ESSENTIALLY DISJOINT OPERATOR RANGES

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DEFINITION

A subspace R of a Banach space E is called an *operator range* (in E) if there is a Banach space Z and a bounded linear operator $T : Z \to E$ such that $R = T(Z)$.

SOME REFERENCES

P.A. Fillmore and J.P. Williams, 1971; R.W. Cross, 1980; V. Shevchik 1982; V. Fonf 1991; A.N. Plichko, 1992; R.W. Cross and V. Shevchik, 1998; D. Kitson and R.M. Timoney 2011; A.F.M. ter Elst and M. Sauter, 2016; T. Oikhberg, 2017; V. P. Fonf, S. Lajara, S. Troyanksi and C. Zanco, 2019.

THEOREM [Saxon and Wilanski, 1977]

Let E be a Banach space. T.F.A.E:

1. E has a proper dense operator range,
2. E has a separable infinite dimensional quotient,
3. E has a closed separable infinite dimensional Y quasicomplemented in E, i.e. there is a closed subspace Z such that $Y \cap Z = \{0\}$ and $\overline{Y + Z} = E$.
4. E has a strictly increasing sequence $\{X_n\}_n$ of closed subspaces such that $\overline{\bigcup_n X_n} = E$,

THEOREM [Shevchik, 1982]
If E is a separable Banach space and R is a proper dense operator range in E, then there is a compact one-to-one and dense-range operator $T : E \to E$ such that $T(E) \cap R = \{0\}$.

THEOREM [Chalendar and Partington, 2005]
If E is a separable Banach space and Y is an infinite dimensional and closed subspace of E, then there is a compact one-to-one and dense-range operator $T : E \to E$ such that $T(Y)$ is a dense subspace of Y.

THEOREM [Chalendar and Partington, 2005]
If E is a separable Banach space and $\{Y_n\}_n$ is a chain (i.e. $Y_n \subset Y_{n+1}$ of infinite-dimensional closed subspaces of E, then there is a compact one-to-one and dense-range operator $T : E \to E$ such that $T(Y_n)$ is a dense subspace of Y_n for all n.

DEFINITIONS

* $\mathcal{R}(E)$ **denotes the family of infinite-codimensional operator ranges in the Banach space** E**.**

In particular, proper and dense operator ranges of E **are contained in** $\mathcal{R}(E)$**.**

* $\mathcal{S}(E)$ **denotes the family made up of all countable unions of elements of** $\mathcal{R}(E)$**.**

* **A subspace** Z **in** E^* **is called:**

▷ **total (over** E**) whenever** $\overline{Z}^{w^*} = E^*$**.**

▷ λ**-norming (over** E**) for some** $\lambda \in (0,1)$ **whenever** $\sup\{|x^*(x)| : x^* \in B_Z\} \geq \lambda \|x\|$ **for all** $x \in E$**.**

* **For** $Y \subset E$**, the set** $Y^\perp$ **denotes the annihilator of** Y **in** E^* **(i.e.** $Y^\perp = \{f \in E^* : f|_Y = 0\}$**) .**

* **A compact operator** $T : E \to E$ **is called nuclear whenever** T **is of the form** $T(x) = \sum_n e_n f_n(x)$ **for certain sequences** $\{e_n\} \subset E$ **and** $\{f_n\} \subset E^*$ **with** $\sum_n \|e_n\| \|f_n\| < \infty$**.**

THEOREM 1 [M. J-S, S. Lajara]
Let E be an (infinite dimensional) separable Banach space and Z be a closed, separable and total subspace of E^*. Then, for any couple of elements $R \in \mathcal{S}(E)$ and $V \in \mathcal{S}(Z)$ there exists a nuclear operator $T : E \to E$ such that

1. $\overline{T(E)} = E$,
2. $T^*(E^*) \subset Z$,
3. $\overline{T^*(Z)} = Z$ **(in particular, T is one-to-one),**
4. $T(E) \cap R = \{0\}$,
5. $T^*(E^*) \cap V = \{0\}$.

