Generalized Mazur maps in a noncommutative setting

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Supported by NSF grant DMS-1900985

Workshop on Banach spaces and Banach lattices II
ICMAT, Madrid

May 10, 2022

Unit spheres/balls

Notation

For a Banach space X, we will denote its unit sphere by

$$\mathscr{S}(X) = \{x \in X \ : \ \|x\| = 1\},$$

Disclaimer

I would normally write S_X, but we will be considering spaces like the Schatten p-class S_p and writing S_{S_p} is just awful.

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and its unit ball by

$$\mathscr{B}(X) = \{x \in X \ : \ \|x\| \leq 1\}$$

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Any two separable infinite-dimensional Banach spaces are homeomorphic.

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For $1 < p < \infty$, if X is uniformly homeomorphic to ℓ_p, then X is isomorphic to ℓ_p.

Theorem (Mazur 1929)

For $1 \leq p, q < \infty$, $\mathscr{S}(L_p[0,1])$ is uniformly homeomorphic to $\mathscr{S}(L_q[0,1])$.

The classical Mazur map

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Definition

For $1 \leq p, q < \infty$ and a measure μ, the Mazur map $M_{p,q} : L_p(\mu) \to L_q(\mu)$ is given by

$$f \quad \mapsto \quad f|f|^{\frac{p-q}{q}} = \mathrm{sign}(f)|f|^{\frac{p}{q}}$$

Theorem (Mazur 1929)

For $1 \leq p, q < \infty$ and a measure μ, the map $M_{p,q}$ is a uniform homeomorphism between $\mathscr{S}(L_p(\mu))$ and $\mathscr{S}(L_q(\mu))$.

Noncommutative Mazur maps

Definition

For $1 \leq p, q < \infty$ and a von Neumann algebra $\mathcal{M}$, the Mazur map $M_{p,q} : L_p(\mathcal{M}) \to L_q(\mathcal{M})$ is given by

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Theorem (Ricard 2015)

For $1 \leq p, q < \infty$ and a von Neumann algebra $\mathcal{M}$, the Mazur map $M_{p,q}$ is $\min\{\frac{p}{q}, 1\}$-Hölder on the unit ball of $L_p(\mathcal{M})$.

In particular, it is a uniform homeomorphism between $\mathscr{S}(L_p(\mathcal{M}))$ and $\mathscr{S}(L_q(\mathcal{M}))$.

Our goal

- To study noncommutative versions of other generalized Mazur maps that have been useful in Banach space theory.
- Use them to get new examples of uniform homeomorphisms between spheres.

The Odell–Schlumprecht characterization

Theorem (Odell–Schlumprecht 1994)

Let E be a Banach space with a 1-unconditional basis. Then $\mathscr{S}(E)$ is uniformly homeomorphic to $\mathscr{S}(\ell_1)$ if and only if E does not contain ℓ_∞^n's uniformly.

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The proof

Is based on two generalizations of the Mazur map:

- A p-convexification one.
- An entropy-based one.

The first generalized Mazur map: p-convexification

Note that

$$\ell_p = \left\{ (x_j) \ : \ \left(|x_j|^p\right) \in \ell_1 \right\} \qquad \|(x_j)\|_{\ell_p} = \|(|x_j|^p)\|_{\ell_1}^{1/p}$$

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Similarly, for a 1-unconditional sequence space E we can define its p-convexification

$$E^{(p)} = \left\{(x_j) \;:\; \left(|x_j|^p\right) \in E\right\} \qquad \|(x_j)\|_{E^{(p)}} = \|(|x_j|^p)\|_{E}^{1/p}$$

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When E is 1-unconditional, the map

$$G_p : (x_j) \mapsto \big(\operatorname{sign}(x_j)|x_j|^p\big)$$

maps $\mathscr{S}(E^{(p)})$ to $\mathscr{S}(E)$

The first generalized Mazur map: convexification

Proposition (Odell–Schlumprecht 1994)

Let $1 < p < \infty$ and let E be a Banach space with a 1-unconditional basis. The generalized Mazur map

$$G_p : (x_j) \mapsto \big(\operatorname{sign}(x_j)|x_j|^p\big)$$

is a uniform homeomorphism between $\mathscr{S}(E^{(p)})$ and $\mathscr{S}(E)$. Moreover, the moduli of uniform continuity of G_p and G_p^{-1} depend only on p.

