$C(K)$-spaces associated with almost disjoint families (Part I)

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Main parts: joint work with Piotr Koszmider (IMPAN, Warsaw)
and with Max Arnott (Lancaster)

Overview

Today:

- Almost disjoint families: What are they, and why are they important in the theory of Banach spaces and lattices?
- $C(K)$-spaces: Some background and recent applications of almost disjoint families.
- Operator theory: Banach spaces with "few operators"; in particular: $C(K)$-spaces with few operators.
- Fundamental facts about the Mrówka space associated with an almost disjoint family.
- Some consequences of the main theorem: Self-maps, decompositions, closed operator ideals, characters.

Tomorrow:

- Outline of the proof of the main theorem (joint work with Koszmider).
- Automatic continuity of homomorphisms from $\mathscr{B}(X)$.
- Uniqueness of algebra norm on all quotients of $\mathscr{B}(X)$ by its closed ideals (joint work with Arnott).

Let $[\mathbb{N}]^\omega$ denote the set of infinite subsets of $\mathbb{N}$.

Definition. A family $\mathcal{A} \subset [\mathbb{N}]^\omega$ is *almost disjoint* if

$$|A \cap B| < \infty \qquad (A, B \in \mathcal{A},\ A \neq B).$$

Key result. (Sierpiński, 1928.) *Uncountable almost disjoint families exist!*
In fact, there are almost disjoint families of cardinality $\mathfrak{c}$*.*

Moreover, they are "easy" to construct — once you know the trick!

Construction 1.
Starting point: We may replace $\mathbb{N}$ with any countably infinite set. We use $\mathbb{Q}$.

For every $r \in \mathbb{R} \setminus \mathbb{Q}$, choose a sequence $(q_n^{(r)})_{n\in\mathbb{N}}$ in $\mathbb{Q}$ which converges to r.

Define $A_r = \{q_n^{(r)} : n \in \mathbb{N}\} \in [\mathbb{Q}]^\omega$.

Suppose that $|A_r \cap A_s| = \infty$ for some $r, s \in \mathbb{R} \setminus \mathbb{Q}$.

Then $(q_n^{(r)})_{n\in\mathbb{N}}$ and $(q_n^{(s)})_{n\in\mathbb{N}}$ have an infinite subsequence in common, so $r = s$.

Hence $|A_r \cap A_s| < \infty$ for $r \neq s$.

Thus $\{A_r : r \in \mathbb{R} \setminus \mathbb{Q}\}$ is an almost disjoint family of cardinality $\mathfrak{c}$ in $[\mathbb{Q}]^\omega$. $\square$

A second construction of an almost disjoint family of cardinality $\mathfrak{c}$

We begin with the ***Cantor cube*** $\{0,1\}^{\mathbb{N}} = \{f : \mathbb{N} \to \{0,1\}\}$.

Given $f \in \{0,1\}^{\mathbb{N}}$ and $n \in \mathbb{N}$, define

$$m_n(f) = 2^{n-1} + \sum_{j=1}^{n-1} f(j) 2^{n-1-j},$$

that is, the number whose binary expansion is

$$1\, f(1)\, f(2) \ldots f(n-2)\, f(n-1).$$

Set $M(f) = \{m_n(f) : n \in \mathbb{N}\} \in [\mathbb{N}]^{\omega}$.

Claim: The family $\mathcal{A} = \{M(f) : f \in \{0,1\}^{\mathbb{N}}\}$ is almost disjoint.

Reason: Suppose that $f, g \in \{0,1\}^{\mathbb{N}}$ are distinct, and let

$$k = \min\{j \in \mathbb{N} : f(j) \neq g(j)\}.$$

Then $m_n(f) = m_n(g)$ for $n \leqslant k$ and $m_n(f) \neq m_n(g)$ for $n > k$, so

$$|M(f) \cap M(g)| = k.$$

In particular, $|\mathcal{A}| = |\{0,1\}^{\mathbb{N}}| = \mathfrak{c}$. $\square$

Almost disjoint families — applications

Almost disjoint families appear in many different branches of mathematics:

- Perhaps their most natural home is **infinite combinatorics**.
- They are closely connected with **foundations/set theory** because often, the construction of an almost disjoint family with particular properties relies on additional axioms (outside ZFC).
  The continuum hypothesis (CH) — or its negation — are the most "obvious" examples.
- Almost disjoint families also play a role in **topology** because every almost disjoint family $\mathcal{A} \subseteq [\mathbb{N}]^\omega$ induces a locally compact Hausdorff space $K_{\mathcal{A}}$.
  This connection goes back to Alexandroff and Urysohn (1920's), who essentially gave our first construction.
- This links almost disjoint families to **functional analysis**: Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and let K be a locally compact Hausdorff space. Then

$$C_0(K) = \Big\{ f\colon K \to \mathbb{K} \ : f \text{ is continuous and } \\ \{t \in K : |f(t)| \geqslant \varepsilon\} \text{ is compact for every } \varepsilon > 0 \Big\}$$

  is a Banach space/Banach lattice/C^*-algebra.

