Invariant subspaces for positive operators on Banach lattices with unconditional basis

F. Javier González-Doña

ICMAT - Universidad Complutense de Madrid

Workshop on Banach spaces and Banach lattices II

Invariant Subspace Problem

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Open problem

(i) *Does every **positive** operator on ℓ_p ($1 \leq p < \infty$) have a non-trivial invariant subspace?*

(ii) *Does every **positive** operator on a Banach lattice have a non-trivial invariant subspace?*

The setting

• If X is a real Banach space with an unconditional basis (e_n), we can define an order $\sum_{n=1}^{\infty} x_n e_n \leq \sum_{n=1}^{\infty} y_n e_n$ if and only if $x_n \leq y_n$ for every $n \in \mathbb{N}$, that turns $(X, \leq)$ into a Banach lattice.

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• In this setting, an operator $T \in \mathcal{L}(X)$ is positive if and only if its associated matrix is positive.

• If X is any Banach lattice, $T \in \mathcal{L}(X)$ is a lattice homomorphism if $T(x \vee y) = Tx \vee Ty$.

The Abramovich, Aliprantis and Burkinshaw Theorem

Theorem (Abramovich, Aliprantis and Burkinshaw, 1994)

Let X be a Banach lattice and $T : X \to X$ be a positive operator. Assume there exists a positive operator $S \in \{T\}'(ST = TS)$ such that:

(i) *There exists $x_0 > 0$ such that $\lim_n \|S^n x_0\|^{1/n} = 0$. (S is locally quasinilpotent at x_0)*

(ii) *S dominates a non-zero compact operator.*

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• Does this result characterize the existence of non-trivial invariant ideals for positive operators?

A counterexample

Theorem (Gallardo-Gutiérrez, GD and Tradacete, 2020)

There exists a Banach lattice X and a positive operator $T : X \to X$ such that has non-trivial invariant subspaces but does not commute with any operator that is locally quasinilpotent. Moreover, we can get X to have its order induced by an unconditional basis and T to be a lattice homomorphism with non-trivial invariant ideals.

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- For example, let $X = \ell_2$ and $T : \ell_2 \to \ell_2$ such that $Te_n = w_n e_{n+1}$, where $w_n = \exp((n+1)^{1/2} - n^{1/2})$.

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- This operator is unitarily equivalent to the shift M_z acting on $H^2(\beta)$, where $\beta = (\beta_n)_n$ and $\beta_n = e^{n^{1/2}}$.
- $\{M_z\}' = \{M_\phi : \phi \in H^\infty\}$ and the multipliers are not locally quasinilpotent operators.

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Corollary (Gallardo-Gutiérrez, GD, Tradacete)

Every lattice homomorphism on ℓ_p $(1 \leq p < \infty)$ *has a non-trivial invariant subspace. Moreover, a non-trivial invariant ideal.*

Lattice Homomorphisms on different Banach Lattices

- Let $\alpha \in [0,1]$ be an irrational number and define $T_\alpha : L^p[0,1] \to L^p[0,1]$ as

$$(T_\alpha f)(t) = t \cdot f(\{t+\alpha\}).$$

These operators are known as **Bishop operators.** Every Bishop operator is a lattice homomorphism on $L^p([0,1])$.

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- It is still unknown if every Bishop operator has a non-trivial invariant subspace.

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Theorem (Grivaux, 2002)

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• Can we extend our methods to show that every tridiagonal operator has a non-trivial invariant ideal?

Tridiagonal Operators

Theorem (Gallardo-Gutiérrez, GD, Tradacete, 2020)

Let X be a Banach lattice whose order is induced by an unconditional basis and let $T : X \to X$ be a positive, tridiagonal operator. Then, T has no non-trivial ideals if and only if both the sub-diagonal and the super-diagonal of its matrix representation have no null elements.

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Band-Diagonal Operators

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A heptadiagonal matrix

Ideals For Band-Diagonal Operators

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Characterize the existence of non-trivial invariant ideals for band-diagonal operators.

