Subspaces of $C(K)$ with unique Hahn-Banach extensions

A joint work with Antonio José Guirao and Vicente Montesinos

Christian Cobollo

Workshop on Banach spaces and Banach lattices II ICMAT (Madrid), 9-13 May 2022

This work is supported by GVA, Project PROMETEO/2021/070.

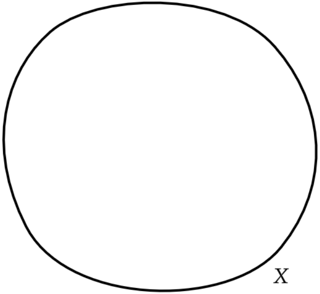
Motivation

R. Phelps

R. R. Phelps, *Uniqueness of Hahn–Banach extensions and unique best approximation* **(1960).**

Motivation

Unique Extension Properties



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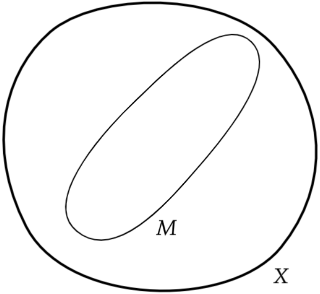

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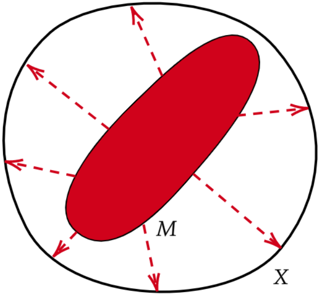
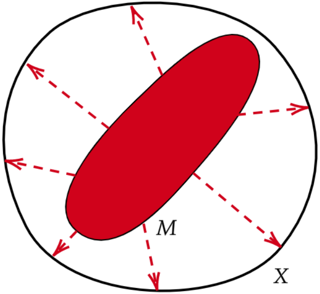


Definition (Phelps, 1960)

$M \hookrightarrow X$ has **property $\boldsymbol{U}$ in $\boldsymbol{X}$** if: every $f^* \in M^*$ has unique norm-preserving extension to X.



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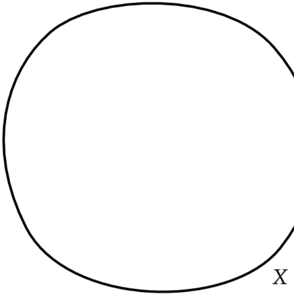

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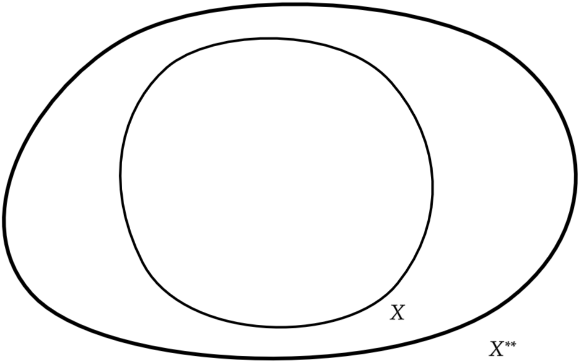

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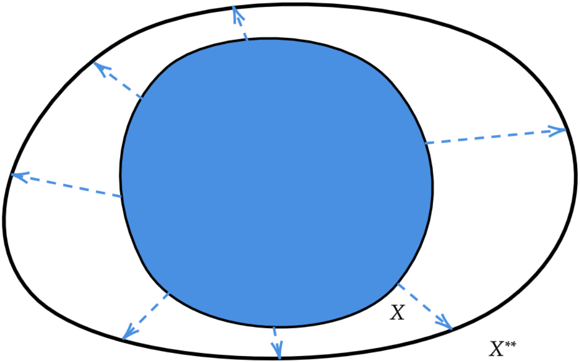
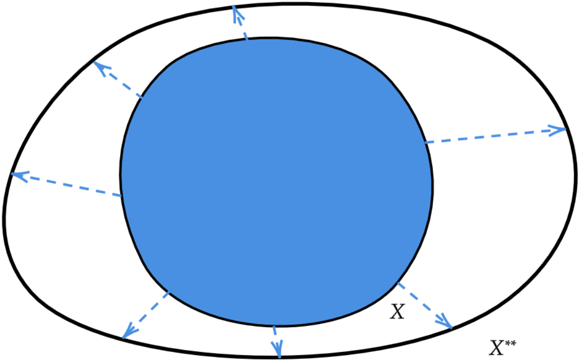


