Variable Lebesgue Spaces $L^{p(t)}(\Omega)$ versus $L_p(\Omega)$ spaces when it comes to Fixed Point Theory

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Introduction

Theorem (Banach Contraction Principle, 1922)

Let (C, d) be a complete m.s. and $T : C \to C$ a contraction. Then there exists a unique $x_0 \in C$ such that $T(x_0) = x_0$

$T : C \to C$ contraction: $\exists k \in (0, 1)$ $d(Tx, Ty) \leq kd(x, y)$ $\forall x, y \in C$.

The result does not hold if $k = 1$?

$T : \mathbb{R} \to \mathbb{R}$, $T(x) = x + a$ $(a \neq 0)$ fails to have a fixed point.

Definition

Let (C, d) be a m.s. A mapping $T : C \to C$ is nonexpansive if

$$d(Tx, Ty) \leq d(x, y) \qquad \forall x, y \in C.$$

Metric Fixed Point Theory for Nonexpansive Mappings

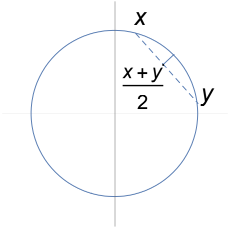
The first positive fixed point results for nonexpansive mappings were obtained in the 1960's in the framework of Banach spaces.

Theorem (D. Göhde, F. Browder, 1965)

Let $(X, \|\cdot\|)$ be a uniformly convex Banach space and $C \subset X$ closed convex and bounded. Let $T : C \to C$ be nonexpansive. Then T has a fixed point.

A Banach space is (UC) if for very $\epsilon \in (0, 2]$, $\exists\ \delta = \delta_X(\epsilon) > 0$ such that

$$\left.\begin{array}{l} \|x\| = 1 \\ \|y\| = 1 \\ \|x - y\| \geq \epsilon \end{array}\right\} \Longrightarrow \left\| \frac{x+y}{2} \right\| \leq 1 - \delta_X(\epsilon)$$



First positive results

Theorem (Kirk, 1965)

Let $(X, \|\cdot\|)$ be a Banach space with normal structure. Let C be a convex weakly compact subset of X and $T : C \to C$ nonexpansive. Then T has a fixed point

$(X, \|\cdot\|)$ has normal structure (NS) if for every convex weakly compact subset C with $\operatorname{diam}(C) > 0$, there exists $x_0 \in C$ such that

$$C \subset B(x_0, r) \qquad \text{for some } 0 < r < \operatorname{diam}(C).$$

Every UC Banach space is reflexive and has NS.

Standard definitions on Metric Fixed Point Theory

- $(X, \|\cdot\|)$ is said to have the FPP if for every closed convex bounded C and for every nonexpansive mapping $T : C \to C$, there is $x \in C$ with $T(x) = x$.

- $(X, \|\cdot\|)$ is said to have the w-FPP if for every convex weakly compact C and for every nonexpansive mapping $T : C \to C$, there is $x \in C$ with $T(x) = x$.

$$\text{FPP} \Rightarrow w\text{-FPP}$$

If X reflexive, both properties are alike.

Negative results: find a counterexample

$(\ell_1, \|\cdot\|_1)$ fails to have the FPP

$$C = \overline{co}(e_n) = \{x := (t_n) : t_n \geq 0, \sum_{n=1}^{\infty} t_n = 1\},\ T : C \to C$$

$$T(t_1, t_2, t_3, \cdots) = (0, t_1, t_2, t_3, \cdots)$$

T is fixed point free and $\|Tx - Ty\|_1 = \|x - y\|_1$.

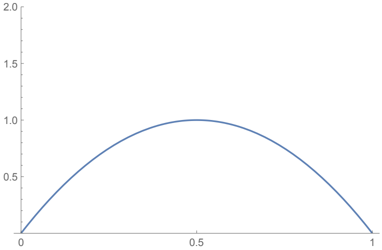
D.E. Alspach, 1981: $L_1[0,1]$ fails to have the w-FPP

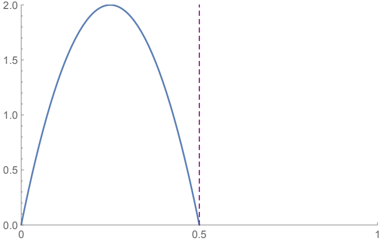
$C = \left\{ f : [0,1] \to [0,1], \int_0^1 f\,dm = \frac{1}{2} \right\}$ convex, w-compact: $T : C \to C$,

