Isometries of combinatorial Banach spaces

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May 13, 2022 - BSBL

[1] Projeto Temático Fapesp 2016/25574-8

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Motivations

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Theorem (Banach-Stone)

For every compact Hausdorff spaces K and L, given an isometry $T : C(K) \to C(L)$, then there are a homeomorphism $\varphi : L \to K$ and $g \in C(L)$ such that $|g(y)| = 1$ and $T(f)(y) = g(y) \cdot (f \circ \varphi)(y)$ for all $f \in C(K)$ and every $y \in L$.

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Theorem (folklore)

For $X = c_0$ or $X = \ell_p$, $1 \leq p < \infty$, $p \neq 2$, given an isometry $T : X \to X$, there are a bijection $\pi : \mathbb{N} \to \mathbb{N}$ and a sequence $(\theta_n)_n$ such that $|\theta_n| = 1$ and $T(e_n) = \theta_n e_{\pi(n)}$ for every $n \in \mathbb{N}$.

Isometries of the Schreier spaces

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Given an isometry $T : X_{\mathcal{S}} \to X_{\mathcal{S}}$, there is a sequence $(\theta_n)_n$ such that $|\theta_n| = 1$ and $T(e_n) = \theta_n e_n$ for every $n \in \mathbb{N}$.

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Given a countable ordinal $\alpha \geq 1$,

$$\mathcal{S}_{\alpha+1} = \{\bigcup_{i=1}^{n} s_i : s_i \in \mathcal{S}_\alpha \text{ and } \{\min s_i : 1 \leq i \leq n\} \in \mathcal{S}\} \cup \{\emptyset\}$$

and

$$S_\alpha := \bigcup_{n\in\mathbb{N}} \mathcal{S}_{\alpha_n} \upharpoonright (\mathbb{N} \setminus n)$$

where $(\alpha_n)_n$ is an increasing sequence converging to α, if α is limit.

Isometry group

We say (Antunes, Beanland) that the group of isometries of X is

- standard if the isometries are those T such that

$$T(e_n) = \theta_n e_{\pi(n)}$$

  for some $|\theta_n| = 1$ and $\pi \in S_\infty$;

- diagonal if the isometries are those T such that

$$T(e_n) = \theta_n e_n$$

  for some $|\theta_n| = 1$.

Combinatorial spaces

Theorem (B, Ferenczi, Tcaciuc, 2020)

For every regular families $\mathcal{F}$ and $\mathcal{G}$, given an isometry $T : X_{\mathcal{F}} \to X_{\mathcal{G}}$, there are a bijection $\pi : \mathbb{N} \to \mathbb{N}$ and a sequence $(\theta_n)_n$ such that $|\theta_n| = 1$ and $T(e_n) = \theta_n e_{\pi(n)}$ for every $n \in \mathbb{N}$.

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Ingredients of the proof:

- Characterization of the extreme points of the dual ball (Gowers):

$$Ext(B_{X^*_{\mathcal{F}}}) = \{\sum_{n \in s} \theta_n e^*_n : s \in \mathcal{F}^{MAX}, |\theta_n| = 1\}.$$

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- Extract by hand the form of each $T^*(e^*_n)$.

Moreover, we also get that in the case of Schreier spaces of any order, the only bijection allowed is the identity.

Combinatorial (nonseparable) spaces

Theorem (B, Piña, 2021)

For every compact and hereditary families $\mathcal{F}$ and $\mathcal{G}$ such that singletons are in the closure of $\mathcal{F}^{MAX}$ and $\mathcal{G}^{MAX}$, given an isometry $T : X_{\mathcal{F}} \to X_{\mathcal{G}}$, there are a bijection $\pi : \Gamma \to \Gamma$ and a sequence $(\theta_\gamma)_{\gamma \in \Gamma}$ such that $|\theta_\gamma| = 1$ and $T(e_\gamma) = \theta_\gamma e_{\pi(\gamma)}$ for every $\gamma \in \Gamma$.

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Ingredients of the proof:

- Same characterization of the extreme points of the dual ball (Gowers):

$$Ext(B_{X_{\mathcal{F}}^*}) = \{\sum_{n\in s} \theta_n e_n^* : s \in \mathcal{F}^{MAX}, |\theta_n| = 1\}.$$

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- Extract the form of each $T^*(e_n^*)$ using w*-continuity of T^*.

Nonseparable examples

Theorem (Lopez-Abad, Todorcevic, 2013)

$c : [\kappa]^{<\omega} \to 2$ *witnesses that* κ *is not* ω*-Erdös iff*

$$\mathcal{F} = \{F \subseteq \kappa : c \restriction_{[F]^k} \text{ is constant for every } k\}$$

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Proposition (B., Lopez-Abad, Todorcevic, 2018)

If $\mathcal{F}$ is a hereditary, compact and large family of finite subsets of $\kappa(+1)$, then

$$\mathcal{G} = \{C \subseteq 2^{\leq\kappa} : C \text{ is a chain and } \{ht_T(t) : t \in C\} \in \mathcal{F}\}$$

is hereditary, compact and large.
($2^{\leq\kappa}$ is the complete binary tree of height $\kappa + 1$)

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* Is every isometry $T : X^*_{\mathcal{F}} \to X^*_{\mathcal{G}}$ w* continuous? Or, equivalently, is every isometry $T : X^*_{\mathcal{F}} \to X^*_{\mathcal{G}}$ the adjoint operator of an isometry $S : X_{\mathcal{G}} \to X_{\mathcal{F}}$?