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Proposition (Radjavi and Troitsky, 2008)

Let X be a Banach lattice with unconditional basis $(e_n)_{n\in\mathbb{N}}$ and let T be a positive operator. Then, T has no non-trivial invariant ideals if and only if for every $i,j\in\mathbb{N}$ with $i\neq j$ there exists $n\in\mathbb{N}$ such that $(T^n e_j)_i > 0$.

Dynamics of Positive Matrices

$$T = \begin{pmatrix} * & * & 0 & 0 & 0 & 0 & \cdots \\ * & * & * & 0 & 0 & 0 & \cdots \\ 0 & * & 0 & * & 0 & 0 & \cdots \\ 0 & 0 & * & * & \color{blue}{0} & 0 & \cdots \\ 0 & 0 & 0 & * & * & * & \cdots \\ 0 & 0 & 0 & 0 & * & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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Dynamics of Positive Matrices

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Dynamics of Positive Matrices vs. Invariant Ideals

Understanding dynamics of Positive Matrices

$\Updownarrow$

Characterizing the existence of non-trivial invariant ideals

The Main Result

Theorem (Gallardo-Gutiérrez, GD, 2021)

Let X be a Banach lattice with unconditional basis, let $T \in \mathcal{L}(X)$ be $2k+1$-diagonal, positive operator and let $A = (a_{i,j})_{i,j \in \mathbb{N}}$ be its associated matrix. Assume that $a_{n+m,n} > 0$ for every $m \in \{1, \cdots, k-1\}$. Then, T has a non-trivial invariant ideal if and only if there exists $n_0 \in \mathbb{N}$ such that $a_{i,j} = 0$ for every $(i,j) \in \{1, \cdots, n_0\} \times \{n_0+1, n_0+2, \cdots\}$.

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Adding Zeros

- By relaxing the hypotheses on the matrix, we obtain positive operators with non-trivial invariant ideals that do not satisfy the previous condition:

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• How does 'adding zeros' modify the existence of non-trivial invariant ideals?

Honeycomb Matrix

• An infinite matrix $A = (a_{n,m})_{n,m\in\mathbb{N}}$ is said to be **k-honeycomb** if there exists $j \in \{0, \cdots, k-1\}$ such that one of the following conditions is satisfied:

(i) $a_{n,m} = 0$ for every $n \in k\mathbb{N} - j$ and $m \in \mathbb{N} \setminus (k\mathbb{N} - j)$.

(ii) $a_{n,m} = 0$ for every $n \in \mathbb{N} \setminus (k\mathbb{N} - j)$ and $m \in k\mathbb{N} - j$.

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$$\begin{pmatrix} * & * & * & * & * & * & \cdots \\ 0 & * & 0 & * & 0 & * & \cdots \\ * & * & * & * & * & * & \cdots \\ 0 & * & 0 & * & 0 & * & \cdots \\ * & * & * & * & * & * & \cdots \\ 0 & * & 0 & * & 0 & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

2-honeycomb matrix

Honeycomb Matrix

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3-honeycomb matrix

k-honeycomb band-diagonal operators

Theorem (Gallardo-Gutiérrez, GD, 2021)

Let X be a Banach lattice with unconditional basis, let $T \in \mathcal{L}(X)$ be a $(2k+1)-$diagonal, positive operator. Assume its associated matrix $A = (a_{n,m})_{n,m\in\mathbb{N}}$ satisfies $a_{i,i+k}a_{i+k,i} > 0$.

