Invariant subspaces for positive operators on Banach lattices with unconditional basis

F. Javier González-Doña

ICMAT - Universidad Complutense de Madrid



Workshop on Banach spaces and Banach lattices II

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Open problem

- (i) Does every positive operator on ℓ_p (1 ≤ p < ∞) have a non-trivial invariant subspace?
- Does every **positive** operator on a Banach lattice have a non-trivial invariant subspace?

• If X is a real Banach space with an unconditional basis (e_n) , we can define an order $\sum_{n=1}^{\infty} x_n e_n \leq \sum_{n=1}^{\infty} y_n e_n$ if and only if $x_n \leq y_n$ for every $n \in \mathbb{N}$, that turns (X, \leq) into a Banach lattice.

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• In this setting, an operator $T \in \mathcal{L}(X)$ is positive if and only if its associated matrix is positive.

• If X is any Banach lattice, $T \in \mathcal{L}(X)$ is a lattice homomorphism if $T(x \lor y) = Tx \lor Ty$.

Theorem (Abramovich, Aliprantis and Burkinshaw, 1994)

Let X be a Banach lattice and $T : X \to X$ be a positive operator. Assume there exists a positive operator $S \in \{T\}'(ST = TS)$ such that:

- (i) There exists $x_0 > 0$ such that $\lim_n ||S^n x_0||^{1/n} = 0$. (S is locally quasinilpotent at x_0)
- (ii) *S* dominates a non-zero compact operator.

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• Does this result characterize the existence of non-trivial invariant ideals for positive operators?

There exists a Banach lattice X and a positive operator $T : X \to X$ such that has non-trivial invariant subspaces but does not commute with any operator that is locally quasinilpotent. Moreover, we can get X to have its order induced by an unconditional basis and T to be a lattice homomorphism with non-trivial invariant ideals.

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• For example, let $X = \ell_2$ and $T : \ell_2 \to \ell_2$ such that $Te_n = w_n e_{n+1}$, where $w_n = \exp((n+1)^{1/2} - n^{1/2})$.

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• $\{M_z\}' = \{M_\phi: \phi \in H^\infty\}$ and the multipliers are not locally quasinilpotent operators.

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Proposition (Gallardo-Gutiérrez, GD, Tradacete, 2020)

Let X be a Banach lattice with unconditional basis and let $T \in \mathcal{L}(X)$ be a positive operator. T is a lattice homomorphism if and only if every row of its matrix representation has at most one positive entry.

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Corollary (Gallardo-Gutiérrez, GD, Tradacete)

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Lattice Homomorphisms on different Banach Lattices

• Let $\alpha \in [0,1]$ be an irrational number and define $T_{\alpha}: L^p[0,1] \rightarrow L^p[0,1]$ as

$$(T_{\alpha}f)(t) = t \cdot f(\{t + \alpha\}).$$

These operators are known as **Bishop operators.** Every Bishop operator is a lattice homomorphism on $L^{p}([0, 1])$.

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Proposition (Kitover and Wickstead, 2007)

There exist Bishop operators that do not have non-trivial invariant sublattices and, in particular, do not have non-trivial invariant ideals.

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There exist Bishop operators that do not have non-trivial invariant sublattices and, in particular, do not have non-trivial invariant ideals.

• It is still unknown if every Bishop operator has a non-trivial invariant subspace.

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Theorem (Grivaux, 2002)

Every positive tridiagonal operator has a non-trivial invariant subspace.

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• Can we extend our methods to show that every tridiagonal operator has a non-trivial invariant ideal?

Let X be a Banach lattice whose order is induced by an unconditional basis and let $T : X \rightarrow X$ be a positive, tridiagonal operator. Then, T has no non-trivial ideals if and only if both the sub-diagonal and the super-diagonal of its matrix representation have no null elements.

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• Let X be a Banach lattice with unconditional basis. An operator $T : X \to X$ is **band-diagonal** (2k + 1-diagonal) if its associated matrix is band-diagonal (2k + 1-diagonal).

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A heptadiagonal matrix

Open problem

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An approach

Characterize the existence of non-trivial invariant ideals for band-diagonal operators.