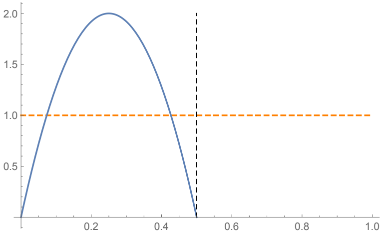
$$Tf(t) = \begin{cases} \min\{2f(2t), 1\} & 0 \leq t \leq \frac{1}{2} \\ \max\{2f(2t-1) - 1, 0\} & \frac{1}{2} < t \leq 1 \end{cases}$$

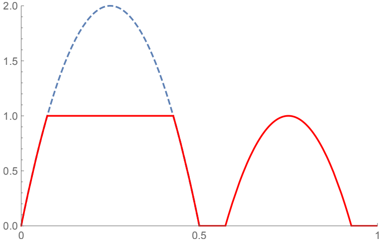
T is fixed point free in C and $\|Tf - Tg\|_1 = \|f - g\|_1 \ \forall f, g \in C$.

Image of $f(t) = -4(t - \frac{1}{2})^2 + 1$ under Alspach's example









Long-standing open problems:

Characterize those Banach spaces which have the FPP or the w-FPP (in case of no-reflexivity). We know that:

- $NS \Rightarrow w - \mathrm{FPP}$ (Kirk's theorem).
- w-FPP is an isometric property: Every Banach space containing an isometric copy of $(L_1[0,1], \|\cdot\|_1)$ fails the w-FPP: ℓ_∞ fails the w-FPP.
- Every separable Banach space can be renormed to have NS (V. Dulst, 1982) and therefore to have w-FPP (even every dual of a separable Banach space can be renormed to have NS).
- w-FPP is not an isomorphic property: Compare $(L_1[0,1], \|\cdot\|_1)$ and V. Dulst's renorming. Additionally, ℓ_∞ can be renormed to have w-FPP.

We know that Reflexivity + "some geometric property "imply the FPP

- FPP is an isometric property: Every Banach space containing an isometric copy of $(\ell_1, \|\cdot\|_1)$ fails the FPP.
- FPP is not an isomorphic property: ℓ_1 can be renormed to have the FPP (P.K. Lin 2008). This shows that there exist some non-reflexive Banach spaces with the FPP.
- ℓ_∞ cannot be renormed to have the FPP.

Long-standing open questions:

- Does every reflexive Banach space fulfil the FPP?
- Does every super-reflexive Banach space fulfil the FPP?
- Does every renorming of a Hilbert space fulfil the FPP?

Ruling out the FPP: counterexample or alternative method

Every Banach space X containing an isomorphic copy of ℓ_1 verifies:
$\exists (x_n) \subset X$, $(\epsilon_n) \subset (0,1)$ (Strong version of James' distortion theorem):

$$(1-\epsilon_k)\sum_{n=k}^{\infty}|t_n| \le \left\|\sum_{n=k}^{\infty} t_n x_n\right\| \le \sum_{n=k}^{\infty}|t_n| \qquad \forall (t_n)\in \ell_1.$$

Definition (Hagler, 1972)

A Banach space $(X, \|\cdot\|)$ contains an asymptotically isometric copy (a.i.c.) of ℓ_1 if there exist $(x_n) \subset X$ and $(\epsilon_n) \subset (0,1)$, $\lim_n \epsilon_n = 0$ such that

$$\sum_{n=1}^{\infty}(1-\epsilon_n)|t_n| \le \left\|\sum_{n=1}^{\infty} t_n x_n\right\| \le \sum_{n=1}^{\infty}|t_n| \qquad \text{for all } (t_n)\in \ell_1.$$

Theorem (P. Dowling, C. Lennard, 1997)

Every Banach space containing an a.i.c. of ℓ_1 fails the FPP.

Fixed Point Property in Lebesgue spaces $L_p(\Omega)$:

Let (Ω, Σ, μ) be non-atomic σ-finite measure space. TFAE:

1) $1 < p < +\infty$.
2) $(L^p(\Omega), \|\cdot\|_p)$ is UC
3) $(L^p(\Omega), \|\cdot\|_p)$ has the FPP.

For the purely atomic case ℓ_p:

- ℓ_p has the FPP $\Leftrightarrow 1 < p < +\infty$
- ℓ_p has the w-FPP $\Leftrightarrow 1 \leq p < +\infty$.

Maurey's result: Reflexive subspaces contained in $L_1(\Omega)$.

Let X be a closed subspace of $L_1(\Omega)$: X is non-reflexive if and only if it contains an isomorphic copy of ℓ_1.

Theorem (B. Maurey, 1981)

If $X \subset L_1(\Omega)$ is reflexive, then $(X, \|\cdot\|_1)$ has the FPP.

Theorem (P. Dowling, C. Lennard, 1997)

Let $X \subset L_1(\Omega)$: X is non-reflexive if and only if it contains an isometrically isomorphic copy of ℓ_1.