The locally compact space induced by an almost disjoint family (functional analytic approach)

Definition. Given an almost disjoint family $\mathcal{A} \subseteq [\mathbb{N}]^\omega$, define

$$X_\mathcal{A} = \overline{\mathrm{span}}\{1_A : A \in \mathcal{A} \cup [\mathbb{N}]^{<\omega}\} \subseteq \ell_\infty,$$

where 1_A is the indicator function of A and $[\mathbb{N}]^{<\omega}$ the set of finite subsets of $\mathbb{N}$.

Check: $X_\mathcal{A}$ is a self-adjoint subalgebra of ℓ_∞.

Gelfand–Naimark Theorem: $X_\mathcal{A} \cong C_0(K_\mathcal{A})$ for some locally compact Hausdorff space $K_\mathcal{A}$.

Origins:

- Banach spaces of the form $X_\mathcal{A}$ were first studied by Johnson and Lindenstrauss (*Israel J. Math.* 1974).
- Locally compact Hausdorff spaces of the form $K_\mathcal{A}$ were first studied by Alexandroff and Urysohn (1920's).

  Terminology: AU-compactum, (Isbell–)Mrówka space, Ψ-space.

Mini-break

Any questions?

The locally compact space induced by an almost disjoint family (continued)

There are other ways of defining the Mrówka space $K_{\mathcal{A}}$:

- I shall give a "hands-on" definition later.
- If you prefer abstract methods, you can view $K_{\mathcal{A}}$ as the Stone space of the Boolean subalgebra of $\mathscr{P}(\mathbb{N})$ generated by $\mathcal{A} \cup [\mathbb{N}]^{<\omega}$.

Remark. $\|1_A - 1_B\|_\infty = 1$ for distinct $A, B \subseteq \mathbb{N}$.

Hence, for an almost disjoint family $\mathcal{A} \subseteq [\mathbb{N}]^\omega$,

$$X_{\mathcal{A}} \text{ is separable } \iff \mathcal{A} \text{ is countable.}$$

Conclusion: In the "interesting" cases, $C_0(K_{\mathcal{A}})$ is non-separable.

Terminology. A ***C(K)-space*** is a Banach space of the form

$$C(K) = \{f\colon K \to \mathbb{K} : f \text{ is continuous}\}$$

for some compact Hausdorff space K.

Note: All the results I shall discuss are of an isomorphic nature, so I shall often use the term "$C(K)$-space" to mean a Banach space which is **isomorphic** to $C(K)$ for some compact Hausdorff space K.

We shall see later that $C_0(K_{\mathcal{A}})$ is isomorphic to a $C(K)$-space.

Fundamental fact. For a compact Hausdorff space K,

$$C(K) \text{ is separable} \iff K \text{ is metrizable.}$$

Theorem. *Let K be a compact metric space.*

(i) *Suppose that K is finite. Then $C(K) \cong \ell_\infty^n$ for $n = |K|$.*

(ii) (Milutin) *Suppose that K is uncountable. Then $C(K) \cong C(\{0,1\}^{\mathbb{N}})$.*

(iii) (Bessaga–Pełczyński, *Studia Math.* 1960) *Suppose that K is countably infinite. Then there is a unique countable ordinal α such that*

$$C(K) \cong C[0, \omega^{\omega^\alpha}],$$

where $[0, \omega^{\omega^\alpha}]$ denotes set of all ordinals not exceeding ω^{ω^α}, endowed with the order topology.

The "modern" way of determining the ordinal α is via the ***Szlenk index*** because

$$\mathrm{Sz}(C[0, \omega^{\omega^\alpha}]) = \omega^{\alpha+1}.$$

This result is due to Samuel (1983).

Long-standing open problem: Understand the complemented subspaces of $C(K)$-spaces.

Positive results:

- c_0 is **prime** (Pełczyński, *Studia Math.* 1960), that is, every complemented, ∞-dimensional subspace of c_0 is isomorphic to c_0.
- More generally, for Γ of arbitrary cardinality, every complemented subspace of $c_0(\Gamma)$ is isomorphic to $c_0(\Delta)$ for some $\Delta \subseteq \Gamma$ (Granero, 1998).
- Every complemented, ∞-dimensional subspace of $C[0, \omega^\omega]$ is isomorphic to either c_0 or $C[0, \omega^\omega]$ (Benyamini, *Israel J. Math.* 1978).
- ℓ_∞ is prime (Lindenstrauss, *Israel J. Math.* 1967).

Long-standing conjecture: Every complemented subspace of a $C(K)$-space is isomorphic to a $C(K)$-space (not necessarily for the same K).

This conjecture has recently been **disproved** by Plebanek and Salguero Alarcón:

Theorem. (Plebanek and Salguero Alarcón, preprint 2021). *There exists an uncountable, almost disjoint family $\mathcal{A} \subseteq [\mathbb{N}]^\omega$ such that $C_0(K_\mathcal{A})$ contains a complemented subspace which is not isomorphic to any $C(K)$-space.*

The conjecture is still open for separable $C(K)$-spaces.