The second generalized Mazur map: entropy-based

Well-known

Entropy of a probability distribution (x_j): $H = -\sum_j x_j \log(x_j)$

Definition

If $X = (\mathbb{R}^n, \|\cdot\|)$ is a 1-unconditional strictly convex norm and and $x \in \mathbb{R}^n$ is a probability vector, we define $F_X(x)$ to be the $y \in \mathscr{B}(X)^+$ minimizing the quantity $-\sum_j x_j \log(y_j)$, that is

$$F_X(x) = \underset{y \in \mathscr{B}(X)^+}{\arg\min} - \sum_j x_j \log(y_j)$$

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Important note for later

Closely related to the relative entropy or Kullback–Leibler divergence $D(x||y) = \sum_j x_j(\log(x_j) - \log(y_j))$

The second generalized Mazur map: entropy-based

Let's look at the special case where $X = \ell_p^n$. Fix a probability vector $x \in \mathbb{R}^n$, consider $y \geq 0$ with $\|y\|_{\ell_p} = 1$. We want to

$$\text{minimize } -\sum_j x_j \log(y_j) \quad \text{subject to } \sum_j y_j^p = 1$$

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by Lagrange multipliers we need to have

$$-\frac{x_j}{y_j} = \lambda p y_j^{p-1} \qquad \text{i.e.} \qquad -x_j = \lambda p y_j^p$$

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Adding up we get $\lambda = -1/p$ and thus $y_j = x_j^{1/p}$ meaning

$$F_{\ell_p^n}\big((x_j)\big) = (x_j^{1/p})$$

The second generalized Mazur map: entropy-based

Proposition (Odell–Schlumprecht 1994)

Let $X = (\mathbb{R}^n, \|\cdot\|)$ *be an uniformly convex and uniformly smooth space with a* 1*-unconditional basis. Then* $F_X : \mathscr{S}(\ell_1^n) \to \mathscr{S}(X)$ *is a uniform homeomorphism; moreover*

- *The modulus of continuity of* F_X *depends only on the modulus of uniform convexity of* X
- *The modulus of continuity of* F_X^{-1} *depends only on the modulus of uniform smoothness of* X.

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- *The modulus of continuity of F_X depends only on the modulus of uniform convexity of X*
- *The modulus of continuity of F_X^{-1} depends only on the modulus of uniform smoothness of X.*

From this finite-dimensional result one can “glue together” the pieces and get a uniform homeomorphism

$$F_X : \mathscr{S}(\ell_1) \to \mathscr{S}(X)$$

when X is infinite-dimensional, uniformly convex and uniformly smooth, and has a 1-unconditional basis.

The question, vague form

Question

Is there a noncommutative version of the Odell–Schlumprecht theorem?

Theorem (Odell–Schlumprecht 1994)

Let E be a Banach space with a 1-unconditional basis. Then $\mathscr{S}(E)$ is uniformly homeomorphic to $\mathscr{S}(\ell_1)$ if and only if E does not contain ℓ_∞^n's uniformly.

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Sub-question

Specifically, what about noncommutative versions of their generalized Mazur maps?

Making the “noncommutative” precise

From ℓ_p, we get its “noncommutative” version

$$S_p = \big\{ T : \ell_2 \to \ell_2 \text{ compact, } \big(s_j(T)\big) \in \ell_p \big\}$$

where $(s_j(T))$ is the sequence of singular values of T (eigenvalues of $|T| = \sqrt{T^*T}$), with norm $\|T\|_{S_p} = \|\big(s_j(T)\big)\|_{\ell_p}$

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Similarly, given a sequence space E, define

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Note

For everything to work well, we ask for E to be 1-symmetric: for any permutation $\pi : \mathbb{N} \to \mathbb{N}$, $\|(a_{\pi(n)})\|_E = \|(a_n)\|_E$

Unitarily invariant ideals (of compact operators)

When E is 1-symmetric, S_E satisfies:

- (Ideal property) When $T \in S_E$, $A, B \in \mathcal{B}(\ell_2)$, $ATB \in S_E$.
- (Unitary invariance) When $T \in S_E$, $U, V \in \mathcal{B}(\ell_2)$ unitaries, $UTV \in S_E$ and
$$\|UTV\|_{S_E} = \|T\|_{S_E}$$

The conjecture

Conjecture (CD)

Let S_E be a unitarily invariant ideal. Then $\mathscr{S}(S_E)$ is uniformly homeomorphic to $\mathscr{S}(S_1)$ if and only if S_E does not contain ℓ_∞^n's uniformly.