THEOREM 2 [M. J-S, S. Lajara]
Let E be a separable Banach space and let Y be a closed infinite dimensional subspace of E. Then, for every $R \in \mathcal{S}(Y)$ and $V \in \mathcal{S}(Y^*)$ and every $\lambda \in (0,1)$, there exists a nuclear one-to-one operator $T : E \to E$ such that

1. $T(E) \subset Y$,
2. $\overline{T(Y)} = Y$,
3. $T(E) \cap R = \{0\}$,
4. $T^*(E^*)|_Y \cap V = \{0\}$,
5. $T^*(E^*)|_Y$ **is λ-norming for Y.**
6. **If Y^* is separable, then $\overline{T^*(E^*)|_Y} = Y^*$.**
7. **If X^* is separable, then $\overline{T^*(E^*)} = E^*$.**

COROLLARY 3 [M. J-S, S. Lajara]
Let E be an (infinite dimensional) separable Banach space. Then, for every $R \in \mathcal{S}(E)$, $V \in \mathcal{S}(E^*)$ and every $\lambda \in (0,1)$, there exists a nuclear one-to-one and dense-range operator $T : E \to E$ such that

1. $T(E) \cap R = \{0\}$,
2. $T^*(E^*)$ **is** λ**-norming for** E,
3. $T^*(E^*) \cap V = \{0\}$,
4. **If** E^* **is separable then** $\overline{T^*(E^*)} = E^*$.

COROLLARY 4 [M. J-S, S. Lajara]
Let E be an (infinite dimensional) separable Banach space. Then, for every $R \in \mathcal{R}(E)$, $V \in \mathcal{R}(E^*)$ and $\varepsilon > 0$, there exists an isomorphism $I : E \to E$ such that $I(R) \cap R = \{0\}$, $I^*(V) \cap V = \{0\}$ and $||I - Id|| < \varepsilon$.

THEOREM 5 [M. J-S, S. Lajara]
Let E be a separable Banach space and let Y be a closed infinite-dimensional and infinite codimensional subspace of E and consider elements $R \in \mathcal{S}(E)$ and $V \in \mathcal{S}(E^*)$ such that

$$R \cap Y \in \mathcal{S}(Y), \quad V|_Y \in \mathcal{S}(Y^*) \quad \text{and} \quad V \cap Y^\perp \in \mathcal{S}(Y^\perp).$$

Then, for any $\lambda \in (0,1)$ there is a nuclear one-to-one dense-range operator $T : E \to E$ such that

1. $\overline{T(Y)} = Y$,
2. $T^*(Y^\perp) \subset Y^\perp$ and $T^*(Y^\perp)$ is w^*-sequentially dense in $Y^\perp$,
3. $T(E) \cap R = \{0\}$,
4. $T^*(E^*) \cap V = \{0\}$,
5. $T^*(E^*)|_Y$ is λ-norming for Y,
6. $T^*(E^*)|_Y \cap V|_Y = \{0\}$.
7. If $(E/Y)^*$ is separable then $\overline{T^*(Y^\perp)} = Y^\perp$.
8. If Y^* is separable then $\overline{T^*(E^*)|_Y} = Y^*$.
9. If E^* is separable then $\overline{T^*(E^*)} = E^*$.

DEFINITION
We will say that the closed subspaces X and Y of a Banach space E are quasicomplementary if $X \cap Y = \{0\}$ and $\overline{X + Y} = E$.

THEOREM 6 [M. J-S, S. Lajara]
If X and Y are closed infinite dimensional quasicomplementary subspaces of a separable Banach space E, then for any $R_1 \in \mathcal{S}(X)$, $R_2 \in \mathcal{S}(Y)$, $V_1 \in \mathcal{S}(X^\perp)$ and $V_2 \in \mathcal{S}(Y^\perp)$ there exists a nuclear one-to-one dense range operator $T : E \to E$ such that

1. $\overline{T(X)} = X$ **and** $\overline{T(Y)} = Y$,
2. $\overline{T^*(X^\perp)}^{w^*} = X^\perp$ **and** $\overline{T^*(Y^\perp)}^{w^*} = Y^\perp$,
3. $T(E) \cap (R_1 + R_2) = \{0\}$ **and** $T^*(E^*) \cap (V_1 + V_2) = \{0\}$.