Further results: Extensions and some questions of Koszmider

Theorem (Plebanek and Salguero Alarcón, preprint 2021).
There exists an uncountable, almost disjoint family $\mathcal{A} \subseteq [\mathbb{N}]^\omega$ admitting an extension

$$\{0\} \longrightarrow c_0 \longrightarrow C_0(K_{\mathcal{A}}) \longrightarrow X \longrightarrow \{0\},$$

in which the Banach space is not isomorphic to a $C(K)$-space.

Theorem (Cabello Sánchez, Castillo, Marciszewski, Plebanek and Salguero Alarcón, *JFA* 2020, assuming MA and $\neg$CH).
Let $\mathcal{A}, \mathcal{B} \subseteq [\mathbb{N}]^\omega$ be uncountable, almost disjoint families with $|\mathcal{A}| = |\mathcal{B}| < \mathfrak{c}$. Then

$$C_0(K_{\mathcal{A}}) \cong C_0(K_{\mathcal{B}}) \quad \textit{and} \quad C_0(K_{\mathcal{A}}) \cong C_0(K_{\mathcal{A}}) \oplus C_0(K_{\mathcal{A}}).$$

Theorem (Cabello Sánchez, Castillo, Marciszewski, Plebanek and Salguero Alarcón, *JFA* 2020, building on Marciszewski and Pol, *JMMA* 2009).
There are $2^{\mathfrak{c}}$ non-isomorphic Banach spaces of the form $C_0(K_{\mathcal{A}})$, where $\mathcal{A} \subseteq [\mathbb{N}]^\omega$ is an almost disjoint family of cardinality $\mathfrak{c}$.

Cabello Sánchez, Castillo, Kalton and Yost (*TAMS* 2003) studied extensions of the form

$$\{0\} \longrightarrow C(K) \longrightarrow Y \longrightarrow X \longrightarrow \{0\}$$

for separable Banach spaces X and either $K = [0,1]$ or $K = [0, \omega^\omega]$.

They also raised an important question in the non-separable case, motivated by Sobczyk's Theorem:

CCKY Question: Let K be a non-metrizable, compact Hausdorff space.
Is it possible that every extension of the form

$$\{0\} \longrightarrow c_0 \longrightarrow X \longrightarrow C(K) \longrightarrow \{0\}$$

splits in the category of Banach spaces?

Conjecture: No!

Theorem (Marciszewski and Plebanek, *JFA* 2018, assuming MA and $\neg$CH).
Let $\mathcal{A} \subseteq [\mathbb{N}]^\omega$ be an uncountable, almost disjoint family with $|\mathcal{A}| < \mathfrak{c}$. Then every extension of the form

$$\{0\} \longrightarrow c_0 \longrightarrow X \longrightarrow C_0(K_{\mathcal{A}}) \longrightarrow \{0\}$$

splits in the category of Banach spaces.

Theorem (Avilés, Marciszewski and Plebanek, *Adv. Math.* 2020, assuming CH).
For every non-metrizable compact Hausdorff space K, there is an extension of the form

$$\{0\} \longrightarrow c_0 \longrightarrow X \longrightarrow C(K) \longrightarrow \{0\}$$

which does not split in the category of Banach spaces.

Any questions?

My motivation: Which operators must exist on an ∞-dimensional Banach space?

Let X be a Banach space over the scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and consider the Banach algebra

$$\mathscr{B}(X) = \{T\colon X \to X : T \underbrace{\text{bounded and linear}}_{\text{“an operator”}}\}.$$

Question: How “small” can $\mathscr{B}(X)$ be?

We always have the identity operator Id.

Hahn–Banach ensures there are “many” finite-rank operators.

However, the ideal of finite-rank operators,

$$\mathscr{F}(X) = \{T \in \mathscr{B}(X) : \dim T[X] < \infty\},$$

is never closed when X is ∞-dimensional.

Theorem. *The following 3 conditions are equivalent for a Banach space X:*

- $\mathscr{F}(X)$ *is closed;*
- $\mathscr{F}(X) = \mathscr{B}(X)$;
- $\dim X < \infty$.

Moreover, when $n := \dim X < \infty$, $\mathscr{B}(X) \cong M_n(\mathbb{K})$, which is simple.

A generalization of the latter fact to arbitrary dimensions: Every non-zero ideal of $\mathscr{B}(X)$ contains $\mathscr{F}(X)$.

Question: Is it possible that an ∞-dimensional Banach space X admits no other operators, so that

$$\mathscr{B}(X) = \mathbb{K}\,\mathrm{Id} + \overline{\mathscr{F}(X)}?$$

Theorem (Argyros–Haydon, *Acta Math.* 2011). Yes!

There is a Banach space X_{AH} which has "very few operators":

$$\mathscr{B}(X_{\mathsf{AH}}) = \mathbb{K}\,\mathrm{Id} + \mathscr{K}(X_{\mathsf{AH}}).$$

Moreover, X_{AH} has a Schauder basis, so that $\mathscr{K}(X_{\mathsf{AH}}) = \overline{\mathscr{F}(X_{\mathsf{AH}})}$, and $X_{\mathsf{AH}}^* \cong \ell_1$.