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Towards a proof

- $[\Rightarrow]$: The Odell–Schlumprecht argument works without changes (based on a result of Enflo).
- $[\Leftarrow]$: Let us try to use generalized Mazur maps:
  - p-Convexification.
  - Entropy-based.

Unitarily invariant matrix norms

Finite-dimensional simplification

To understand a Mazur map defined on S_E, by density it suffices to understand it on S_E^n.

Definition

A norm $|||\cdot|||$ on $M_n = M_n(\mathbb{C})$ is called *unitarily invariant* if for any unitaries $U, V \in M_n$ and $A \in M_n$ we have

$$|||UAV||| = |||A|||$$

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Theorem

Unitarily invariant norms on M_n correspond to 1-symmetric norms on $\mathbb{R}^n$ (via the singular values).

Some properties of unitarily invariant matrix norms

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$|||ABC||| \leq \|A\|_\infty \, |||B||| \, \|C\|_\infty$ (actually equivalent to unitary invariance)

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Hölder's inequality

If $1/p + 1/q = 1/r$, then $|||\,|AB|^r|||^{1/r} \leq |||\,|A|^p|||^{1/p} \, |||\,|B|^q|||^{1/q}$.

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"Interpolation"

If $T : M_n \to M_n$ is contractive with respect to the operator and trace norms, then it is contractive with respect to any unitarily invariant norm.

p-convexification Mazur map, noncommutative setting

Remark

When E is 1-symmetric we have

$$\|A\|_{S_{E(p)}} = \| |A|^p \|_{S_E}^{1/p}.$$

Definition

Define $G_p : S_{E(p)} \to S_E$ by $G_p(x) = u|x|^p$, where x has polar representation $x = u|x|$.

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Theorem (CD)

Let $3 \leq p < \infty$, and let E be a 1-symmetric sequence space. Then the map G_p is a uniform homeomorphism between $\mathscr{S}(S_{E(p)})$ and $\mathscr{S}(S_E)$. Moreover, the moduli of continuity of G_p and G_p^{-1} depend only on p.

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The proof follows closely the strategies of [Ricard 2015].

The overall structure of the proof

STEPS:

1. Prove it for positive elements.
2. Reduction to self-adjoint elements.
3. Reduction to commutator estimate.
4. Breaking commutators into pieces.
5. Technical estimates.

Most of the proof is based on [Ricard 2015], replacing Hölder/Interpolation by their versions for unitarily invariant matrix norms.

The 2×2 trick

If $x \in S_E$, define the self-adjoint element $\tilde{x} = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix}$ and observe that

$$2 \|x\|_{S_E} \geq \|\tilde{x}\|_{S_E} \geq \|x\|_{S_E} .$$

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Reduction to self-adjoint case

x, y with polar decompositions $x = u|x|$ and $y = v|y|$. Now define

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They are selfadjoint with polar decompositions

$$\tilde{x} = \tilde{u}|\tilde{x}| = \begin{pmatrix} 0 & u \\ u^* & 0 \end{pmatrix} \cdot \begin{pmatrix} u|x|u^* & 0 \\ 0 & |x| \end{pmatrix}, \tilde{y} = \tilde{v}|\tilde{y}| = \begin{pmatrix} 0 & v \\ v^* & 0 \end{pmatrix} \cdot \begin{pmatrix} v|y|v^* & 0 \\ 0 & |y| \end{pmatrix}.$$

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Therefore

$$\tilde{u}|\tilde{x}|^p = \begin{pmatrix} 0 & u|x|^p \\ |x|^p u^* & 0 \end{pmatrix} \quad \text{and} \quad \tilde{v}|\tilde{y}|^p = \begin{pmatrix} 0 & v|y|^p \\ |y|^p v^* & 0 \end{pmatrix},$$

Reduction to self-adjoint case, contd.