Definition (Sullivan, 1977)

X is **Hahn–Banach Smooth** if: X has property $\boldsymbol{U}$ in X^{**}.

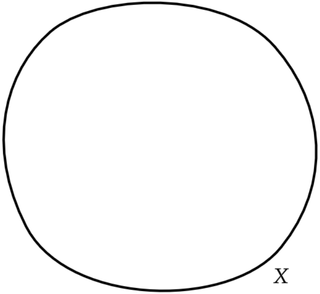


C. C., A. J. Guirao, and V. Montesinos, *A remark on totally smooth renormings* **(2020)**.

Theorem

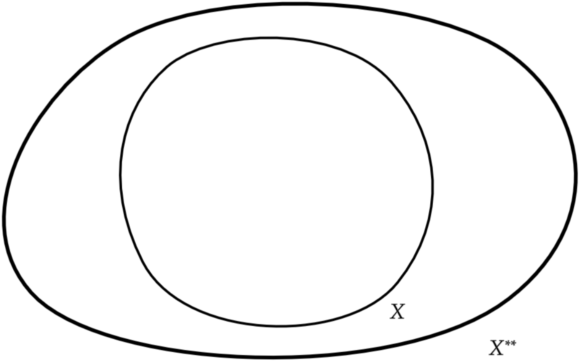
Let $(X, \|\cdot\|)$ be a Banach space. TFAE:

1. X renormable s.t. X has property U in X^{**};
2. X^* renormable s.t. w^*-w-Kadets–Klee norm;
3. X^* renormable dual LUR norm;
4. X renormable s.t. **every** $M \hookrightarrow \boldsymbol{X}$ has property U in X^{**}.


X

Motivation

Unique Extension Properties in $C(K)$



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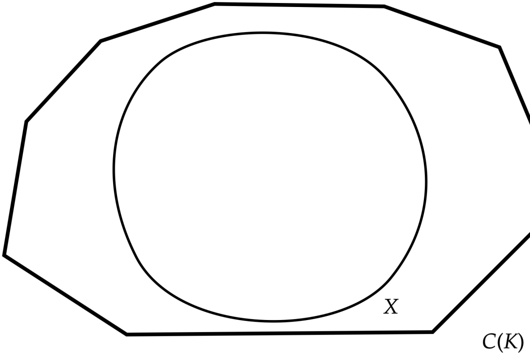
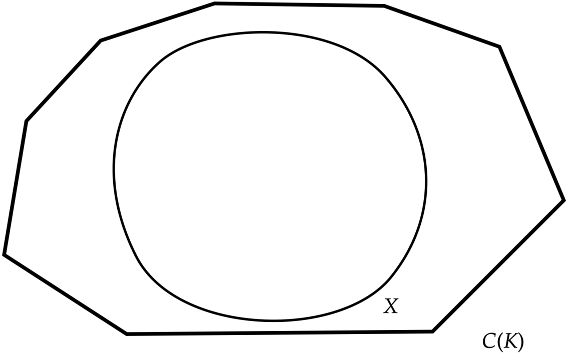


Example

Take $K_X := (B_{X^*}, w^*)$. Then X can be considered as a linear closed subspace of $C(K_X)$.



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Also $\|x^*\| = \|\delta_{x^*}\| = \| - \delta_{-x^*}\| = 1$.

$\implies$ δ_{x^*} and $-\delta_{-x^*}$ are two different HB-extentions of x^*.