Key ingredients in the construction of the Argyros–Haydon space

This result was the culmination of a long sequence of constructions of "exotic" Banach spaces initiated by Tsirelson's construction of a Banach space that does not contain c_0 or ℓ_p for $1 \leqslant p < \infty$ (*Funct. Anal. Appl.* 1972).

Other key contributions:

- Bourgain and Delbaen found a completely new approach to constructing preduals of ℓ_1 (*Acta Math.* 1980).
- Schlumprecht modified Tsirelson's construction to obtain an arbitrarily distortable Banach space (*Israel J. Math.* 1991).
- Gowers and Maurey modified Schlumprecht's contruction to obtain a hereditarily indecomposable Banach space X (*JAMS* 1993).
  This implies that X admits "few operators" in the sense that

  $$\mathscr{B}(X) = \mathbb{C}\,\mathrm{Id} + \mathscr{S}(X),$$

  where $\mathscr{S}(X)$ is the ideal of strictly singular operators.

  Definition. An operator $S\colon X \to Y$ is ***strictly singular*** if the restriction of S to any ∞-dimensional subspace of X is not an isomorphic embedding.

  In other words, for every $\varepsilon > 0$, every ∞-dimensional subspace of X contains a unit vector x such that $\|Sx\| < \varepsilon$.

Earliest (?) example:

Theorem (Shelah, *Israel J. Math.* 1978, assuming $\diamondsuit$).
There is a non-separable Banach space X such that

$$\mathscr{B}(X) = \mathbb{K}\,\mathrm{Id} + \mathscr{X}(X),$$

where $\mathscr{X}(X) = \{T \in \mathscr{B}(X) : T[X] \text{ is separable}\}$.

Further results:

- Shelah and Steprāns (*PAMS* 1988): example within ZFC.
- Wark (*JLMS* 2001): reflexive example.
- Wark (*Studia Math.* 2018): uniformly convex example.

Connection with our topic: ***C*(*K*)-spaces with few operators**.

Note: The results I shall discuss are of an isomorphic nature, so in this context a "$C(K)$-space" can be any Banach space which is isomorphic to $C(K)$ for a compact Hausdorff space K.

What does it mean that a $C(K)$-space has "few operators"?

Theorem (Pełczyński, *Bull. Acad. Pol. Sci.* 1965).
The following 3 conditions are equivalent for an operator $T\colon C(K) \to X$, where K is a compact Hausdorff space and X is a Banach space:

- *T is weakly compact;*
- *T is strictly singular;*
- *T does not fix a copy of c_0.*

Terminology. An operator $T\colon C(K) \to X$ ***fixes a copy of*** c_0 if there is an operator $R\colon c_0 \to C(K)$ such that TR is an isomorphic embedding.

Remark (Koszmider–L, 2021). The "few operator" results of Gowers–Maurey and Argyros–Haydon have no direct analogues for $C(K)$-spaces because

$$\dim \frac{\mathscr{B}(C(K))}{\mathscr{W}(C(K))} = \infty$$

whenever K is infinite.
In fact $\mathscr{B}(C(K))/\mathscr{W}(C(K))$ is non-separable.

Starting point. Every $C(K)$-space admits ***multiplication operators***:

$$M_f : g \mapsto fg, \;\; C(K) \to C(K),$$

where $f \in C(K)$.

Theorem (Koszmider, *Math. Ann.* 2004, assuming CH; Plebanek, *Top. Appl.* 2004, within ZFC). *There is an infinite compact Hausdorff space K such that*

$$\mathscr{B}(C(K)) = \{M_f : f \in C(K)\} + \mathscr{W}(C(K)).$$

Note: $C(K)$ is a ***Grothendieck space***, that is, every weak*-convergent sequence in $C(K)^*$ converges weakly.

Consequence: $C(K)$ does not contain any complemented subspaces isomorphic to c_0.

Koszmider's second $C(K)$-space with few operators: the Mrówka space

Theorem (Koszmider, *PAMS* 2005, assuming CH; Koszmider–L, *Adv. Math.* 2021, within ZFC).
There is an uncountable, almost disjoint family $\mathcal{A} \subseteq [\mathbb{N}]^\omega$ *such that*

$$\mathscr{B}(C_0(K_\mathcal{A})) = \mathbb{K}\,\mathrm{Id} + \mathscr{X}(C_0(K_\mathcal{A})).$$

I shall outline the proof of this result in tomorrow's lecture.

Note: It is an exact analogue of Shelah's space among $C(K)$-spaces.

