{c_0}$ and its dual

Let $X = \left(\bigoplus_{n=1}^{\infty} \ell_2^n\right)_{c_0}$. Note: $X^* = \left(\bigoplus_{n=1}^{\infty} \ell_2^n\right)_{\ell_1}$.

**Theorem** (Arnott–L, in preparation). *Every quotient algebra of $\mathscr{B}(X)$ and of $\mathscr{B}(X^*)$ by a closed ideal has a unique algebra norm.*

Key ingredient:

**Theorem** (L–Loy–Read, *JFA* 2004). *$\mathscr{B}(X)$ constains exactly four closed ideals, namely*

$$\{0\} \subset \mathscr{K}(X) \subset \overline{\mathscr{G}_{c_0}(X)} \subset \mathscr{B}(X),$$

*where* $\mathscr{G}_{c_0}(X) = \left\{X \xrightarrow{S} c_0 \xrightarrow{T} X : S \in \mathscr{B}(X, c_0) \text{ and } T \in \mathscr{B}(c_0, X)\right\}$.

- $X \cong X \oplus X$, giving maximality by Johnson's Theorem.
- Minimality for $\mathscr{B}(X)$ follows from Yood's Theorem.
- Ware handled $\mathscr{B}(X)/\mathscr{K}(X)$.
- Only minimality of the quotient norm of $\mathscr{B}(X)/\overline{\mathscr{G}_{c_0}(X)}$ remains. It follows from the previous result because the proof of LLR shows that $\mathrm{Id}_X$ factors through every operator $T \in \mathscr{B}(X) \setminus \overline{\mathscr{G}_{c_0}(X)}$ with control over the norms.

**Theorem** (L–Schlumprecht–Zsák, *J. Op. Th.* 2006). *$\mathscr{B}(X^*)$ contains exactly four closed ideals, namely*

$$\{0\} \subset \mathscr{K}(X^*) \subset \overline{\mathscr{G}_{\ell_1}(X^*)} \subset \mathscr{B}(X^*).$$

## New results III: The direct sum $\left(\bigoplus_{n=1}^{\infty} \ell_2^n\right)_{c_0} \oplus c_0(\Gamma)$

**Theorem** (Arnott–L). *Let* $\Gamma$ *be an uncountable set, and set*

$$Y = \left(\bigoplus_{n=1}^{\infty} \ell_2^n\right)_{c_0} \oplus c_0(\Gamma).$$

*Every quotient algebra of* $\mathscr{B}(Y)$ *by a closed ideal has a unique algebra norm.*

Key ingredient: The classification of the closed operator ideals on $Y$ (Arnott–L, *JMAA* 2021).

**Remark.** We have similar results for the dual space

$$Y^* = \left(\bigoplus_{n=1}^{\infty} \ell_2^n\right)_{\ell_1} \oplus \ell_1(\Gamma).$$

# The lattice of closed operator ideals on $Y = \left(\bigoplus_{n=1}^{\infty} \ell_2^n\right)_{c_0} \oplus c_0(\Gamma)$

$$
\begin{tikzcd}
 & \{0\} \arrow[d] & \\
 & \mathscr{K}(Y) \arrow[d] & \\
 & \overline{\mathscr{G}_{c_0}}(Y) \arrow[dl] \arrow[dr] & \\
\mathscr{K}_{\aleph_1}(Y) \arrow[d] & & \mathscr{J}_{\aleph_2}(Y) \arrow[d] \arrow[dll] \\
\mathscr{K}_{\aleph_2}(Y) \arrow[d] & & \mathscr{J}_{\aleph_3}(Y) \arrow[d] \arrow[dll] \\
\mathscr{K}_{\aleph_3}(Y) \arrow[d] & & \mathscr{J}_{\aleph_4}(Y) \arrow[d] \\
\vdots \arrow[d] & & \vdots \arrow[d] \\
\mathscr{K}_{\Gamma}(Y) \arrow[dr] & & \mathscr{J}_{\Gamma^+}(Y) \arrow[dl] \\
 & \mathscr{B}(Y) &
\end{tikzcd}
$$

## Explanation of the notation used in the diagram

- An arrow from an ideal $\mathscr{I}$ to an ideal $\mathscr{J}$ means that $\mathscr{I} \subsetneqq \mathscr{J}$, with no closed ideals of $\mathscr{B}(Y)$ strictly contained in between $\mathscr{I}$ and $\mathscr{J}$.
- $\mathscr{K}_\kappa(Y)$ denotes the ideal of $\kappa$-compact operators for a cardinal $\kappa \geqslant \aleph_1$.
- As before, set $X = \left(\bigoplus_{n=1}^\infty \ell_2^n\right)_{c_0}$. For a cardinal $\aleph_2 \leqslant \kappa \leqslant \Gamma^+$,

$$\mathscr{J}_\kappa(Y) = \left\{ \begin{pmatrix} T_{1,1} & T_{1,2} \\ T_{2,1} & T_{2,2} \end{pmatrix} \in \mathscr{B}(Y) : T_{1,1} \in \overline{\mathscr{G}_{c_0}}(X), \right.$$
$$\left. T_{1,2} \in \mathscr{B}(c_0(\Gamma), X),\ T_{2,1} \in \mathscr{B}(X, c_0(\Gamma)),\ T_{2,2} \in \mathscr{K}_\kappa(c_0(\Gamma)) \right\}.$$

Outline of proof strategy:

- Calkin algebra: Use Ware's results for direct sums.
- Other quotients: Use the fact that the quotient algebra splits as a direct sum

$$\frac{\mathscr{B}(X)}{\mathscr{I}} \oplus \frac{\mathscr{B}(c_0(\Gamma))}{\mathscr{J}},$$

together with uniform factorization of idempotents.

# The end

Thank you!