Duality of the subspace:

Duality of the subspace: A linear subspace M has property U in X iff $M^{\perp}$ is Haar (Chebyshev) subspace in X^*.

Property U inside $C(K)$

Phelps' Approach

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Proposition

Let $X \hookrightarrow C(K)$, $\dim(X) = n$. If X has property U in $C(K)$, then for every $x \in X$, $|x(t)| = \|x\|$ for at most n points $t \in K$.

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Let $A \subset K$ closed set. Then the closed ideal $I_A := \{f \in C(K) : f(a) = 0, \text{ for every } a \in A\}$ has property U in $C(K)$.

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Notice $X \hookrightarrow C(K)$ is an M-ideal if and only if $X = I_A$ for some closed $A \subset K$.

Property U inside $C(K)$

Definition

Let $T : X \to Y$, it is a U-embedding of X into Y if T is a linear isometry whose range has property U in Y.

$$X \xrightarrow{\quad T \quad} C(K)$$

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Let $T : X \to Y$ U-embedding, $A \subset S_{X^*}$. Then, $T^*_{|\mathrm{HB}(A)} : \mathrm{HB}(A) \to A$ is a w^*-w^*-homeomorphism.

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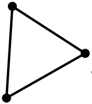

$\mathfrak{F}(f)$

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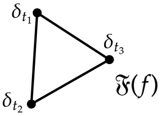

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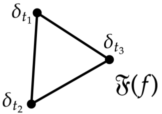

Simplex
(Inf. Dimension)

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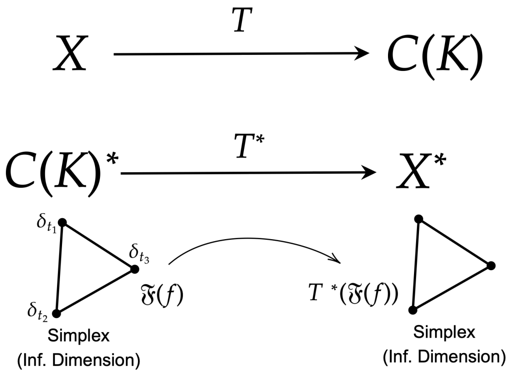

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Let $T : X \to Y$, it is a U-embedding of X into Y if T is a linear isometry whose range has property U in Y.

Proposition (Necessary condition 1)

Let $T : X \to C(K)$ be a U-embedding. Then, B_{X^*} is a **simplexoid.**

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$$\begin{tikzcd} X \arrow[r, "T"] & C(K) \\ C(K)^* \arrow[r, "T^*"] & X^* \\ K \arrow[u, "\delta^K"] & \end{tikzcd}$$

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Looking for U-embeddings into $C(K)$

Necessary conditions

$$\mathcal{B}(X, C(K)) \longleftrightarrow C\Big(K, \big(X^*, w^*\big)\Big)$$

$$T \qquad\qquad\qquad\qquad F$$

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$F : K \to X^*$ w^*-continuous map $\implies T_F x(t) := \langle x, F(t)\rangle$.

Lemma

Let K compact, X Banach space. Let $F \in C(K, (X^*, w^*))$. TFAE:

1. T_F is an isometry into.
2. $-F(K) \cup F(K)$ is a James boundary for X.
3. $F(K)$ is a 1-norming subset of S_{X^*}.
4. $F(K) \subset B_{X^*}$ and Ext $B_{X^*} \subset -F(K) \cup F(K)$.

