Stopping-time spaces and their operators

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IM UJ / MU AV ČR

Workshop on Banach spaces and Banach lattices II, ICMAT (Madrid)
joint work with R. Lechner (Linz), `arXiv:2112.12534`

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- $0 \hookrightarrow \mathcal{K}(\ell_2) \hookrightarrow \mathcal{B}(\ell_2)$ (Calkin).
- other classical spaces:
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  - $0 \hookrightarrow \mathcal{K}(X) \hookrightarrow \mathcal{X}^{\aleph_0}(X) \hookrightarrow \mathcal{X}^{\aleph_1}(X) \hookrightarrow \ldots \hookrightarrow \mathcal{B}(X)$,
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- $Z = X_{\mathrm{AH}} \oplus$ suitably constructed subspace (K.–Laustsen).

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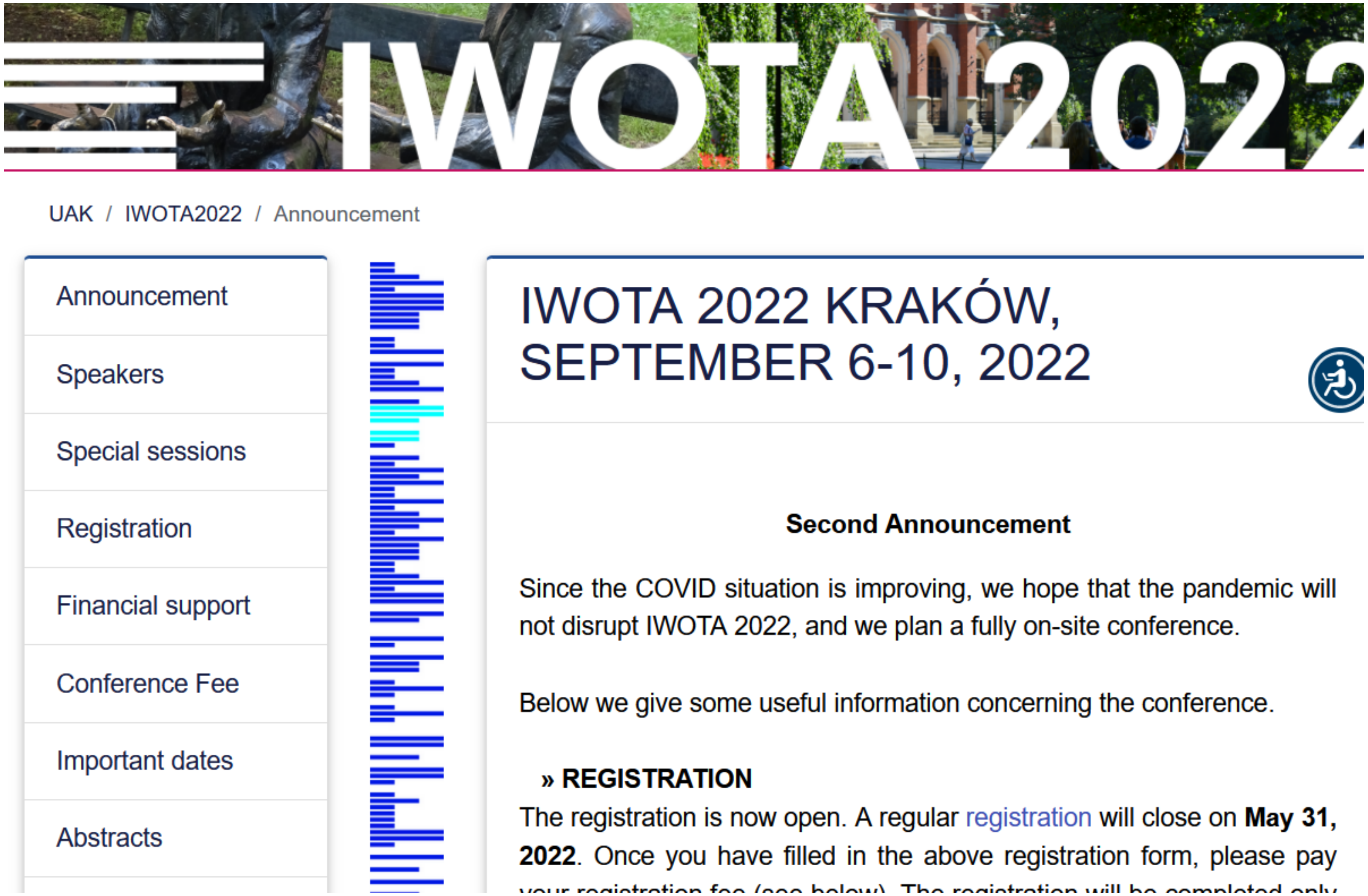
- c_0, ℓ_p (here $p = \infty$ is included, btw. $\ell_\infty \cong L_\infty$);
- $L_p[0,1]$ for $p \in [1, \infty]$.
- $c_0(\Gamma), \ell_p(\Gamma)$ for $p \in [1, \infty)$
- ℓ_∞ / c_0, $\ell_\infty^c(\Gamma)$ for any set Γ (but not every $L_\infty(\mu)$ is in this class!)
- c_0- and ℓ_p-sums of ℓ_2^ns or ℓ_∞^ns as well as more general sums.
- Lorentz sequence spaces
  determined by a decreasing, non-summable sequence and $p \in [1, \infty)$.
- certain Orlicz spaces.
- $C[0,1]$, $C[0, \omega^\omega]$, $C[0, \omega_1]$, and the list goes on.