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Proposition (Radjavi and Troitsky, 2008)

Let X be a Banach lattice with unconditional basis $(e_n)_{n \in \mathbb{N}}$ and let T be a positive operator. Then, T has no non-trivial invariant ideals if and only if for every $i, j \in \mathbb{N}$ with $i \neq j$ there exists $n \in \mathbb{N}$ such that $(T^n e_j)_i > 0$.

$$\mathcal{T} = \begin{pmatrix} * & * & 0 & 0 & 0 & 0 & \cdots \\ * & * & * & 0 & 0 & 0 & \cdots \\ 0 & * & 0 & * & 0 & 0 & \cdots \\ 0 & 0 & * & * & 0 & 0 & \cdots \\ 0 & 0 & 0 & * & * & * & \cdots \\ 0 & 0 & 0 & 0 & * & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

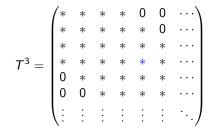
$$\mathcal{T}^{2} = \begin{pmatrix} * & * & * & 0 & 0 & 0 & \cdots \\ * & * & * & * & 0 & 0 & \cdots \\ * & * & * & * & 0 & 0 & \cdots \\ 0 & * & * & * & 0 & 0 & \cdots \\ 0 & 0 & * & * & * & * & \cdots \\ 0 & 0 & 0 & * & * & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\mathcal{T}^{3} = \begin{pmatrix} * & * & * & * & 0 & 0 & \cdots \\ * & * & * & * & 0 & 0 & \cdots \\ * & * & * & * & 0 & 0 & \cdots \\ * & * & * & * & 0 & 0 & \cdots \\ 0 & * & * & * & * & * & \cdots \\ 0 & 0 & * & * & * & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\mathcal{T}^{5} = \begin{pmatrix} * & * & * & * & 0 & 0 & \cdots \\ * & * & * & * & 0 & 0 & \cdots \\ * & * & * & * & 0 & 0 & \cdots \\ * & * & * & * & * & * & \cdots \\ * & * & * & * & * & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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Let X be a Banach lattice with unconditional basis, let $T \in \mathcal{L}(X)$ be 2k + 1-diagonal, positive operator and let $A = (a_{i,j})_{i,j \in \mathbb{N}}$ be its associated matrix. Assume that $a_{n+m,n} > 0$ for every $m \in \{1, \dots, k-1\}$. Then, T has a non-trivial invariant ideal if and only if there exists $n_0 \in \mathbb{N}$ such that $a_{i,j} = 0$ for every $(i,j) \in \{1, \dots, n_0\} \times \{n_0 + 1, n_0 + 2, \dots\}$.

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• By relaxing the hypotheses on the matrix, we obtain positive operators with non-trivial invariant ideals that do not satisfy the previous condition:

$$\mathcal{T} = \begin{pmatrix} * & 0 & * & 0 & * & 0 & \cdots \\ 0 & * & 0 & * & 0 & * & \cdots \\ * & 0 & * & 0 & * & 0 & \cdots \\ 0 & * & 0 & * & 0 & * & \cdots \\ * & 0 & * & 0 & * & 0 & \cdots \\ 0 & * & 0 & * & 0 & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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• How does 'adding zeros' modify the existence of non-trivial invariant ideals?

Honeycomb Matrix

• An infinite matrix $A = (a_{n,m})_{n,m \in \mathbb{N}}$ is said to be **k-honeycomb** if there exists $j \in \{0, \dots, k-1\}$ such that one of the following conditions is satisfied:

(i)
$$a_{n,m} = 0$$
 for every $n \in k\mathbb{N} - j$ and $m \in \mathbb{N} \setminus (k\mathbb{N} - j)$.

(ii) $a_{n,m} = 0$ for every $n \in \mathbb{N} \setminus (k\mathbb{N} - j)$ and $m \in k\mathbb{N} - j$.