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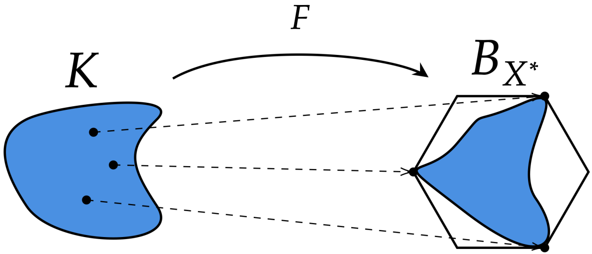

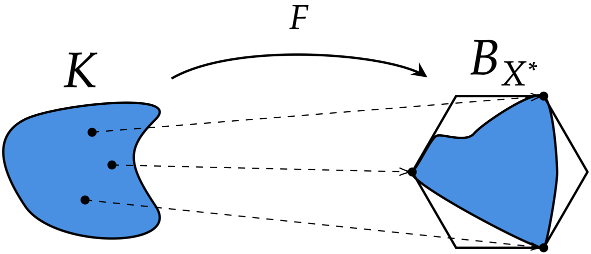

F
K
B_{X^*}

Looking for U-embeddings into $C(K)$

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Remark

Let $F \in C(K,(X^*,w^*))$ and $t \in K$ such that $\|F(t)\| = 1$. Then $\delta_t \in C(K)^*$ is a Hahn–Banach extension of $F(t)$.

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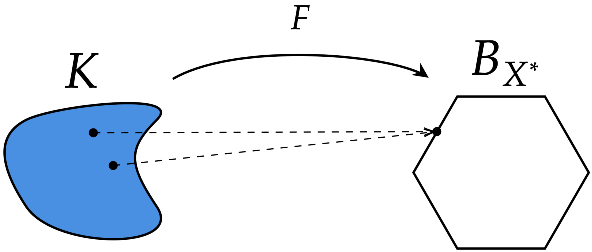

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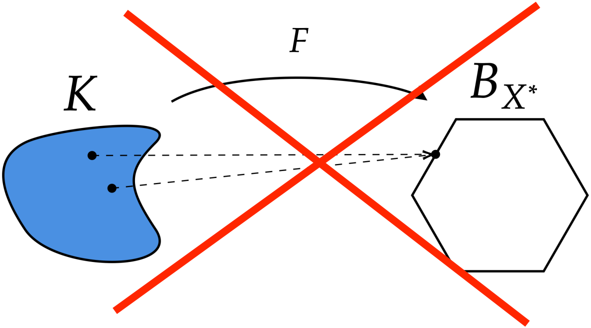

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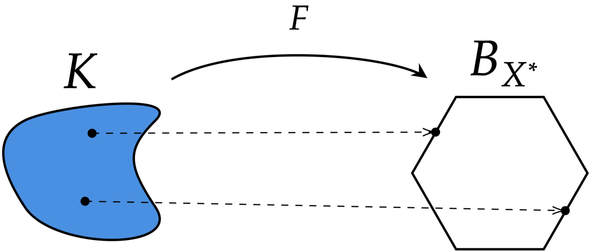

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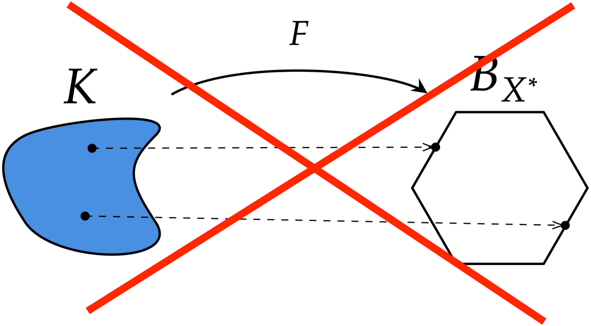

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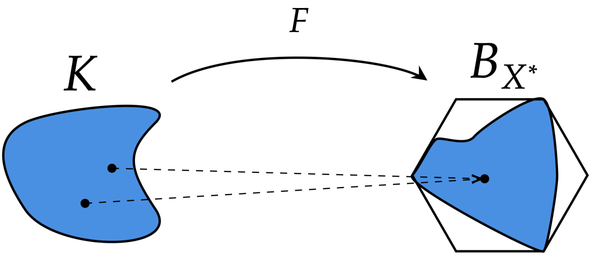

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Proposition (Necessary condition 2)

Let $T : X \to C(K)$ be a U-embedding. Then, if $t, s \in K$ $t \neq s$ such that $F(t) = \pm F(s)$, then $\|F(t)\| < 1$.