A shameless advertisement: IWOTA2022 in Kraków, September 6-10, 2022

iwota2022.urk.edu.pl



You may be interested in the panel session in Special Session *Operator ideals and operators on Banach spaces* run by Niels Laustsen, Kevin Beanland, and I.

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- Buehler considered the space S^2 that may be viewed as a certain convexification of S^1 and proved that ℓ^p does not embed therein for $p \in [1,2)$.
- S^2 is a member of a broader scale of spaces S^p $(p \in [1,\infty))$.
  Schechtman:
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- S^1 contains isomorphic copies Orlicz sequence spaces ℓ^M for a rather wide class of Orlicz functions M.

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The above results provide yet another example of resemblance between S^1 and L^1 as Enflo & Starbird proved that I_{L^1} factors through every operator $T\colon L^1 \to L^1$ that fixes a copy of L^1.

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- A *stopping time* is an $\mathbb{N} \cup \{\infty\}$-valued random variable T on $(\Omega, \mathcal{F}, P)$ s.t. for each n, we have $[T = n] \in \mathcal{F}_n$. $\mathcal{T}$ denotes the family of all stopping times on $(\Omega, \mathcal{F}, P)$ w.r.t. $\mathbb{F}$.

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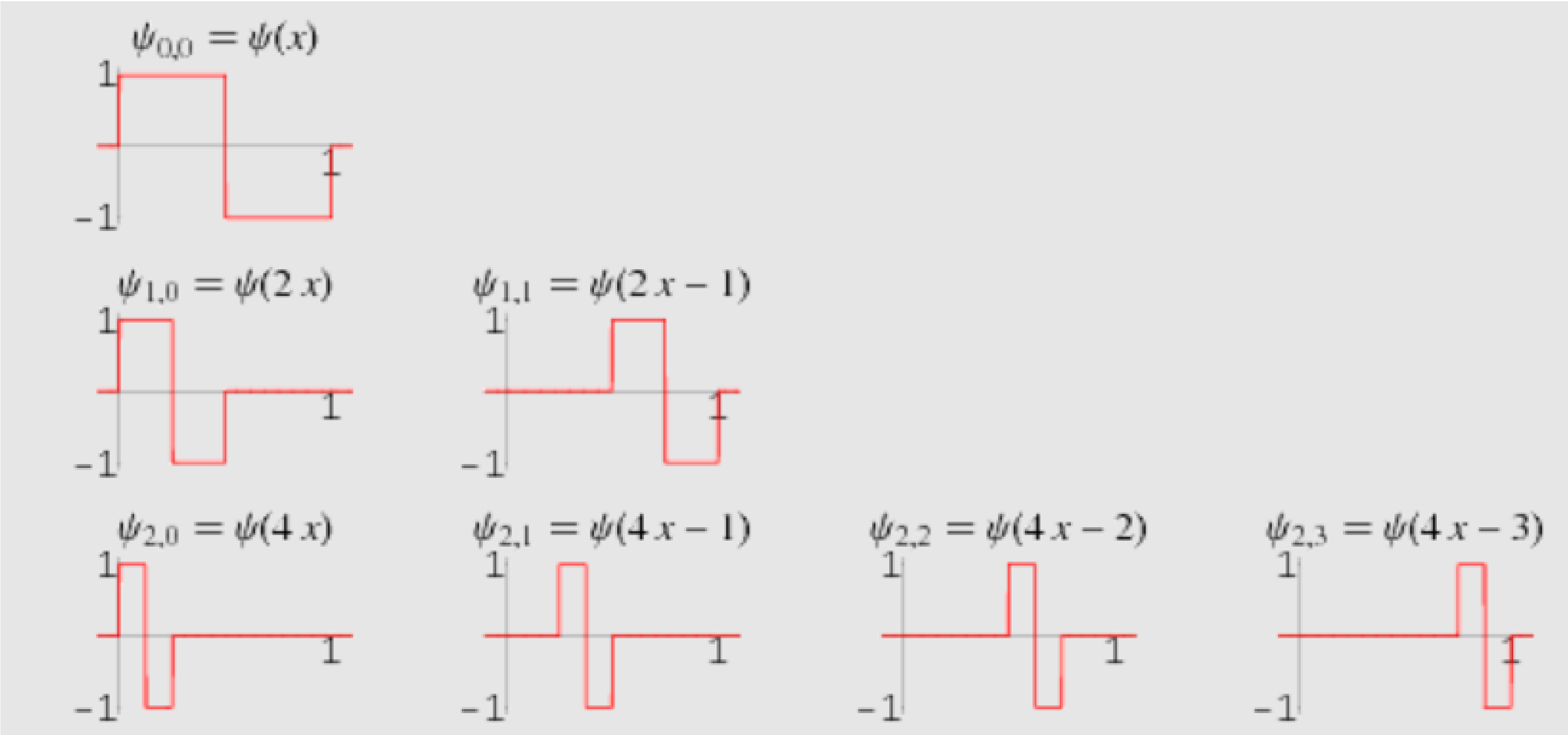
In a sense only the 'dyadic-tree filtration' is useful.

Haar/Rademacher functions



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$$\|((a_t)_{t\in 2^{<\omega}}\|_{S^p} = \sup\Big\{\big(\sum_{s\in\mathcal{A}} |a_s|^p\big)^{1/p} : \mathcal{A} \subset 2^{<\omega} \text{ is an antichain}\Big\}.$$

- The standard u.v.b. $(e_t)_{t\in 2^{<\omega}}$ of c_{00} is then a 1-unconditional basis of S^p.

$2^{<\omega}$: the binary tree, which indexes the Haar basis $(h_t)_{t\in 2^{<\omega}}$ of $C(\Delta)$.

- B may be viewed as the space with a minimal 1-unconditional basis that dominates the Haar basis in $C(\Delta)$.
  It is the completion $c_{00}(2^{<\omega})$ (span of $\{h_t : t \in 2^{<\omega}\}$ in $C(\Delta)$) w.r.t.

$$\|(a_t)_{t\in 2^{<\omega}}\|_B = \sup\Big\{\Big\| \sum_{t\in 2^{<\omega}} a_t\varepsilon_t h_t\Big\|_{C(\Delta)} : t \in 2^{<\omega}, |\varepsilon_t| = 1\Big\}.$$

- The norm in B may be isometrically realised as

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- If $(e_t^*)_{t\in 2^{<\omega}}$ denotes the coordinate functionals associated to $(e_t)_{t\in 2^{<\omega}}$, then B may be viewed as a subspace of $D = (S^1)^*$ spanned by $(e_t^*)_{t\in 2^{<\omega}}$.