Given $x, y \in S_E$ define

$$\tilde{x} = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \quad \text{and} \quad \tilde{y} = \begin{pmatrix} 0 & y \\ y^* & 0 \end{pmatrix}.$$

Then

$$\|x - y\|_{S_E} \quad \sim \quad \|\tilde{x} - \tilde{y}\|_{S_E}$$

and

$$\|G_p(x) - G_p(y)\|_{S_E} \quad \sim \quad \|G_p(\tilde{x}) - G_p(\tilde{y})\|_{S_E}$$

Reduction to commutator estimate

We use the commutator notation $[x, b] = xb - bx$.

Given $x, y \in S_E$ put

$$z = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

So that

$$[z, b] = \begin{pmatrix} 0 & x - y \\ 0 & 0 \end{pmatrix}$$

and thus

$$\big\|[z, b]\big\|_{S_E} = \big\|x - y\big\|_{S_E} \qquad \text{and} \qquad \big\|[G_p(z), b]\big\|_{S_E} = \big\|G_p(x) - G_p(y)\big\|_{S_E}.$$

Breaking commutators into pieces

Let $x \in S^n_{E^{(p)}}$ with $x = x^*$ and $b \in M_n$.

Write $e_+ = 1_{[0,\infty)}(x)$ and $e_- = 1_{(-\infty,0)}(x)$ and put $b_{\pm,\pm} = e_\pm b e_\pm$.

Then

$$\big[G_p(x), b\big] = \\ \big[x_+^p, b_{+,+}\big] - \big[x_-^p, b_{-,-}\big] + \big(x_+^p b_{+,-} + b_{+,-} x_-^p\big) - \big(x_-^p b_{-,+} + b_{-,+} x_+^p\big),$$

so we need to estimate the two types of terms.

Technical estimates, I

Lemma (CD)

Let $p \geq 1$*,* $x \in S^n_{E^{(p)}}$ *with* $x \geq 0$ *and* $b \in M_n$*. Then*

$$\left\| [x, b] \right\|_{S_{E^{(p)}}} \leq 4 \cdot 2^{1/p} \left\| \left[x^p, b \right] \right\|_{S_E}^{1/p},$$

$$\left\| \left[x^p, b \right] \right\|_{S_E} \leq 4 \cdot 3p \cdot 2 \, \|x\|_{S_{E^{(p)}}}^{p-1} \left\| [x, b] \right\|_{S_{E^{(p)}}},$$

Technical estimates, II

Lemma (CD)

(a) *Let* $p \geq 1$. *There exists a constant* C *such that for any* $x, y \in S^n_{E^{(p)}}$ *with* $x, y \geq 0$ *and* $b \in M_n$ *we have*

$$\left\| x^p b + b y^p \right\|_{S_E} \leq C \left\| x \right\|^{p-1}_{S_{E^{(p)}}} \left\| xb + by \right\|_{S_{E^{(p)}}}$$

(b) *If* $p \geq 3$, *then there exists a constant* C_p *such that for any* $x, y \in S^n_{E^{(p)}}$ *with* $x, y \geq 0$ *and* $b \in M_n$ *we have*

$$\left\| xb + by \right\|_{S_{E^{(p)}}} \leq C_p \left\| b \right\|^{1-1/p}_{\infty} \left\| x^p b + b y^p \right\|^{1/p}_{S_E}.$$

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Lemma (CD)

(a) *Let* $p \geq 1$. *There exists a constant* C *such that for any* $x, y \in S^n_{E(p)}$ *with* $x, y \geq 0$ *and* $b \in M_n$ *we have*

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p-convexification Mazur map, noncommutative setting

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What about Ricard's approach?

For a fixed $x \geq 0$, he uses that the spaces $L_p(x^\alpha)$ given by

$$\|b\|_{L_p(x^\alpha)} = \|x^\alpha b + bx^\alpha\|_{L_p}$$

interpolate in p and α [Ricard–Xu 2011].