If X is a closed subspace of $\subset L_1(\Omega)$, then:

X is reflexive if and only if $(X, \|\cdot\|_1)$ has the FPP

Variable Lebesgue Spaces $L^{p(\cdot)}(\Omega)$

Let (Ω, Σ, μ) be a σ-finite measurable space and

$p : \Omega \to [1, +\infty] : t \to p(t)$ measurable function.

Let $\Omega_{<\infty} = \{t : p(t) < +\infty\}$, $\Omega_\infty = \{t \in \Omega : p(t) = +\infty\}$.

$$\rho(f) = \int_{\Omega_{<\infty}} |f(t)|^{p(t)} d\mu + \|f 1_{\Omega_\infty}\|_\infty$$

$$L^{p(\cdot)}(\Omega) := \{f \text{ measurable} : \exists \lambda > 0 : \rho(f/\lambda) < +\infty\}$$

$$\|f\|_{p(\cdot)} = \inf \left\{\lambda > 0 : \rho\left(\frac{f}{\lambda}\right) \leq 1\right\}$$

$(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ is a Banach space

If $p(t) \equiv p$ then $(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)}) = (L_p(\Omega), \|\cdot\|_p)$.

References related to Variable Lebesgue Spaces

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Reflexivity in Variable Lebesgue Spaces

Assume that (Ω, Σ, μ) is a **non-atomic** measure, $p : \Omega \to [1, +\infty)$ measurable.

$$p_- := \operatorname*{ess\,inf}_{t\in\Omega} p(t), \qquad p_+ := \operatorname*{ess\,sup}_{t\in\Omega} p(t)$$

Theorem (Lukes, Pick, Pokorny, 2011)

The following are equivalent:

- $(L^{p(\cdot)}, \|\cdot\|_{p(\cdot)})$ *is reflexive.*
- $1 < p_- \leq p_+ < +\infty$.
- $(L^{p(\cdot)}, \|\cdot\|_{p(\cdot)})$ *is uniformly convex.*

The atomic-case: Nakano sequence spaces

Assume $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$. In this case $p = (p_n) \subset [1, +\infty)$, $\rho(x) = \sum_{n=1}^{\infty} |x(n)|^{p_n}$ and

$$\|x\|_{(p_n)} = \inf \left\{ \lambda > 0 : \sum_{n=1}^{\infty} \left(\frac{|x(n)|}{\lambda} \right)^{p_n} \leq 1 \right\}$$

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Theorem

Let $(p_n) \subset [1, +\infty)$. TFAE:

1) $1 < \liminf_n p_n \leq \limsup_n p_n < +\infty$.
2) $(\ell^{p_n}, \|\cdot\|_{(p_n)})$ *is reflexive.*

Analysis of the w-FPP in Variable Lebesgue Spaces

Theorem (A necessary condition: $p_+ < +\infty$)

If $p_+ = +\infty$, then $(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ contains an isometric copy of $(\ell_\infty, \|\cdot\|_\infty)$. As a consequence:

If $p_+ = +\infty$, $(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ fails the w-FPP, because it contains an isometric copy of $(L_1[0,1], \|\cdot\|_1)$.

Examples: $L^{1+x}[0,+\infty)$, $L^{1+\frac{1}{1-x}}[0,1]$ fail the w-FPP.

Theorem (A sufficient condition)

If $p_+ < +\infty$ and the set $\{t \in \Omega : p(t) = 1\}$ can be split as a purely atomic set and a negligible set at most, then $L^{p(\cdot)}(\Omega)$ has NS.

A characterization of the w-FPP is VLS

Theorem

Let (Ω, Σ, μ) be a σ-finite measure space and let $p : \Omega \to [1, +\infty)$ be measurable. TFAE:

a) $L^{p(\cdot)}(\Omega)$ *satisfies the w-FPP.*

b) $L^{p(\cdot)}(\Omega)$ *does not contain isometrically* $(L^1[0, 1], \| \cdot \|_1)$.

c) $p_+ < +\infty$ *and the set* $\{t \in \Omega : p(t) = 1\}$ *can be split, at the most, into a purely atomic set and a negligible set.*

The FPP in Variable Lebesgue Spaces

If $L^{p(\cdot)}(\Omega)$ is reflexive $\Rightarrow$ $L^{p(\cdot)}(\Omega)$ has the FPP

Theorem (Maurey's result can be generalized)

Assume that $p_+ < \infty$ and let X be a closed subspace of $L^{p(\cdot)}(\Omega)$:

If X is reflexive, then $(X, \|\cdot\|_{p(\cdot)})$ has the FPP.

Is there a converse of Maurey's result in the variable case?