DEFINITION

Let $R \in \mathcal{R}(E)$ **and let** $A : Z \to E$ **be a bounded operator such that** $A(Z) = R$**. Clearly,** $R = \bigcup_{m\in\mathbb{N}} mA(B_Z)$ **(being** B_Z **the closed unit ball of** Z**).**

We define $R^+ := \bigcup_{m\in\mathbb{N}} m\overline{A(B_Z)}$ **(which is in** $\mathcal{R}(E)$**).**

For $R = \bigcup_n R_n \in \mathcal{S}(E)$**, we define similarly** $R^+ := \bigcup_n R_n^+$ **(which is in** $\mathcal{S}(E)$**).**

THEOREM 7 [M. J-S, S. Lajara]
Let E be an (infinite dimensional) separable Banach space and $\{Y_n\}_n$ a chain of closed subspaces of E such that Y_1, Y_n/Y_{n-1} and E/Z are infinite dimensional for all $n > 1$, where $Z = \overline{\cup_n Y_n}$, and consider sets $R \in \mathcal{S}(E)$ and $W \in \mathcal{S}(E^*)$ such that

(a) $R^+ \cap Y_n \in \mathcal{S}(Y_n)$ **for all** $n \geq 1$,

(b) $W \cap Z^\perp \in \mathcal{S}(Z^\perp)$,

(c) $W|_{Y_1} \in \mathcal{S}(Y_1^*)$ **and** $W|_{Y_n} \cap Y_{n-1}^\perp|_{Y_n} \in \mathcal{S}(Y_{n-1}^\perp|_{Y_n})$ **if** $n \geq 2$.

Then, there is a nuclear and one-to-one operator $T : E \to E$ such that

1. $\overline{T(Y_n)} = Y_n$,
2. $T(E) \cap R = \{0\}$,
3. $\overline{T(E)} = E$,
4. $T^*(E^*)|_{Y_n} \cap W|_{Y_n} = \{0\}$ **for all** $n \geq 1$,
5. $T^*(E^*) \cap W = \{0\}$,
6. **If** Y_n^* **is separable then** $\overline{T^*(E^*)|_{Y_k}} = Y_k^*$ **for all** $k \leq n$,
7. **If** E^* **is separable then** $\overline{T^*(E^*)} = E^*$.

PROOFS. SOME IDEAS

▷ **The compact operators defined in the proofs can be written as a finite or infinite sum of operator of the form** $T(x) = \sum_n 2^{-n} e_n f_n(x)$ **for** $x \in E$ **and certain sequences** $\{e_n\}_n \subset B_E$ **and** $\{f_n\}_n \subset B_{E^*}$ **. So** $T(B_E) \subset \overline{\mathrm{conv}}(\{\pm 2^{-n} e_n\}_n)$ **and** $T^*(B_{E^*}) \subset \overline{\mathrm{conv}}(\{\pm 2^{-n} f_n\}_n)$**.**

▷ **We are seeking sequences** $\{x_n\}_n \subset E$ **and** $\{x_n^*\}_n \subset E^*$ **with** $\lim_n x_n = 0$, $\lim_n x_n^* = 0$, **satisfying** $\overline{\mathrm{conv}}(\{\pm x_n\}_n) \cap R = \{0\}$ **and** $\overline{\mathrm{conv}}(\{\pm x_n^*\}_n) \cap W = \{0\}$ **and additional properties (minimality, miminality with respect to a subspace, M-basis, total,** λ**-norming condition for the span of these sets, etc...), where** R **and** W **are the given operator ranges.**

▷ **Fonf, 1991: Let** V **a closed convex and bounded set is a Banach space** E **such that** $\mathrm{codim}(\mathrm{span}\, V) = \infty$**. Then, for every pair of sequences** $\{v_n\}_n \subset S_E$ **and** $\{\varepsilon_n\}_n \subset (0, \infty)$**, there are sequences** $\{w_n\}_n \subset S_E$ **and** $\{\gamma_n\}_n \subset (0, \infty)$ **satisfying** $\sum_n \gamma_n < 1$, $\|v_n - w_n\| < \varepsilon_n$ **for all** $n \geq 1$ **and**

$$\overline{\mathrm{conv}}(\{\pm \gamma_n w_n\}_n) \cap V = \{0\}.$$

PROOFS. SOME IDEAS

▷**The compact operator defined in the proof of Theorem 7 (assuming that** $\overline{\cup_n Y_n} = E$**) can be written as a sum** $T(x) = \sum_n T_n(x)$**, where each** T_n **is of the form** $T_n(x) = \sum_k f_k(x) e_k$ **for certain sequences** $\{e_k\}_k \subset B_{Y_n}$ **and** $\{f_k\}_k \subset B_{Y_{n-1}^{\perp}}$ **converging to** 0 **such that** $T_n(B_E) \subset \overline{\text{conv}}(\{\pm e_k\}_k) \subset Y_n$ **and** $T_n^*(B_{E^*}) \subset \overline{\text{conv}}(\{\pm f_k\}_k) \subset Y_{n-1}^{\perp}$**.**

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THANK YOU