Transfinite almost square Banach spaces

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Overview

- Background
- Transfinite almost square spaces
- The renorming problem
- More examples
  - Direct sums
  - Tensor products
  - Ultraproducts
- Octahedral norms

Definition (G. Godefroy and B. Maurey)

X is **octahedral** if, for every finite-dimensional $Y \subset X$ and $\varepsilon > 0$, there is $x \in S_X$ such that

$$\|y + rx\| \geq (1 - \varepsilon)(\|y\| + |r|) \ \forall y \in Y \text{ and } r \in \mathbb{R}.$$

Examples: ℓ_1, $L_1[0,1]$ and $C[0,1]$.

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Theorem (G. Godefroy, 1989)

Tfae:

(i) $\ell_1 \subset X$,

(ii) *X admits an equivalent* **octahedral** *norm.*

Definition (S. Ciaci, J. Langemets and A. Lissitsin)

X is $< \kappa$-**octahedral** if, for every $Y \subset X$ with $\text{dens}(Y) < \kappa$ and $\varepsilon > 0$, there is $x \in S_X$ such that

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Theorem (A. Avilés, G. Martínez-Cervantes and A. Rueda Zoca, 2021)

If $\kappa > \aleph_0$, then tfae:

(i) $\ell_1(\kappa) \subset X$,

(ii) *X admits an equivalent $< \kappa$-"**rigid octahedral**" norm.*

Definition (T. A. Abrahamsen, J. Langemets and V. Lima)

X is **almost square** (**ASQ**) if, for every finite-dimensional $Y \subset X$ and $\varepsilon > 0$, there is $x \in S_X$ such that

$$\|y + rx\| \leq (1 + \varepsilon) \max\{\|y\|, |r|\} \; \forall y \in Y \text{ and } r \in \mathbb{R}.$$

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Theorem (J. Becerra Guerrero, G. López-Pérez and A. Rueda Zoca, 2016)

Tfae:

(i) $c_0 \subset X$,

(ii) *X admits an equivalent* **ASQ** *norm.*

Question

What is the connection between the containment of $c_0(\kappa)$ and transfinite versions of **ASQ** equivalent renorming?

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- X is $< \kappa$-**square** ($\mathbf{SQ}_{<\kappa}$) if, for every $Y \subset X$ with $\text{dens}(Y) < \kappa$, there is $x \in S_X$ such that

$$\|y + rx\| = \max\{\|y\|, |r|\} \ \forall y \in Y \text{ and } r \in \mathbb{R}.$$

Examples: $c_0(\kappa)$ and $\ell_\infty^c(\kappa)$.

- Let $\mathscr{F}$ be a non-principal ultrafilter over $\mathbb{N}$. Define

$$\|\!| x \|\!| := \max\left\{ \left| \lim_{\mathscr{F}} x(n) \right|, \sup_{n\in\mathbb{N}} \left| x(n) - \lim_{\mathscr{F}} x(m) \right| \right\}.$$

Then $(\ell_\infty, \|\!| \cdot \|\!|)$ is $\mathbf{SQ}_{<\aleph_0}$, but it is not $\mathbf{ASQ}_{<\aleph_1}$.

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- Define

$$X_n := \left\{f \in S_{C(\mathbb{R}^n)} : f(x) = -f(-x)\ \forall x \in S_{\mathbb{R}^n}\right\}.$$

  Then $c_0(\mathbb{N}, X_n)$ and $\ell_\infty(\mathbb{N}, X_n)$ are $\mathbf{SQ}_{<\aleph_0}$.

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  Then $c_0(\mathbb{N}, X_n)$ and $\ell_\infty(\mathbb{N}, X_n)$ are $\mathbf{SQ}_{<\aleph_0}$.

- If X is a space of almost universal disposition for Banach spaces of density character $< \kappa$, then X is $\mathbf{ASQ}_{<\kappa}$.

Proposition (A. Avilés, G. Martínez-Cervantes and A. Rueda Zoca)

Let X be a locally compact Hausdroff space. $C_0(X)$ is **ASQ** *if, and only if, X is non-compact.*

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Theorem

Let X be T_4 locally compact and $\kappa > \aleph_0$. Tfae:

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(iii) *If* $\mathscr{K}$ *is a family consisting of* $< \kappa$ *many compact sets in* X, *then* $\bigcup \mathscr{K}$ *is not dense in* X.