Remarks. Let $\mathcal{A} \subseteq [\mathbb{N}]^\omega$ be an almost disjoint family. Then:

- $C_0(K_\mathcal{A})$ contains a complemented copy of c_0, so it is not a Grothendieck space.
- $C_0(K_\mathcal{A}) \cong C(\alpha K_\mathcal{A})$, where $\alpha K_\mathcal{A}$ is the one-point compactification of $K_\mathcal{A}$.
  Proof. Write $C_0(K_\mathcal{A}) \cong X \oplus c_0$ for some Banach space X. Then
  $$C(\alpha K_\mathcal{A}) \cong C_0(K_\mathcal{A}) \oplus \mathbb{K} \cong X \oplus c_0 \oplus \mathbb{K} \cong X \oplus c_0 \cong C_0(K_\mathcal{A}). \qquad \square$$
  In particular, $C_0(K_\mathcal{A})$ is isomorphic to a $C(K)$-space.
- Every separable subspace of $C_0(K_\mathcal{A})$ is contained in a subspace isomorphic to c_0, so

$$\begin{aligned}\mathscr{X}(C_0(K_\mathcal{A})) = \big\{C_0(K_\mathcal{A}) \xrightarrow{S} c_0 \xrightarrow{T} C_0(K_\mathcal{A}) : S \in \mathscr{B}(C_0(K_\mathcal{A}), c_0),\\ T \in \mathscr{B}(c_0, C_0(K_\mathcal{A}))\big\} =: \mathscr{G}_{c_0}(C_0(K_\mathcal{A})).\end{aligned}$$

Any questions?

Fundamental facts about the Mrówka space $K_{\mathcal{A}}$

Let $\mathcal{A} \subseteq [\mathbb{N}]^\omega$ be an almost disjoint family.

- The **standard representation** of $K_{\mathcal{A}}$:

$$K_{\mathcal{A}} = \{x_n : n \in \mathbb{N}\} \cup \{y_A : A \in \mathcal{A}\},$$

  where x_n is isolated for every $n \in \mathbb{N}$, and $x_n \underset{A \ni n \to \infty}{\longrightarrow} y_A$ for every $A \in \mathcal{A}$.

  More precisely, the sets

$$U(A, F) = \{x_n : n \in A \setminus F\} \cup \{y_A\}, \quad \text{where} \quad F \in [\mathbb{N}]^{<\omega},$$

  form a neighbourhood basis at y_A.
- $K_{\mathcal{A}}$ is compact if and only if $\mathcal{A}$ and $\mathbb{N} \setminus \bigcup \mathcal{A}$ are both finite.
- $\{x_n : n \in \mathbb{N}\}$ is the set of isolated points of $K_{\mathcal{A}}$.
  It is dense in $K_{\mathcal{A}}$, so $K_{\mathcal{A}}$ is separable. Hence

$$K_{\mathcal{A}} \text{ metrizable} \iff K_{\mathcal{A}} \text{ second countable} \iff \mathcal{A} \text{ countable.}$$

- The subspace $K_{\mathcal{A}} \setminus \{x_n : n \in \mathbb{N}\} = \{y_A : A \in \mathcal{A}\}$ is closed and discrete.
- Consequence: $K_{\mathcal{A}}$ is ***scattered***; that is, every non-empty subset L contains a relatively isolated point.

  Proof. (i) L contains a relatively isolated point if $x_n \in L$ for some $n \in \mathbb{N}$.
  (ii) Suppose that $y_A \in L$ is **not** isolated in L for some $A \in \mathcal{A}$. Then $x_n \in L$ for infinitely many $n \in A$, so (i) applies. □

Fundamental facts about $C_0(K_{\mathcal{A}})$

Recall: the locally compact Hausdorff space $K_{\mathcal{A}}$ is scattered. This has a number of important consequences for the Banach space $C_0(K_{\mathcal{A}})$.

- $C_0(K_{\mathcal{A}})$ is an **Asplund space**.

  That is, every separable subspace of $C_0(K_{\mathcal{A}})$ has separable dual.

- **Rudin's Theorem:** $C_0(K_{\mathcal{A}})^* \cong \ell_1(K_{\mathcal{A}})$.

  Consequence: For $T \in \mathscr{B}(C_0(K_{\mathcal{A}}))$, $f \in C_0(K_{\mathcal{A}})$ and $s \in K_{\mathcal{A}}$,

  $$T(f)(s) = \langle T(f), \delta_s \rangle = \langle f, T^*(\delta_s) \rangle = \sum_{t \in K_{\mathcal{A}}} T^*(\delta_s)(\{t\}) f(t),$$

  where $\delta_s \in C_0(K_{\mathcal{A}})^*$ is the point evaluation at s.

  We think of $\big(T^*(\delta_s)(\{t\})\big)_{s,t \in K_{\mathcal{A}}}$ as a "**matrix representation**" of T.

- **LPP Theorem** (Lotz–Peck–Porta, *Proc. Edinb. Math. Soc.* 1979).
  $C_0(K_{\mathcal{A}})$ is saturated with complemented copies of c_0.

  That is, every closed, ∞-dimensional subspace of $C_0(K_{\mathcal{A}})$ contains a subspace which is isomorphic to c_0 and complemented in $C_0(K_{\mathcal{A}})$.