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Remark

In particular: $-F(K) \cap F(K) \cap S_{X^*} = \emptyset$.

Looking for U-embeddings into $C(K)$

Necessary conditions

Proposition (Necessary condition 3)

Let $T : X \to C(K)$ be a U-embedding. Put $E_1^+ := F_T(K) \cap \mathrm{Ext}\, B_{X^*}$. If $t \in K$ such that $\delta_t \notin \overline{\mathrm{HB}_T(E_1^+)}^{w^*}$, then $\|F(t)\| < 1$.

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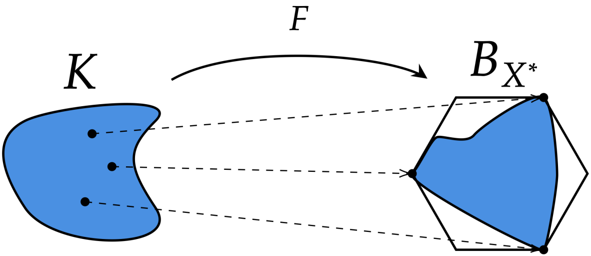

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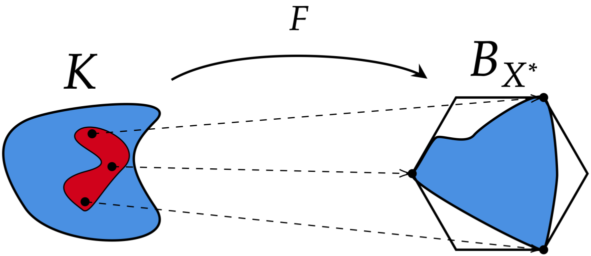

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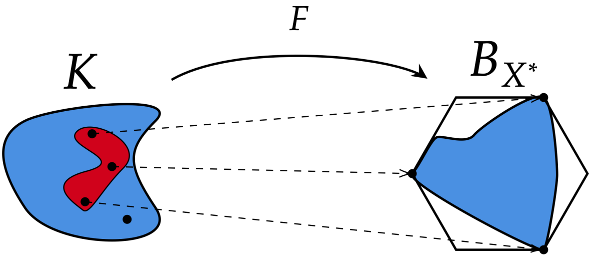

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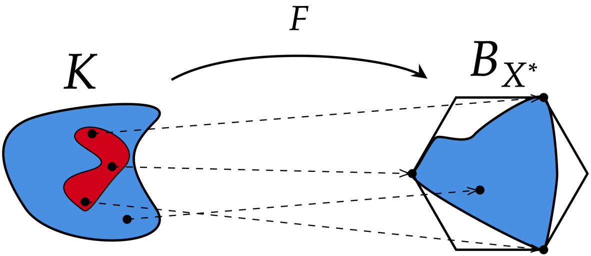

Looking for U-embeddings into $C(K)$

Necessary conditions

Necessary condition: B_{X^*} a simplexoid.

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Necessary conditions

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Definition

Given a Banach space X, a w^*-closed set $E \subset B_{X^*}$ will be called U-suitable if it satisfies the following properties:

1. $-E \cap E \cap S_{X^*} = \emptyset$.
2. $-E \cup E = \overline{\text{Ext } B_{X^*}}^{w^*}$.

Moreover, we will say E is **proper U-suitable** set if:

$$\overline{\text{Ext } \mathfrak{F}(x)}^{w^*} \cap E = \mathfrak{F}(x) \cap E, \quad \text{for every } x \in S_X.$$

Looking for U-embeddings into $C(K)$

Necessary conditions

Proposition

Let X be a Banach space such that $(\overline{\text{Ext } B_{X^*}}^{w^*} \cap S_{X^*}, w^*)$ is connected. Then, X cannot be U-embedded into a $C(K)$-space.