Let E be a space with a 1-subsymmetric Schauder basis.

- For $\mathcal{A} \subseteq 2^{<\omega}$, let $P_{\mathcal{A}}$ denote the projection on $\mathcal{A}$.
- S^E and B^E as completions of c_{00} with respect to the norms of $(a_t)_{t \in 2^{<\omega}} \in c_{00}(2^{<\omega})$:

$$\begin{aligned} \|(a_t)_{t \in 2^{<\omega}}\|_{S^E} &= \sup\left\{ \|P_{\mathcal{A}}(a_t)_{t \in 2^{<\omega}}\|_E : \mathcal{A} \subset 2^{<\omega} \text{ is an antichain} \right\}, \\ \|(a_t)_{t \in 2^{<\omega}}\|_{B^E} &= \sup\left\{ \|P_{\mathcal{A}}(a_t)_{t \in 2^{<\omega}}\|_E : \mathcal{A} \subset 2^{<\omega} \text{ is a branch} \right\}, \end{aligned}$$

General case

Let E be a space with a 1-subsymmetric Schauder basis.

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- B^{ℓ^1} corresponds to B. We put $D^E = (S^E)^*$.
- For a space E with a 1-subsymmetric basis indexed by $2^{<\omega}$ we denote by
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  - $(e_t)_{t\in 2^{<\omega}}$ the standard Schauder basis in S^E or B^E,
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Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a dual pair of B. spaces. $((e_\gamma, f_\gamma))_{\gamma\in\Gamma}$ in $X \times Y$:

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- $(x_n)_{n=1}^\infty$ is *C-dominated* by $(\xi_n)_{n=1}^\infty$ if for all scalar sequences $(a_n)_{n=1}^\infty$

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and for all $(a_n)_{n=1}^\infty$ s.t. $\sum_{n=1}^\infty a_n \xi_n$ converges w.r.t $\sigma(X, Y)$ we have

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Let E be a space with a normalised basis indexed by $2^{<\omega}$, say $(e_t)_{t\in 2^{<\omega}}$. Then $(e_t)_{t\in 2^{<\omega}}$ is

- 1-*unconditional*, whenever for all finitely supported sequences of scalars $(a_t)_{t\in 2^{<\omega}}$ and $(\gamma_t)_{t\in 2^{<\omega}}$ one has

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- 1-*spreading*, if $(e_t)_{t\in 2^{<\omega}}$ is 1-equivalent to each of its increasing subsequences (with respect to the standard linear order), *i.e.*, if for all finitely supported sequences of scalars $(a_t)_{t\in 2^{<\omega}}$

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- 1-*subsymmetric* if it is 1-unconditional as well as 1-spreading.

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- is *almost annihilating* for a set $\mathcal{A} \subset \mathcal{B}(X)$, if for $T \in \mathcal{A}$ and $\eta > 0$ there are $(x_n)_{n=1}^{\infty}$ in X and $(y_n)_{n=1}^{\infty}$ in Y, respectively, such that

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  - (i) $(x_n)_{n=1}^{\infty}$ is dominated by $(e_n)_{n=1}^{\infty}$;
  - (ii) $(y_n)_{n=1}^{\infty}$ is dominated by $(f_n)_{n=1}^{\infty}$;
  - (iii) $\inf_{n\in\mathbb{N}} \langle x_n, y_n \rangle \geqslant 1$;
  - (iv) $\sup_{n\in\mathbb{N}} \langle Tx_n, y_n \rangle \leqslant \eta$.