Complemented and uncomplemented copies of c_0 in $C_0(K_{\mathcal{A}})$

Let $\mathcal{A} \subseteq [\mathbb{N}]^\omega$ be an almost disjoint family.

If $\mathcal{A}$ is countable, then $C_0(K_{\mathcal{A}})$ is separable.
Then Sobczyk's Theorem implies that all copies of c_0 are complemented.

Suppose $\mathcal{A}$ is uncountable. Then:

- The canonical copy of c_0 is **not** complemented in $X_{\mathcal{A}} \cong C_0(K_{\mathcal{A}})$.
- More generally, suppose that $T \in \mathscr{B}(C_0(K_{\mathcal{A}}))$ satisfies $T1_{\{x_n\}} = 0$ for every $n \in \mathbb{N}$. Then $\mathcal{A}$ contains a subset $\mathcal{A}'$ of the same cardinality as $\mathcal{A}$ such that $T1_{U(A,F)} = 0$ for every $A \in \mathcal{A}'$ and $F \in [\mathbb{N}]^{<\omega}$. (Horváth–L, work in progress).
- On the other hand, it is easy to construct a complemented copy of c_0 in $C_0(K_{\mathcal{A}})$.

Constructing a complemented copy of c_0 in $C_0(K_{\mathcal{A}})$

Let $\mathcal{A} \subseteq [\mathbb{N}]^\omega$ be an almost disjoint family.

Take $A = \{a_1 < a_2 < \cdots\} \in \mathcal{A}$.

Define $Ue_n = 1_{\{x_{a_n}\}}$ for each $n \in \mathbb{N}$, where (e_n) is the standard basis for c_0.

Extend by linearity and continuity to a linear isometry $U\colon c_0 \to C_0(K_{\mathcal{A}})$.

Define $Vf = \big(f(x_{a_n}) - f(y_A)\big)_{n\in\mathbb{N}}$ for each $f \in C_0(K_{\mathcal{A}})$.

Note: $Vf \in c_0$ because $x_{a_n} \to y_A$ as $n \to \infty$.

Hence $V\colon C_0(K_{\mathcal{A}}) \to c_0$.

Clearly V is linear and bounded with norm 2.

For $k \in \mathbb{N}$,

$$VUe_k = \big(1_{\{x_{a_k}\}}(x_{a_n}) - 1_{\{x_{a_k}\}}(y_A)\big)_{n\in\mathbb{N}} = (\delta_{k,n})_{n\in\mathbb{N}} = e_k,$$

so $VU = \mathrm{Id}_{c_0}$.

Therefore $UV \in \mathscr{B}(C_0(K_{\mathcal{A}}))$ is a projection onto a subspace isomorphic to c_0.

$\square$

Any questions?

Recall:

Theorem (Koszmider, *PAMS* 2005, assuming CH; Koszmider–L, *Adv. Math.* 2021, within ZFC).
There is an uncountable, almost disjoint family $\mathcal{A} \subseteq [\mathbb{N}]^\omega$ such that

$$\mathscr{B}(C_0(K_{\mathcal{A}})) = \mathbb{K}\,\mathrm{Id} + \mathscr{X}(C_0(K_{\mathcal{A}})). \qquad (*)$$

In the next few slides, we assume that $\mathcal{A} \subseteq [\mathbb{N}]^\omega$ is an uncountable, almost disjoint family such that $(*)$ is satisfied. We shall say that "$\mathcal{A}$ admits few operators".

Theorem. (Koszmider–L). Let $\phi\colon K_{\mathcal{A}} \to K_{\mathcal{A}}$ be a continuous self-map. Then:

- either ϕ has countable range,
- or ϕ fixes all but countably many points of $K_{\mathcal{A}}$.

Theorem (Koszmider 2005). *$C_0(K_{\mathcal{A}})$ has no non-trivial decompositions. More precisely, suppose that $C_0(K_{\mathcal{A}}) = X \oplus Y$ for some closed, ∞-dimensional subspaces X and Y. Then $X \cong C_0(K_{\mathcal{A}})$ and $Y \cong c_0$, or vice versa.*

Proof: $C_0(K_{\mathcal{A}})$ has no non-trivial decompositions

Proof. Let $P \in \mathscr{B}(C_0(K_{\mathcal{A}}))$ be a projection with

$$P[C_0(K_{\mathcal{A}})] = X \quad \text{and} \quad \ker P = Y, \quad \text{where} \quad \dim X = \dim Y = \infty.$$

By $(*)$, $P = \lambda \operatorname{Id} + S$, where $\lambda \in \mathbb{K}$ and $S \in \mathscr{X}(C_0(K_{\mathcal{A}})])$.
Then

$$\lambda \operatorname{Id} + S = P = P^2 = \lambda^2 \operatorname{Id} + 2\lambda S + S^2,$$

so

$$(\lambda - \lambda^2) \operatorname{Id} = (2\lambda - 1)S + S^2 \in \mathscr{X}(C_0(K_{\mathcal{A}})).$$

Hence $\lambda^2 = \lambda$, that is, $\lambda = 0$ or $\lambda = 1$.

