Isometries of combinatorial Banach spaces

Christina Brech¹

Instituto de Matemática e Estatística Universidade de São Paulo

May 13, 2022 - BSBL

¹Projeto Temático Fapesp 2016/25574-8

May 12: Celebrating Women in Math



Nicole Tomczak-Jaegermann



Ofelia Alas



Mirna Dzamonja

May 12: Celebrating Women in Math



Yolanda Moreno



Claribet Piña



Alejandra Cáceres

Motivations

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Theorem (Banach-Stone)

For every compact Hausdorff spaces K and L, given an isometry $T : C(K) \to C(L)$, then there are a homeomorphism $\varphi : L \to K$ and $g \in C(L)$ such that |g(y)| = 1 and $T(f)(y) = g(y) \cdot (f \circ \varphi)(y)$ for all $f \in C(K)$ and every $y \in L$.

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Theorem (folklore)

For $X = c_0$ or $X = \ell_p$, $1 \le p < \infty$, $p \ne 2$, given an isometry $T : X \to X$, there are a bijection $\pi : \mathbb{N} \to \mathbb{N}$ and a sequence $(\theta_n)_n$ such that $|\theta_n| = 1$ and $T(e_n) = \theta_n e_{\pi(n)}$ for every $n \in \mathbb{N}$.

Isometries of the Schreier spaces

Theorem

Given an isometry $T : X_S \to X_S$, there is a sequence $(\theta_n)_n$ such that $|\theta_n| = 1$ and $T(e_n) = \theta_n e_n$ for every $n \in \mathbb{N}$.

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Theorem (Antunes, Beanland, Chu, 2020)

The same holds for Schreier spaces of larger finite order.

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Given a countable ordinal $\alpha \geq 1$,

$$\mathcal{S}_{lpha+1} = \{\bigcup_{i=1}^{n} s_i : s_i \in \mathcal{S}_{lpha} \text{ and } \{\min s_i : 1 \le i \le n\} \in \mathcal{S}\} \cup \{\emptyset\}$$

and

$$\mathcal{S}_{lpha} := igcup_{n \in \mathbb{N}} \mathcal{S}_{lpha_n} \upharpoonright (\mathbb{N} \setminus n)$$

where $(\alpha_n)_n$ is an increasing sequence converging to α , if α is limit.

Isometry group

We say (Antunes, Beanland) that the group of isometries of X is

• standard if the isometries are those T such that

$$T(e_n) = \theta_n e_{\pi(n)}$$

for some $|\theta_n| = 1$ and $\pi \in S_\infty$;

• diagonal if the isometries are those T such that

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for some $|\theta_n| = 1$.

Theorem (B, Ferenczi, Tcaciuc, 2020)

For every regular families \mathcal{F} and \mathcal{G} , given an isometry $T : X_{\mathcal{F}} \to X_{\mathcal{G}}$, there are a bijection $\pi : \mathbb{N} \to \mathbb{N}$ and a sequence $(\theta_n)_n$ such that $|\theta_n| = 1$ and $T(e_n) = \theta_n e_{\pi(n)}$ for every $n \in \mathbb{N}$.

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Ingredients of the proof:

• Characterization of the extreme points of the dual ball (Gowers):

$$Ext(B_{X_{\mathcal{F}}^*}) = \{\sum_{n \in s} \theta_n e_n^* : s \in \mathcal{F}^{MAX}, |\theta_n| = 1\}$$

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• Extract by hand the form of each $T^*(e_n^*)$.

Moreover, we also get that in the case of Schreier spaces of any order, the only bijection allowed is the identity.

Combinatorial (nonseparable) spaces

Theorem (B, Piña, 2021)

For every compact and hereditary families \mathcal{F} and \mathcal{G} such that singletons are in the closure of \mathcal{F}^{MAX} and \mathcal{G}^{MAX} , given an isometry $T : X_{\mathcal{F}} \to X_{\mathcal{G}}$, there are a bijection $\pi : \Gamma \to \Gamma$ and a sequence $(\theta_{\gamma})_{\gamma \in \Gamma}$ such that $|\theta_{\gamma}| = 1$ and $T(e_{\gamma}) = \theta_{\gamma} e_{\pi(\gamma)}$ for every $\gamma \in \Gamma$.