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2-honeycomb matrix

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3-honeycomb matrix

Let X be a Banach lattice with unconditional basis, let $T \in \mathcal{L}(X)$ be a (2k + 1)-diagonal, positive operator. Assume its associated matrix $A = (a_{n,m})_{n,m \in \mathbb{N}}$ satisfies $a_{i,i+k}a_{i+k,i} > 0$.

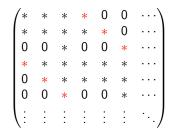
- (i) If k = 2, 3, then T has a non-trivial invariant ideals if and only if A is k-honeycomb.
- (ii) If $k \ge 4$, T can be chosen to have non-trivial invariant ideals and to not be k'-honeycomb for any $k' \in \mathbb{N}$.

$$T = \begin{pmatrix} * & * & * & 0 & 0 & 0 & \cdots \\ 0 & * & 0 & * & 0 & 0 & \cdots \\ * & * & * & * & * & 0 & \cdots \\ 0 & * & 0 & * & 0 & * & \cdots \\ 0 & 0 & * & 0 & * & \cdots \\ 0 & 0 & 0 & * & 0 & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

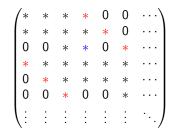
T is 2-honeycomb, it has non-trivial invariant ideals.

$$\mathcal{T} = \begin{pmatrix} * & * & * & 0 & 0 & 0 & \cdots \\ 0 & * & * & * & 0 & 0 & \cdots \\ * & * & * & * & * & 0 & \cdots \\ 0 & * & 0 & * & 0 & * & \cdots \\ 0 & 0 & * & 0 & * & \cdots \\ 0 & 0 & 0 & * & 0 & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

 ${\cal T}$ is **no longer** 2-honeycomb, it has **no** non-trivial invariant ideals.



T is 3-honeycomb, it has non-trivial invariant ideals.



T is **no longer** 3-honeycomb, it has **no** non-trivial invariant ideals.

Let X be a Banach lattice with unconditional basis and let $T \in \mathcal{L}(X)$ be a band-diagonal operator. Let $A = (a_{,m})_{n,m\in\mathbb{N}}$ be its associated matrix and assume that, for every $n \in \mathbb{N}$, there exists $j_n \in \mathbb{N}$ such that

$$a_{n,n+j_n}a_{n+j_n,n}>0$$

and $a_{n,n+m} = a_{n+m,n} = 0$ for every $n \in \mathbb{N}$ and $m \neq 0, k_n$. Then, T has a non-trivial invariant subspace.

$$T = \begin{pmatrix} * & * & 0 & 0 & 0 & 0 & \cdots \\ * & * & 0 & * & 0 & 0 & \cdots \\ 0 & 0 & * & * & 0 & 0 & \cdots \\ 0 & * & * & * & 0 & * & \cdots \\ 0 & 0 & 0 & 0 & * & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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$$x^*(T^nx) = \int_{\mathbb{R}} t^n d\mu(t) \qquad n \ge 0.$$

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Theorem (Atzmon and Godefroy, 2001)

Let X be a real Banach space and let $T \in \mathcal{L}(X)$ be an operator that admits a moment sequence. Then, T has a non-trivial invariant subspace.

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Theorem (Atzmon and Godefroy, 2001)

Let X be a real Banach space and let $T \in \mathcal{L}(X)$ be an operator that admits a moment sequence. Then, T has a non-trivial invariant subspace.

- The hypothesis on the matrix coefficients allows us to construct moment sequences for operators verifying the statement.
- Finally, the complex Banach space case follows from the real one.

To Sum Up

• We obtain characterizations for the existence of non-trivial invariant ideals for positive band-diagonal operators.

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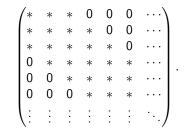
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• We are able to prove the existence of non-trivial invariant subspaces for positive, band-diagonal operators with no invariant ideals.

To Sum Up

• We obtain characterizations for the existence of non-trivial invariant ideals for positive band-diagonal operators.

- We are able to prove the existence of non-trivial invariant subspaces for positive, band-diagonal operators with no invariant ideals.
- Nevertheless, we still do not know if every positive, pentadiagonal operator has a non-trivial invariant subspace



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