Suppose that $\lambda = 0$, so $X = P[C_0(K_{\mathcal{A}})]$ is separable.

Then X is contained in a subspace isomorphic to c_0, and obviously complemented in it.

Hence, by a theorem of Pełczyński, $X \cong c_0$.

By the LPP Theorem, we have $Y \cong c_0 \oplus Z$ for some subspace Z. Hence

$$C_0(K_{\mathcal{A}}) = X \oplus Y \cong c_0 \oplus c_0 \oplus Z \cong c_0 \oplus Z \cong Y.$$

The proof for $\lambda = 1$ is similar, just with X and Y switching roles. $\square$

Consequences of main theorem: Closed ideal structure

Theorem. (Kania and Kochanek, *J. Op. Th.* 2014; Brooker, unpublished). *Let $\mathcal{A} \subseteq [\mathbb{N}]^\omega$ be an uncountable, almost disjoint family which admits few operators. Then $\mathscr{B}(C_0(K_\mathcal{A}))$ contains exactly four closed ideals:*

$$\{0\} \subset \mathscr{K}(C_0(K_\mathcal{A})) \subset \mathscr{X}(C_0(K_\mathcal{A})) \subset \mathscr{B}(C_0(K_\mathcal{A})).$$

Proof.

- $C(K)$-spaces have the AP, so $\mathscr{K}(C_0(K_\mathcal{A})) = \overline{\mathscr{F}(C_0(K_\mathcal{A}))}$ is the smallest non-zero closed ideal of $\mathscr{B}(C_0(K_\mathcal{A}))$.
- $\mathscr{K}(X) \subseteq \mathscr{X}(X)$ for all Banach spaces X.
- Take a projection $P \in \mathscr{B}(C_0(K_\mathcal{A}))$ with $P[C_0(K_\mathcal{A})] \cong c_0$. Then

$$P \in \mathscr{X}(C_0(K_\mathcal{A})) \setminus \mathscr{K}(C_0(K_\mathcal{A})).$$

- $\mathscr{X}(C_0(K_\mathcal{A}))$ is proper because $C_0(K_\mathcal{A})$ is non-separable. Hence it has codimension 1 in $\mathscr{B}(C_0(K_\mathcal{A}))$ by hypotehsis, so $\mathscr{X}(C_0(K_\mathcal{A}))$ is maximal.
- It remains to show that $\mathscr{X}(C_0(K_\mathcal{A}))$ is contained in the ideal generated by any non-compact $T \in \mathscr{B}(C_0(K_\mathcal{A}))$.

  This will follow from the fact that Id_{c_0} factors through T.

Proof: Kania–Kochanek's ideal classification (continued)

Aim: To show that Id_{c_0} factors through every non-compact op. on $C_0(K_{\mathcal{A}})$.

First step: $\mathscr{K}(C_0(K_{\mathcal{A}})) = \mathscr{W}(C_0(K_{\mathcal{A}}))$.

Proof. Since $C_0(K_{\mathcal{A}})^* \cong \ell_1(K_{\mathcal{A}})$, it has the Schur property, so

$$\begin{aligned}
T \in \mathscr{K}(C_0(K_{\mathcal{A}})) &\iff T^* \in \mathscr{K}(C_0(K_{\mathcal{A}})^*) && \text{(Schauder's Thm)}\\
&\iff T^* \in \mathscr{W}(C_0(K_{\mathcal{A}})^*) && \text{(Schur property)}\\
&\iff T \in \mathscr{W}(C_0(K_{\mathcal{A}})) && \text{(Gantmacher's Thm)}\\
&\iff T \text{ does not fix } c_0 && \text{(Pełczyński's Thm).} \qquad \square
\end{aligned}$$

Hence, for $T \in \mathscr{B}(C_0(K_{\mathcal{A}})) \setminus \mathscr{K}(C_0(K_{\mathcal{A}}))$, we can find $R\colon c_0 \to C_0(K_{\mathcal{A}})$ such that TR is an isomorphic embedding.

Then $TR[C_0(K_{\mathcal{A}})]$ is closed and ∞-dimensional.

By the LPP Theorem: $TR[C_0(K_{\mathcal{A}})]$ contains a subspace Z such that:

- there is an isomorphism $U\colon c_0 \to Z$;
- there is a projection $P\colon C_0(K_{\mathcal{A}}) \to Z$.

Let $W = (TR)^{-1}[Z] \subseteq c_0$.

The restriction of TR to W is an isomorphism onto Z, with inverse $S\colon Z \to W$.

Conclusion: $\mathrm{Id}_{c_0} = (U^{-1}P)T(RJ_W SU)$, where $J_W\colon W \to c_0$ is the inclusion. $\square$

Mini-break

Any questions?