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• Extract the form of each $T^*(e_n^*)$ using w*-continuity of T^* .

Nonseparable examples

Theorem (Lopez-Abad, Todorcevic, 2013) $c : [\kappa]^{<\omega} \rightarrow 2$ witnesses that κ is not ω -Erdös iff

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Proposition (B., Lopez-Abad, Todorcevic, 2018) If \mathcal{F} is a hereditary, compact and large family of finite subsets of $\kappa(+1)$, then

$$\mathcal{G}=\{C\subseteq 2^{\leq\kappa}: C ext{ is a chain and } \{ht_{\mathcal{T}}(t): t\in C\}\in \mathcal{F}\}$$

is hereditary, compact and large.

 $(2^{\leq \kappa}$ is the complete binary tree of height $\kappa + 1)$

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- * (Godefroy, 1978) Yes, for spaces whose dual do not contain ℓ_1 .

Proposition

If the Cantor-Bendixson ranks of the singletons are different for both families, then the only bijection $\pi: \kappa \to \kappa$ which induces an isometry is the identity.

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- There are compact families on $\ensuremath{\mathbb{N}}$ which cannot be permuted onto any regular one.
- There are homeomorphic compact families on ℕ which cannot be permuted one to the other.

Theorem (Beauzamy, Casazza, 1980's(?); Antunes, Beanland, 2022) Given an isometry $T : T[\frac{1}{k}, S] \to T[\frac{1}{k}, S]$, there are a bijection $\pi : \{1, \ldots, k\} \to \{1, \ldots, k\}$ and a sequence $(\theta_n)_n$ such that $|\theta_n| = 1$ and $T(e_n) = \theta_n e_{\pi(n)}$ for $n \le k$ and $T(e_n) = \theta_n e_n$ for n > k. Theorem (Beauzamy, Casazza, 1980's(?); Antunes, Beanland, 2022) Given an isometry $T : T[\frac{1}{k}, S] \to T[\frac{1}{k}, S]$, there are a bijection $\pi : \{1, \ldots, k\} \to \{1, \ldots, k\}$ and a sequence $(\theta_n)_n$ such that $|\theta_n| = 1$ and $T(e_n) = \theta_n e_{\pi(n)}$ for $n \le k$ and $T(e_n) = \theta_n e_n$ for n > k.

* Do similar results hold for the nonseparable Tsirelson-type examples from BLT?

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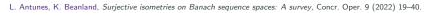
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- * Do combinatorial spaces have?

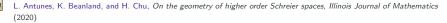
- * X has the Mazur-Ulam property if, for every Y, any isometry from S_X onto S_Y extends to a linear isometry from X onto Y.
- * c_0 and ℓ_p 's do have the Mazur-Ulam property and so does Tsirelson space.
- * Do combinatorial spaces have?
- * At least, does every isometry between the spheres of two combinatorial spaces extend to a linear isometry?

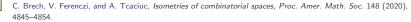
Questions

- * Do similar results hold for ℓ_p -norms?
- * Are these results instances of a more general result?

Main References







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05-09 December 2022, São Paulo

Brazilian Workshop in Banach Spaces

Butantã Edition

This workshop will focus on the following directions:

- Ramsey theory and set theory;
- · Homological theory, lattices and interpolation;
- · Operator theory and dynamics of operators;
- Nonlinear theory on Banach spaces.

The program includes four mini-courses:

Piotr Koszmider (Polish Academy of Sciences).

Étienne Matheron (Université d'Artois).

Eva Pernecka (Czech T. U. in Prague).

Noé De Rancourt (Charles University). [to be confirmed]

Main Speakers

Frédéric Bayart (U. Clermont Auvergne) Geraldo Botelho (U. F. Uberdándia) Bruno Braga (U. Virginia & PUC Rio de Janeiro) Yolanda Moreno (U. Extremadura) Sofia Ortega Castillo (U. Guadalajara) Grzegorz Piebanek (Wrocław U.) Pedro Tradacele (ICMAT - Madrid)

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