(i) *If $k = 2, 3$, then T has a non-trivial invariant ideals if and only if A is $k-$honeycomb.*

(ii) *If $k \geq 4$, T can be chosen to have non-trivial invariant ideals and to not be k'-honeycomb for any $k' \in \mathbb{N}$.*

An Example

$$T = \begin{pmatrix} * & * & {\color{red}*} & 0 & 0 & 0 & \cdots \\ 0 & * & 0 & {\color{red}*} & 0 & 0 & \cdots \\ {\color{red}*} & * & * & * & {\color{red}*} & 0 & \cdots \\ 0 & {\color{red}*} & 0 & * & 0 & {\color{red}*} & \cdots \\ 0 & 0 & {\color{red}*} & * & * & * & \cdots \\ 0 & 0 & 0 & {\color{red}*} & 0 & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

T is 2-honeycomb, it has non-trivial invariant ideals.

An Example

$$T = \begin{pmatrix} * & * & * & 0 & 0 & 0 & \cdots \\ 0 & * & * & * & 0 & 0 & \cdots \\ * & * & * & * & * & 0 & \cdots \\ 0 & * & 0 & * & 0 & * & \cdots \\ 0 & 0 & * & * & * & * & \cdots \\ 0 & 0 & 0 & * & 0 & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

T is **no longer** 2-honeycomb, it has **no** non-trivial invariant ideals.

An Example

$$\begin{pmatrix} * & * & * & * & 0 & 0 & \cdots \\ * & * & * & * & * & 0 & \cdots \\ 0 & 0 & * & 0 & 0 & * & \cdots \\ * & * & * & * & * & * & \cdots \\ 0 & * & * & * & * & * & \cdots \\ 0 & 0 & * & 0 & 0 & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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T is **no longer** 3-honeycomb, it has **no** non-trivial invariant ideals.

A Generalization of Grivaux's Theorem

Theorem (Gallardo-Gutiérrez,GD,2021)

Let X be a Banach lattice with unconditional basis and let $T \in \mathcal{L}(X)$ be a band-diagonal operator. Let $A = (a_{,m})_{n,m\in\mathbb{N}}$ be its associated matrix and assume that, for every $n \in \mathbb{N}$, there exists $j_n \in \mathbb{N}$ such that

$$a_{n,n+j_n} a_{n+j_n,n} > 0$$

and $a_{n,n+m} = a_{n+m,n} = 0$ for every $n \in \mathbb{N}$ and $m \neq 0, k_n$. Then, T has a non-trivial invariant subspace.

$$T = \begin{pmatrix} * & * & 0 & 0 & 0 & 0 & \cdots \\ * & * & 0 & * & 0 & 0 & \cdots \\ 0 & 0 & * & * & 0 & 0 & \cdots \\ 0 & * & * & * & 0 & * & \cdots \\ 0 & 0 & 0 & 0 & * & * & \cdots \\ 0 & 0 & 0 & * & * & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The Key Idea

- The methods to prove last result are based on Operator Theory techniques.

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An operator $T \in \mathcal{L}(X)$ has a moment sequence if there exist $x \in X \setminus \{0\}$ and $x^* \in X^* \setminus \{0\}$ and a non-negative Borel measure μ on $\mathbb{R}$ such that

$$x^*(T^n x) = \int_{\mathbb{R}} t^n d\mu(t) \qquad n \geq 0.$$

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Theorem (Atzmon and Godefroy, 2001)

Let X be a real Banach space and let $T \in \mathcal{L}(X)$ be an operator that admits a moment sequence. Then, T has a non-trivial invariant subspace.

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• The hypothesis on the matrix coefficients allows us to construct moment sequences for operators verifying the statement.

• Finally, the complex Banach space case follows from the real one.

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• We are able to prove the existence of non-trivial invariant subspaces for positive, band-diagonal operators with no invariant ideals.

• Nevertheless, we still do not know if every positive, pentadiagonal operator has a non-trivial invariant subspace

$$\begin{pmatrix} * & * & * & 0 & 0 & 0 & \cdots \\ * & * & * & * & 0 & 0 & \cdots \\ * & * & * & * & * & 0 & \cdots \\ 0 & * & * & * & * & * & \cdots \\ 0 & 0 & * & * & * & * & \cdots \\ 0 & 0 & 0 & * & * & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

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