Consequences of main theorem: The existence of a character

Let $\mathcal{A} \subseteq [\mathbb{N}]^\omega$ be an uncountable, almost disjoint family such that

$$\mathscr{B}(C_0(K_\mathcal{A})) = \mathbb{K}\,\mathrm{Id} + \mathscr{X}(C_0(K_\mathcal{A})). \qquad (*)$$

Then the ideal $\mathscr{X}(C_0(K_\mathcal{A}))$ has codimension 1 in $\mathscr{B}(C_0(K_\mathcal{A}))$.

Such ideals play a special role in the theory of Banach algebras.

Definition. Let $\mathscr{B}$ be a Banach algebra. A non-zero algebra homomorphism (= a non-zero multiplicative linear functional) $\varphi\colon \mathscr{B} \to \mathbb{K}$ is a ***character***.

Remarks. Let $\varphi\colon \mathscr{B} \to \mathbb{K}$ be a character. Then:

- φ is continuous.
- $\ker\varphi$ is a closed, maximal ideal of codimension 1 in $\mathscr{B}$.
- Conversely, suppose that $\mathscr{I}$ is a closed ideal of codimension 1 in $\mathscr{B}$. Then

  $$\mathscr{B}/\mathscr{I} \cong \mathbb{K},$$

  and so the quotient map induces a character on $\mathscr{B}$.

Characters on $\mathscr{B}(X)$ are "unusual" due to following result:

Theorem. *Let X be a Banach space which contains a complemented subspace isomorphic to $X \oplus X$. Then $\mathscr{B}(X)$ has no closed ideals of finite codimension, and therefore no characters.*

However, characters are known to exist on $\mathscr{B}(X)$ in a number of cases:

- $X = J_p$ (the p^{th} quasi-reflexive James space) for $1 < p < \infty$ (Berkson–Porta, *JFA* 1969; Edelstein–Mityagin, *Funct. Anal. Appl.* 1970).
- $X = C[0, \omega_1]$ (Edelstein–Mityagin, *Funct. Anal. Appl.* 1970; implicit in Semadeni, *Bull. Acad. Pol. Sci.* 1960).
- X being any of the previously described "few operator" spaces of Argyros–Haydon, Gowers–Maurey, Shelah, Steprāns, Wark.

Remark. Let $\psi\colon \mathscr{B}(X_{\mathsf{AH}}) \to \mathbb{K}$ be the character, where X_{AH} is the Banach space of Argyros–Haydon. Then

$$\ker \psi = \overline{\mathscr{F}(X_{\mathsf{AH}})}$$

is as small as it can be.

Consequences of the main theorem: A character on $\mathscr{B}(C(K))$ with smallest possible kernel

Let $\mathcal{A} \subseteq [\mathbb{N}]^\omega$ be an uncountable, almost disjoint family which admits few operators, and recall that $\mathscr{B}(C_0(K_\mathcal{A}))$ admits a character φ with

$$\ker\varphi = \mathscr{X}(C_0(K_\mathcal{A})) = \mathscr{G}_{c_0}(C_0(K_\mathcal{A}))$$
$$= \{C_0(K_\mathcal{A}) \xrightarrow{S} c_0 \xrightarrow{T} C_0(K_\mathcal{A}) : S \in \mathscr{B}(C_0(K_\mathcal{A}), c_0),\ T \in \mathscr{B}(c_0, C_0(K_\mathcal{A}))\}.$$

Analogous to the Argyros–Haydon space, this is the smallest the kernel of a character can be in the context of $C(K)$-spaces. More precisely, we have:

Theorem (Koszmider–L). *Let K be an infinite compact Hausdorff space such that $\mathscr{B}(C(K))$ admits a character $\psi\colon \mathscr{B}(C(K)) \to \mathbb{K}$. Then*

$$\mathscr{G}_{c_0}(C(K)) \subseteq \ker\psi.$$

Corollary. *$\mathscr{B}(C_0(K_\mathcal{A}))$ admits only one character.*

Proof of the theorem

Set-up: K is an infinite compact Hausdorff space and $\psi\colon \mathscr{B}(C(K)) \to \mathbb{K}$ is a character. We seek to prove that

$$\mathscr{G}_{c_0}(C(K)) \subseteq \ker\psi.$$

Take $S \in \mathscr{G}_{c_0}(C(K))$. We consider two cases:

Case 1: $C(K)$ is a Grothendieck space. Then:

- Diestel (1972): S is weakly compact because it has separable range.
- Pełczyński: S is strictly singular.
- L (*Proc. Edinb. Math. Soc.* 2002): S belongs to *every* maximal ideal of $\mathscr{B}(C(K))$.
- Hence $S \in \ker\psi$.

Case 2: $C(K)$ is **not** a Grothendieck space. Then:

- Schachermayer (*Diss. Math.* 1982): There is a projection $P \in \mathscr{B}(C(K))$ with $P[C(K)] \cong c_0$.
- $\mathscr{B}(c_0)$ admits no characters, so $\psi(P) = 0$.
- Write $S = TPR$ for some $R, T \in \mathscr{B}(C(K))$. Then

$$\psi(S) = \psi(T)\psi(P)\psi(R) = 0. \qquad \square$$

To be continued tomorrow...

Thank you!