Theorem

Let $p : \Omega \to [1, +\infty]$ be measurable.

$L^{p(\cdot)}(\Omega)$ is not reflexive $\Leftrightarrow$ it contains an isomorphic copy of ℓ_1.

The absense of a.i.c. of ℓ_1 except for the trivial case

Question: Under non-reflexivity in $L^{p(\cdot)}(\Omega)$, is it possible to find an asymptotically isometric copy of ℓ_1 (as in the $L_1(\Omega)$ case)?

Theorem

Let $L^{p(\cdot)}(\Omega)$ be nonreflexive. Then the following are equivalent:

1) $L^{p(\cdot)}(\Omega)$ *contains an asymptotically isometric copy of* ℓ_1.
2) $L^{p(\cdot)}(\Omega)$ *contains an isometric copy of* ℓ_1.

Question: Are FPP and reflexivity equivalent under the framework of Variable Lebesgue spaces (in the same way that both concepts are equivalent in L_p-spaces)?

FPP and reflexivity are not equivalent in the variable context

- There exist some nonreflexive Variable Lebesgue Spaces $(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ with the FPP.

  (In contrast with the situation in the classical Lebesgue $(L^p(\Omega), \|\cdot\|_p)$ family).

- Consequence: There are some classical nonreflexive Banach spaces that fulfil the FPP (without any renorming procedure)

Near-infinity concentrated norms and the FPP

Definition

Let X B.s. with a Schauder basis and let $\|\cdot\|$ be a norm on X. The norm $\|\cdot\|$ is called near-infinity concentrated (n.i.c.) if:

1. For every $\epsilon > 0$ there exists some $k \in \mathbb{N}$ such that

$$\left[\|x\| + \limsup_n \|x_n\|\right](1-\epsilon) \leq \limsup_n \|x + x_n\|$$

   whenever $k \leq x$ and $(x_n)_n$ is a block basic sequence (b.b.s.).

2. For every $k \in \mathbb{N}$, $\exists\ F_k : (0, +\infty) \to [0, +\infty)$ with
   - $\lim_{\lambda \to 0^+} \frac{F_k(\lambda)}{\lambda} = 0$ and satisfying
   - $\limsup_n \|x_n + \lambda z\| \leq \limsup_n \|x_n\| + F_k(\lambda)\|z\|$

   $\forall$ b.b.s. (x_n) with $\liminf_n \|x_n\| \geq 1$, $\forall \lambda \in (0, +\infty)$, $\forall z \in X$ with $z \leq k$, $\|z\| \leq 1$

- E. Castillo-Santos, P. Dowling, H. Fetter, M. Japón, C. Lennard, B Sims, B. Turett, *Near-infinity concentrated norms and the fixed point property for nonexpansive maps on closed, bounded, convex sets.* J. Funct. Anal. 275 (2018), no. 3, 559-576.

Theorem (2018)

If X has a boundedly complete Schauder basis and $\|\cdot\|$ is near-infinity concentrated, then $(X, \|\cdot\|)$ has the FPP.

Theorem

Let $(p_n) \subset (1, +\infty)$ with $\lim_n p_n = 1$.

Then $(\ell_{p_n}, \|\cdot\|_{p_n})$ has a near-infinity concentrated norm.

Consequently:

The space $(\ell_{p_n}, \|\cdot\|_{p_n})$ is a nonreflexive VLS and it has the FPP.

Some open questions related the w-FPP:

We have already proved that

$L^{p(\cdot)}(\Omega)$ has the w-FPP $\Leftrightarrow$ it does NOT contain $L_1[0,1]$ isometrically

Open question: Let X be a Banach space or a function Banach lattice:

Does X have the w-FPP $\Leftrightarrow$ it does not contain $L_1[0,1]$ isometrically?

Remark: The shortage of examples of fixed point free nonexpansive mappings on weakly compact subsets.

Can new examples arise beyond the backdrop of the baker transform and the $L_1[0,1]$ space?

Some open questions related the FPP:

Consider the purely atomic case and let (p_n) be a bounded sequence in $(1, +\infty)$ with $\liminf_n p_n = 1$ (and therefore ℓ^{p_n} is nonreflexive):

We proved that $(\ell^{p_n}, \|\cdot\|_{p_n})$ has the FPP if $\lim_n p_n = 1$.

Questions:

- Does $(\ell^{p_n}, \|\cdot\|_{p_n})$ have the FPP for $\liminf_n p_n = 1$?

  Conjecture:

- Does $L^{p(\cdot)}(\Omega)$ have the FPP $\Leftrightarrow$ it does NOT contain $(\ell_1, \|\cdot\|_1)$ isometrically?

Some of the main references

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Thank you very much for all your attention.