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Corollary

No Gâteaux smooth Banach space can be U-embedded into a $C(K)$ space.

Construction of embeddings with unique extensions

Theorem

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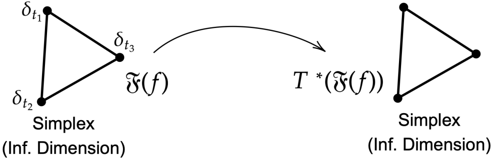

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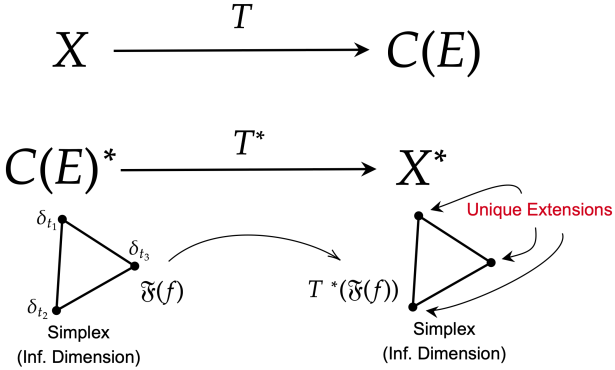

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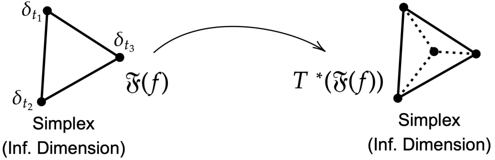

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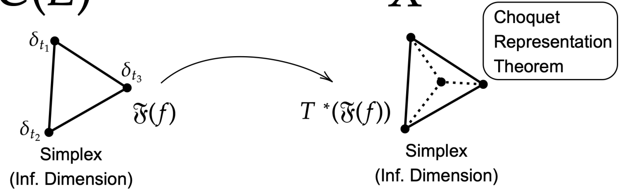

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A finite-dimensional Banach space X can be U-embedded into a $C(K)$ if and only if its dual unit ball is a simplexoid and admits a proper U-suitable set.

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Corollary

A two-dimensional Banach space X can be U-embedded into a $C(K)$ if and only if it is not Gâteaux smooth.

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2. $T : C(K) \to C(S) \quad \iff \quad h : K \to S$.

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Theorem

Let K and S be compact spaces and isometric embedding $T\colon C(K) \to C(S)$. TFAE:

1. T is a U-embedding;
2. There exist a closed $S_0 \subset S$, an homeomorphism $h\colon K \to S_0$ and a continuous function $\varepsilon\colon K \to \{\pm 1\}$ such that $F_T(h(t)) = \varepsilon(t)\delta_t^K$ for every $t \in K$ and $\|F_T(s)\| < 1$ for every $s \in S \setminus S_0$;

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3. There exists a continuous embedding $h\colon K \to S$ admitting a Pełczyńsky's norm-one extension operator with $h(K)$ a G_δ set.

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The “canonical way”, once more, fails.

Proposition

Let $h : S \to K$ continuous and onto. Then, the composition operator $h^\circ : C(K) \to C(S)$ is a U-embedding iff h is a homeomorphism

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However:

Proposition

Let $K \subset S$ be a G_δ set, and $r : S \to K$ a retraction. Then there exist a U-embedding from $C(K)$ into $C(S)$.

Some References

- **C. C., A. J. Guirao, and V. Montesinos,** *A remark on totally smooth renormings* **(2020).**
- **A. Pełczyński,** *Linear extensions, linear averagings, and their applications to linear topological classification of spaces of continuous functions* **(1968).**
- **R. R. Phelps,** *Uniqueness of Hahn–Banach extensions and unique best approximation* **(1960).**
- **R. R. Phelps,** *Lectures on Choquet's theorem* **(2001).**

The End

Thanks For Your Attention!