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* In the BFT version, we start directly with an isometry $T : X_{\mathcal{F}}^* \to X_{\mathcal{G}}^*$, while in BP we need it to be w*-continuous.
* Is every isometry $T : X_{\mathcal{F}}^* \to X_{\mathcal{G}}^*$ w* continuous? Or, equivalently, is every isometry $T : X_{\mathcal{F}}^* \to X_{\mathcal{G}}^*$ the adjoint operator of an isometry $S : X_{\mathcal{G}} \to X_{\mathcal{F}}$?
* (Godefroy, 1978) Yes, for spaces whose dual do not contain ℓ_1.

More results by BP

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If the Cantor-Bendixson ranks of the singletons are different for both families, then the only bijection $\pi : \kappa \to \kappa$ which induces an isometry is the identity.

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Theorem

If $\pi : \mathbb{N} \to \mathbb{N}$ *is a bijection inducing an isometry between* $X_{\mathcal{F}}$ *and* $X_{\mathcal{G}}$ *where* $\mathcal{F}$ *and* $\mathcal{G}$ *are spreading families, then* $\mathcal{F} = \mathcal{G}$*.*

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- There are compact families on $\mathbb{N}$ which cannot be permuted onto any regular one.

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- There are compact families on $\mathbb{N}$ which cannot be permuted onto any regular one.
- There are homeomorphic compact families on $\mathbb{N}$ which cannot be permuted one to the other.

Tsirelson spaces

Theorem (Beauzamy, Casazza, 1980's(?); Antunes, Beanland, 2022)

Given an isometry $T : T[\frac{1}{k}, \mathcal{S}] \to T[\frac{1}{k}, \mathcal{S}]$*, there are a bijection* $\pi : \{1, \dots, k\} \to \{1, \dots, k\}$ *and a sequence* $(\theta_n)_n$ *such that* $|\theta_n| = 1$ *and* $T(e_n) = \theta_n e_{\pi(n)}$ *for* $n \leq k$ *and* $T(e_n) = \theta_n e_n$ *for* $n > k$*.*

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* Do similar results hold for the nonseparable Tsirelson-type examples from BLT?

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* c_0 and ℓ_p's do have the Mazur-Ulam property and so does Tsirelson space.
* Do combinatorial spaces have?
* At least, does every isometry between the spheres of two combinatorial spaces extend to a linear isometry?

Questions

* Do similar results hold for ℓ_p-norms?
* Are these results instances of a more general result?

Main References

L. Antunes, K. Beanland, *Surjective isometries on Banach sequence spaces: A survey*, Concr. Oper. 9 (2022) 19–40.

L. Antunes, K. Beanland, and H. Chu, *On the geometry of higher order Schreier spaces*, *Illinois Journal of Mathematics* (2020)

C. Brech, V. Ferenczi, and A. Tcaciuc, *Isometries of combinatorial spaces*, *Proc. Amer. Math. Soc.* 148 (2020), 4845–4854.

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W. Gowers, *Must an "explicitly defined" Banach space contain c_0 or ℓ_p?*, Feb. 17, 2009, Gowers's Weblog: Mathematics related discussions.

J. Lopez-Abad and S. Todorcevic, *Positional graphs and conditional structure of weakly null sequences*, Adv. Math. **242** (2013), 163–186.

This workshop will focus on the following directions:

- Ramsey theory and set theory;
- Homological theory, lattices and interpolation;
- Operator theory and dynamics of operators;
- Nonlinear theory on Banach spaces.

The program includes four mini-courses:

Piotr Koszmider (Polish Academy of Sciences).

Étienne Matheron (Université d'Artois).

Eva Pernecka (Czech T. U. in Prague).

Noé De Rancourt (Charles University). [to be confirmed]

Main Speakers

Frédéric Bayart (U. Clermont Auvergne)
Geraldo Botelho (U. F. Uberlândia)
Bruno Braga (U. Virginia & PUC Rio de Janeiro)
Yolanda Moreno (U. Extremadura)
Sofía Ortega Castillo (U. Guadalajara)
Grzegorz Plebanek (Wroclaw U.)
Pedro Tradacete (ICMAT - Madrid)

Scientific Committee

Dana Bartosova (U. Florida)
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Eloi Medina Galego (U. São Paulo)
Sophie Grivaux (U. Lille)

Let's thank the organizers of the workshop!

* Pedro Tradacete - ICMAT
* David de Hevia - ICMAT
* Enrique García - ICMAT
* Antonio Avilés - Universidad de Murcia