Theorem. Let $(X, Y, \langle\cdot,\cdot\rangle)$ be a dual pair of Banach spaces. Suppose that

- $(e_n)_{n=1}^{\infty}$ be a basis for X w.r.t. the $\sigma(X, Y)$ topology,
- $(f_n)_{n=1}^{\infty}$ be a basis for space Y w.r.t. the $\sigma(Y, X)$ topology,
- there is $c > 0$ s.t.

$$c\|x\| \leqslant \sup_{\|y\|\leqslant 1} \langle x, y\rangle \leqslant \|x\| \qquad (x \in X). \tag{2}$$

If

- $((e_n, f_n))_{n=1}^{\infty}$ almost annihilates

$$\mathcal{M}_X = \{T\colon I \neq ATB\ (A, B \in \mathcal{B}(X))$$

- has the positive factorisation property

then $\mathcal{M}_X$ is the unique maximal ideal of $\mathcal{B}(X)$.

Proof

- Let $0 < \eta < 1$, $S \in \mathcal{M}_X$, $T \in \mathcal{B}(X)$ and $S + T \notin \mathcal{M}_X$.
- Then there are $A, B \in \mathcal{B}(X)$ s.t. $I_X = B(S+T)A$.
- Since BSA must be in $\mathcal{M}_X$ and $((e_n, f_n))_{n=1}^{\infty}$ is almost annihilating for $\mathcal{M}_X$, we can find sequences $(x_n)_{n=1}^{\infty}$ in X dominated by $(e_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ in Y dominated by $(f_n)_{n=1}^{\infty}$ s.t.

$$\langle x_n, y_n \rangle \geqslant 1 \quad \text{and} \quad \langle BSAx_n, y_n \rangle \leqslant \eta \qquad (n \in \mathbb{N}).$$

- $1 \leqslant \langle x_n, y_n \rangle = \langle B(S+T)Ax_n, y_n \rangle \leqslant \eta + \langle BTAx_n y_n \rangle \qquad (n \in \mathbb{N}).$
- We set $L \colon X \to X$ and $R \colon Y \to Y$ as the linear extensions of

$$Le_n = x_n \quad \text{and} \quad Rf_n = y_n \qquad (n \in \mathbb{N}).$$

- Next, put $U = R^* BTAL$, where R^* denotes the unique operator such that $\langle R^* x, y \rangle = \langle x, Ry \rangle$ for all $x \in X$, $y \in Y$; $R^* \in \mathcal{B}(X)$ and

$$\inf_n \langle Ue_n, f_n \rangle = \inf_n \langle BTAx_n, y_n \rangle \geqslant 1 - \eta > 0.$$

- $((e_n, f_n))_{n=1}^{\infty}$ has the positive factorisation prop. so $U \notin \mathcal{M}_X$, so $T \notin \mathcal{M}_X$.
- $\mathcal{M}_X$ is closed under addition (hence the unique maximal ideal of $\mathcal{B}(X)$).

Theorem Let $(X, Y, \langle \cdot, \cdot \rangle)$ be a dual pair of Banach spaces. Suppose that

- $(e_n)_{n=1}^{\infty}$ is a basis of X with respect to the topology $\sigma(X, Y)$,
- $(f_n)_{n=1}^{\infty}$ is a basis of Y with respect to the topology $\sigma(Y, X)$,
- the system $((e_n, f_n))_{n=1}^{\infty}$ is biorthogonal,
- there exists $c > 0$ such that

$$c\|x\| \leqslant \sup_{\|y\| \leqslant 1} \langle x, y \rangle \leqslant \|x\| \qquad (x \in X), \tag{3}$$

- $((e_n, f_n))_{n=1}^{\infty}$ is strategically supporting,
- $((e_n, f_n))_{n=1}^{\infty}$ has the positive factorisation property.

Then $\mathcal{M}_X$ is the unique maximal ideal of $\mathcal{B}(X)$.

X is either S^E or B^E with basis $(e_s)_{s\in 2^{<\omega}}$ and coordinate functionals $(e_s^*)_{s\in 2^{<\omega}}$.