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Question

If $c_0(\kappa) \subset X$, then X admits an equivalent $\mathbf{SQ}_{<\kappa}$ norm?

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Let X be a dual space. If $c_0 \subset X$, then X admits an equivalent $\mathbf{SQ}_{<\aleph_0}$ norm.

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If $c_0 \subset X$, then $\ell_\infty \subset X$ and it is complemented (ℓ_∞ is 1-injective), i.e. $X = \ell_\infty \oplus Z$.

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If $c_0 \subset X$, then $\ell_\infty \subset X$ and it is complemented (ℓ_∞ is 1-injective), i.e. $X = \ell_\infty \oplus Z$. Endow X with an equivalent norm such that $X = (\ell_\infty, |||\cdot|||) \oplus_\infty Z$, where $(\ell_\infty, |||\cdot|||)$ is $\mathbf{SQ}_{<\aleph_0}$.

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If $c_0 \subset X$, then $\ell_\infty \subset X$ and it is complemented (ℓ_∞ is 1-injective), i.e. $X = \ell_\infty \oplus Z$. Endow X with an equivalent norm such that $X = (\ell_\infty, |||\cdot|||) \oplus_\infty Z$, where $(\ell_\infty, |||\cdot|||)$ is $\mathbf{SQ}_{<\aleph_0}$. Since being $\mathbf{SQ}_{<\aleph_0}$ is passed by one component through ∞-sums, the claim follows. □

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Corollary

Assume **ZF+AD**. *If $\kappa \in \{\aleph_1, \aleph_2\}$, then $\ell_\infty(\kappa)$ admits an equivalent* $\mathbf{SQ}_{<\kappa}$ *norm.*

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Assume **ZF+AD**. *If $\kappa \in \{\aleph_1, \aleph_2\}$, then $\ell_\infty(\kappa)$ admits an equivalent $\mathbf{SQ}_{<\kappa}$ norm.*

Theorem

Let $cf(\kappa) > \aleph_0$ and $dens(X) = \kappa$. If $c_0(\kappa) \subset X$, then X admits an equivalent $\mathbf{SQ}_{<\kappa}$ norm.

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Proposition

Let $\{X_\alpha : \alpha \in \mathscr{A}\}$ *be a family. If, for every* $\varepsilon > 0$*, there is* $\beta \in \mathscr{A}$ *such that* X_β *is "*ε*-*$\mathbf{ASQ}_{<\kappa}$*", then* $\ell_\infty(\mathscr{A}, X_\alpha)$ *is* $\mathbf{ASQ}_{<\kappa}$*.*

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Corollary

*$X \oplus_\infty Y$ is **(A)SQ**$_{<\kappa}$ if and only if either X or Y is **(A)SQ**$_{<\kappa}$.*

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Proposition

Let $\{X_\alpha : \alpha \in \mathscr{A}\}$ *be an uncountable family, then* $c_0(\mathscr{A}, X_\alpha)$ *is* $\mathbf{SQ}_{<\kappa}$*.*

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Let $\kappa > \aleph_0$*. If* X *and* Y *are* **(A)SQ**$_{<\kappa}$*, then* $X\widehat{\otimes}_\pi Y$ *is* **(A)SQ**$_{<\kappa}$*.*

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$$\text{Denote } c_{0,\mathscr{F}}(\mathscr{A}, X_\alpha) := \left\{ x \in \ell_\infty(\mathscr{A}, X_\alpha) : \lim_{\mathscr{F}} x(\alpha) = 0 \right\}$$

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Proposition

Let $\{X_\alpha : \alpha \in \mathscr{A}\}$ be an infinite family of **ASQ**$_{<\kappa}$ *spaces and $\mathscr{F}$ a non-principal ultrafilter in $\mathscr{A}$. If $\mathscr{F}$ is not $\aleph_1$-complete, then $\ell_\infty(\mathscr{A}, X_\alpha)/c_{0,\mathscr{F}}(\mathscr{A}, X_\alpha)$ is* **SQ**$_{<\kappa}$

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If X is $\mathbf{SQ}_{<\kappa}$ *and $\kappa > \aleph_0$, then X^* is $< \kappa$-***"rigid octahedral"**.

Thank you for your attention!

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