Free Complex Banach Lattices

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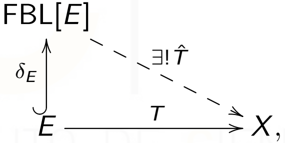
Introduction

Recall the definition of the **free Banach lattice** generated by a **real** Banach space E: a pair $(\mathrm{FBL}[E], \delta_E)$, where $\mathrm{FBL}[E]$ is a Banach lattice and δ_E is an isometric embedding, such that for any Banach lattice X and any operator $T : E \to X$ there exists a unique lattice homomorphism $\hat{T} : \mathrm{FBL}[E] \to X$ making the following diagram commutative:



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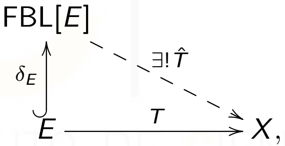

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Our objective is to construct an analogous object in the complex setting.

Motivation

Why are we interested in investigating free complex Banach lattices? They could be a useful tool in order to studying the **Complemented Subspace Problem**.

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Proposition 1.1

If E is a Banach space C_1-isomorphic to a C_2-complemented subspace of a Banach lattice, then $\delta_E(E)$ is C_1C_2-complemented in $FBL[E]$.

Interesting situation: the **contractive** (1-complemented) case.

Motivation

Every 1-complemented subspace of an L_p-space ($1 \leq p < \infty$) is an L_p-space (Douglas 1965, Andô 1966, Bernau-Lacey 1974).

Every 1-complemented subspace of a separable $\mathcal{C}(K)$-space is isomorphic to a $\mathcal{C}(K)$-space (Benyamini 1973).

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Free Banach lattices provide a (not very operative) criterium to identify whether a Banach space is isomorphic to a Banach lattice.

Proposition 1.2

A Banach space E is isomorphic to a Banach lattice if and only if there is an ideal $I \subset FBL[E]$ such that $FBL[E] = I \oplus \delta_E(E)$.

By *an ideal I* of Banach lattice X we mean a (closed) sublattice which is solid, that is, if $|x| \leq |y|$ for some $y \in I$, then $x \in I$.

Some definitions

Let X be a Banach lattice. For every pair $x, y \in X$, we may define

$$|(x,y)| = \sup\{x \cos\theta + y \sin\theta \ : \ \theta \in [0, 2\pi]\} \in X.$$

A **complex Banach lattice** Z is the complexification of a real Banach lattice X, that is, $Z = X \oplus iX$, endowed with the norm $\|x + iy\| = \||(x,y)|\|_X$, $x + iy \in Z$.

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Given a complex Banach space E, its dual E^* and $E^*_{\mathbb{R}}$ may be isometrically identified by means of the map $z^* \in E^* \mapsto \mathfrak{Re}\, z^* \in E^*_{\mathbb{R}}$.

Description of $\mathrm{FBL}_{\mathbb{C}}[E]$

Let E be a complex Banach space. Let us fix any complex Banach lattice $X_{\mathbb{C}}$ and any operator $T : E \to X_{\mathbb{C}}$. We write $Tz = \mathfrak{Re}\, Tz + i\mathfrak{Im}\, Tz$.

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$$\delta_E(z) = \delta_{E_{\mathbb{R}}}(z) - i\delta_{E_{\mathbb{R}}}(iz) \text{ for every } z \in E.$$

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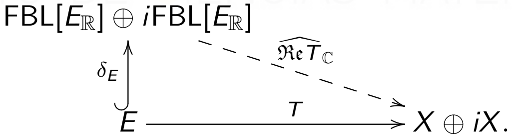

Note that for every $z \in E$ and for every $z^* \in E^*$ we have that

$$\delta_E(z)(\mathfrak{Re}\, z^*) = z^*(z).$$

Hence,

$$\begin{aligned}
&|||\delta_E(z)|||_{\mathrm{FBL}[E_{\mathbb{R}}]} = \\
= \;& \sup\left\{ \sum_{j=1}^m |\delta_E(z)(x^*)| \;:\; (x_j^*)_{j=1}^m \subset E_{\mathbb{R}}^*,\ \sup_{x \in B_E} \sum_{j=1}^m |x_j^*(x)| \leq 1 \right\} \\
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We have to renorm $\mathrm{FBL}[E_{\mathbb{R}}]$ with:

$$\|f\|_{\mathrm{FBL}_{\mathbb{C}}[E]} = \sup\left\{ \sum_{j=1}^m |f(\mathfrak{Re} z_j^*)| \ : \ (z_j^*)_{j=1}^m \subset E^*, \ \sup_{z\in B_E} \sum_{j=1}^m |z_j^*(z)| \leq 1 \right\}.$$

Complex conjugates and $\mathrm{FBL}_{\mathbb{C}}[E]$

In contrast to the real case, we know that $\mathrm{FBL}_{\mathbb{C}}[E] \stackrel{\text{lat}}{=} \mathrm{FBL}_{\mathbb{C}}[F]$ does not imply that E and F are isometric (even isomorphic).

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We have a *partial* converse result to the previous proposition:

Proposition 2.2 (complex version of Oikhberg et alii)

Let E, F be complex Banach spaces which have smooth dual. If $\mathrm{FBL}_{\mathbb{C}}[E]$ is lattice isometric to $\mathrm{FBL}_{\mathbb{C}}[F]$, then E is isometric to F or $\overline{F}$.

Another construction of $\mathrm{FBL}_{\mathbb{C}}[E]$

A **complex vector lattice** Z is the complexification of a real vector lattice X, that is, $Z = X \oplus iX$, such that for every $x + iy \in Z$ we have that $|(x, y)| \in Z$.

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Proposition 2.3 (complex version of Troitsky 2019)

Let E be a complex Banach space. Let $L_{\mathbb{C}} = L \oplus iL$ the complex vector sublattice of $\mathbb{R}^{E^*_{\mathbb{R}}} \oplus i\mathbb{R}^{E^*_{\mathbb{R}}}$ generated by $\{\delta_E(x) : x \in E\}$. There exists a maximal lattice seminorm ν on L such that $\nu(|\delta_E(x)|) \leq \|x\|$ for every $x \in L$. The function ν is a lattice norm and the norm completion of $L_{\mathbb{C}}$ respect to $\nu(|\cdot|)$ is $FBL_{\mathbb{C}}[E]$.

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It is possible to contruct the $\mathrm{FVL}_{\mathbb{C}}(A)$ and $\mathrm{FBL}_{\mathbb{C}}(A)$ (Baker 1968, De Pagter-Wickstead 2015).

Thank you for your attention.