Entropy-based Mazur map, noncommutative setting

Definition (Quantum relative entropy)

If $\rho \in M_n$ is a state (that is, $\rho \geq 0$ and $\mathrm{tr}(\rho) = 1$), and $\sigma \in M_n^+$, we define

$$D(\rho||\sigma) = \begin{cases} \mathrm{tr}[\rho(\log\rho - \log\sigma)] & \text{if } \mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma), \\ +\infty & \text{otherwise,} \end{cases}$$

where the support of an operator $A \in M_n$ is defined as the orthogonal complement of its kernel, $\mathrm{supp}(A) = \ker(A)^\perp$.

Notice the similarity of the term $-\rho\log\sigma$ with the expression used to define the Odell–Schlumprecht map.

Some properties of quantum relative entropy

Monotonicity

If $\rho \in M_n$ is a state, $0 \leq \sigma \leq \sigma'$, then $D(\rho||\sigma') \leq D(\rho||\sigma)$.

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Joint convexity

$\lambda_1, \dots, \lambda_m \geq 0$, $\sum_j \lambda_j = 1$ then

$$\sum_{j=1}^{m} \lambda_j D(\rho_j||\sigma_j) \geq D\left(\sum_{j=1}^{m} \lambda_j \rho_j \middle|\middle| \sum_{j=1}^{m} \lambda_j \sigma_j\right)$$

Entropy-based noncommutative generalized Mazur map

Definition (CD)

If $X = (M_n, |||\cdot|||)$ is a unitarily invariant strictly convex norm and and $\rho \in M_n$ is a state, we define $F_X(\rho)$ to be the $\sigma \in \mathscr{B}(X)^+$ minimizing the relative entropy with respect to ρ, that is,

$$F_X(\rho) = \arg\min_{\sigma \in \mathscr{B}(X)^+} D(\rho || \sigma).$$

Odell–Schlumprecht ideas + Quantum relative entropy

Proposition (CD)

Let $X = (M_n, |||\cdot|||)$ be an unitarily invariant matrix norm which is uniformly convex and uniformly smooth. Then $F_X : \mathscr{S}(S_1^n)^+ \to \mathscr{S}(X)^+$ is a uniform homeomorphism; moreover

- *The modulus of continuity of F_X depends only on the modulus of uniform convexity of X*
- *The modulus of continuity of F_X^{-1} depends only on the modulus of uniform smoothness of X.*

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The remaining challenge

Check that we also get a uniform homeomorphism between the whole spheres.

A big issue with the 2×2 trick

For the previous generalized Mazur map G_p,

$$G_p\left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\right) = \begin{pmatrix} G_p(x) & 0 \\ 0 & G_p(y) \end{pmatrix}$$

However, it is not obvious whether

$$F_X\left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\right) = \begin{pmatrix} F_X(x) & 0 \\ 0 & F_X(y) \end{pmatrix}$$

(or something similar, technically this doesn't quite make sense because F_X is only defined on states).

A way around it: complex interpolation

Theorem (Daher, Kalton 1995)

Let X_0, X_1 *be an interpolation pair with one of them being uniformly convex. Then for any* $\theta, \eta \in (0,1)$*,* $\mathscr{S}(X_\theta)$ *and* $\mathscr{S}(X_\eta)$ *are uniformly homeomorphic.*

Theorem

Let E_0, E_1 *be an interpolation pair of 1-symmetric sequence spaces. Then* $(S_{E_0}, S_{E_1})_\theta = S_{(E_0,E_1)_\theta}$*.*

Proof of the conjecture

Adapt Daher's proof of the Odell–Schlumprecht theorem: still uses G_p, but replaces the entropy-based Mazur map by the complex interpolation argument.

The main theorem

Theorem (CD)

Let E be a 1-unconditional sequence space. The following are equivalent:

1. (a) $\mathscr{S}(S_E)$ *is uniformly homeomorphic to* $\mathscr{S}(S_1)$.
2. (b) S_E *does not contain* ℓ_∞^n*'s uniformly.*
3. (c) E *does not contain* ℓ_∞^n*'s uniformly.*

THANKS!