Definition Let $C \geqslant 1$ and consider the following two-player game $\mathrm{Rep}_{(X,(e_s))}(C)$. For $t \in 2^{<\omega}$, turn t is played out in three steps.

1. Player (I) chooses $\eta_t > 0$, $W_t \in \mathrm{cof}(X)$, and $G_t \in \mathrm{cof}_{w^*}(X^*)$,
2. Player (II) chooses a finite subset E_t of $2^{<\omega}$ and sequences of non-negative real numbers $(\lambda_s^{(t)})_{s\in E_t}$, $(\mu_s^{(t)})_{s\in E_t}$ satisfying $\sum_{s\in E_t}\lambda_s^{(t)}\mu_s^{(t)} = 1$.
3. Player (I) chooses $(\varepsilon_s^{(t)})_{s\in E_t}$ in $\{-1,1\}^{E_t}$.

Player (II) has a winning strategy in $\mathrm{Rep}_{(X,(e_s))}(C)$ if he can force the following properties on the result: For all $t \in 2^{<\omega}$ setting:

$$b_t = \sum_{s\in E_t}\varepsilon_s^{(t)}\lambda_s^{(t)}e_s \qquad \text{and} \qquad b_t^* = \sum_{s\in E_t}\varepsilon_s^{(t)}\mu_s^{(t)}e_s^*$$

(i) the sequences $(b_t)_{t\in 2^{<\omega}}$ and $(e_t)_{t\in 2^{<\omega}}$ are $(1/\sqrt{C},\sqrt{C})$-equivalent,

(ii) the sequences $(b_t^*)_{t\in 2^{<\omega}}$ and $(e_t^*)_{t\in 2^{<\omega}}$ are $(1/\sqrt{C},\sqrt{C})$-equivalent,

(iii) for all $t \in \mathbb{N}$ we have $\mathrm{dist}(b_t, W_t) < \eta_t$, and

(iv) for all $t \in \mathbb{N}$ we have $\mathrm{dist}(b_t^*, G_t) < \eta_t$.

$(e_s)_{s\in 2^{<\omega}}$ is *C-strategically reproducible in X* if for every $\eta > 0$ player II has a winning strategy in the game $\mathrm{Rep}_{(X,(e_s))}(C+\eta)$.

Hard work

Thm 1. $(e_t)_{t\in 2^{<\omega}}$ is 1-strategically reproducible both in S^E and B^E.

Thm 2. If $1 \leqslant p, p' \leqslant \infty$, $1 < q, q' < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$, then $((e_{ij}, f_{ij})\colon i, j \in \mathbb{N})$ has the positive factorisation property in $\ell^p(\ell^q) \times \ell^{p'}(\ell^{q'})$.

Thm 3. Let $1 \leqslant p, q \leqslant \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$. Then $((e_{ij}, f_{ij})\colon i, j \in \mathbb{N})$ is strategically supporting in $\ell^p(\ell^q) \times \ell^{p'}(\ell^{q'})$.

Cor. Let $1 \leqslant p \leqslant \infty$ and $1 < q < \infty$. Then $\mathcal{M}_{\ell^p(\ell^q)}$ is the unique closed proper maximal ideal of $\mathcal{B}(\ell^p(\ell^q))$.

Third main theorem Let X be any Banach space and let E denote a space with a normalised 1-subsymmetric Schauder basis. Then $\mathcal{M}_X$ is the unique maximal ideal of $\mathcal{B}(X)$ in the following cases:

- $X = S^E$, $X = B^E$;
- $X = \ell^p(S^E)$ or $X = \ell^p(B^E)$, $1 \leqslant p \leqslant \infty$;
- $X = D^E = (S^E)^*$, whenever the 1-subsymmetric Schauder basis of E is incomparably non-c_0 on antichains.

Consequently, the spaces $\ell^p(S^E)$, $\ell^p(B^E)$ $(1 \leqslant p \leqslant \infty)$ are primary.