On uniform Mazur intersection property

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- It is not hard to see that Euclidean spaces have the MIP.
- Indeed, Mazur showed that any Banach space with a Fréchet smooth norm has the MIP.

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- He also asked whether the sufficient condition (a) is also necessary. To date, this remains an open question.

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- We will return to this result later again!

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  - (a) There exist Banach spaces with MIP which are not Asplund.
  - (b) Indeed, every Banach space isometrically embeds in a Banach space with the MIP.
  - (c) There exist (under the Continuum Hypothesis) Asplund spaces without any equivalent MIP renorming.

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- Soon after the 1978 paper, there appeared several papers dealing with the property that every compact (weakly compact/finite dimensional compact) convex set is an intersection of balls.
- A unified approach to these different intersection properties was presented by the speaker (1992) and independently by Chen & Lin (1998).

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Definition

We say that a Banach space X has the Uniform Mazur Intersection Property (UMIP) if for every $\varepsilon > 0$, there is $K > 0$ such that whenever a closed convex set $C \subseteq X$ and a point $p \in X$ are such that $\mathrm{diam}(C) \leq 1/\varepsilon$ and $d(p, C) \geq \varepsilon$, there is a closed ball $B \subseteq X$ of radius $\leq K$ such that $C \subseteq B$ and $d(p, B) \geq \varepsilon/2$.

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- Characterisations similar to Giles, Gregory & Sims were also obtained, but an analogue of the w*-denting point criterion was missing.
- This is the main object of our present talk.

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- *A slice of $B(X)$ determined by $f \in S(X^*)$ is a set of the form*

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- *For $x \in S(X)$, $S(B(X^*), x, \delta)$ is called a w*-slice of $B(X^*)$.*
- *We say that $x \in S(X)$ is a denting point of $B(X)$ if for every $\varepsilon > 0$, x is contained in a slice of $B(X)$ of diameter less than ε.*
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  If the slices are determined by the same functional, x *is a strongly exposed point.*
- *A w*-denting (strongly exposed) point of* $B(X^*)$ *is defined similarly.*

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- *For $\varepsilon, \delta > 0$, define*

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- For $x \in S(X)$, $D(x) = \{f \in S(X^*) : f(x) = 1\}$.
- The set valued map D is called the *duality map* and any selection of D is called a *support mapping*.

MIP Vs. UMIP

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The duality map is quasicontinuous, i.e., $\forall\ f \in S(X^*)$, $\varepsilon > 0$, $\exists\ x \in S(X)$, $\delta > 0$ *s.t.* $D(S(X) \cap B(x,\delta)) \subseteq B(f,\varepsilon)$.	*The duality map is uniformly quasicontinuous, i.e.,* $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$ *s.t.* $\forall\ f \in S(X^*)$, $\exists\ x \in S(X)$ *s.t.* $D(S(X) \cap B(x,\delta)) \subseteq B(f,\varepsilon)$.

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Every support mapping maps dense subsets of $S(X)$ to dense subsets of $S(X^*)$.	$\forall\ \varepsilon > 0$, $\exists\ \delta > 0$ s.t. every support mapping maps a δ-net in $S(X)$ to an ε-net in $S(X^*)$.

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$\forall\ \varepsilon > 0$, $D(M_\varepsilon(X))$ is norm dense in $S(X^*)$.	$\forall\ \varepsilon > 0$, $\exists\ \delta > 0$ s.t. $\forall\ f \in S(X^*)$, $\exists\ x \in M_{\varepsilon,\delta}(X)$ s.t. $D(x) \subseteq B(f,\varepsilon)$.

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- and showed that a Banach space X has the MIP if and only if every $f \in S(X^*)$ is a w*-semidenting point of $B(X^*)$.

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Proof

Lemma (Parallel Hyperplane Lemma)

For a nls X, $f, g \in S(X^*)$ and $\varepsilon > 0$, if $\{x \in B(X) : f(x) > \varepsilon\} \subseteq \{x \in X : g(x) > 0\}$, *then* $\|f - g\| \leq 2\varepsilon$.

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Then $\|f - tg\| = \|h\| \leq \varepsilon$.

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Now, if $y \in \{x \in B(X) : f(x) > \varepsilon\}$, then $g(y) > 0$ and

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Thus,

$$\|f - g\| \leq \|f - tg\| + |1 - t| \leq \varepsilon + \varepsilon = 2\varepsilon.$$

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And if $g \in S$ and $x \in A$, then $g(x_0 - x) \leq \|x_0 - x\| \leq r$ and hence, $g(x) \geq g(x_0) - r > 0$.

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Thus, $S \subseteq \{f \in B(X^*) : f(x) > 0 \;\forall\; x \in A\}$.

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Conversely, let $d = d(0, A) > 0$ and let $x_0 \in S_X$ and $0 < \varepsilon < 1$ be such that

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By the proof of the Parallel Hyperplane Lemma, $\forall \ x \in A$, $\exists \ t \in \mathbb{R}$ such that $1 - \varepsilon \leq t \leq 1 + \varepsilon$ and $\|tx/\|x\| - x_0\| \leq \varepsilon$.

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Then for $\lambda \geq M/(1-\varepsilon)$,

$$\begin{aligned} \|x - \lambda x_0\| &\leq \left\|x - \frac{\|x\|}{t} x_0\right\| + \left|\frac{\|x\|}{t} - \lambda\right| \leq \frac{\varepsilon\|x\|}{t} + \lambda - \frac{\|x\|}{t} \\ &= \lambda - \frac{\|x\|}{t}(1-\varepsilon) \leq \lambda - \frac{d(1-\varepsilon)}{1+\varepsilon}. \end{aligned}$$

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$\implies A \subseteq B\left[\lambda x_0, \lambda - \dfrac{d(1-\varepsilon)}{1+\varepsilon}\right]$ and clearly, $0 \notin B\left[\lambda x_0, \lambda - \dfrac{d(1-\varepsilon)}{1+\varepsilon}\right]$.

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For a Banach space X, the following are equivalent :

(a) *X has the UMIP.*

(b) *Every $f \in S(X^*)$ is uniformly w*-semidenting, i.e., given $\varepsilon > 0$, $\exists$ $0 < \delta < 1$ such that for any $f \in S(X^*)$, $\exists$ $x \in S(X)$ such that*

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By PH Lemma, $\|f - g/\|g\|\| \leq 2\varepsilon/3$. It follows that,

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$(b) \Rightarrow (a)$. Let $\varepsilon > 0$ be given. Let $L = 1/\varepsilon + \varepsilon$. Choose $0 < \delta < 1$ for $\varepsilon/4L$ obtained from (b). Let $K = L/\delta + 1$. We will show that this K works.

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By separation theorem, there exists $f \in S(X^*)$ such that $\inf f(D) \geq \varepsilon/2$.

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It follows that if $g \in B(X^*) \cap B(f, \varepsilon/4L)$ and $z \in D$, then $g(z) \geq \varepsilon/4 > 0$. Therefore,

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By the proof of Theorem 2, for $K \geq \lambda \geq L/\delta$,

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More on UMIP

- The following observations are from Whitfield & Zizler (1987):

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- They also produce an example to show that in general, UMIP $\not\Rightarrow$ MIP.

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References

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- J. H. M. Whitfield, V. Zizler, *Uniform Mazur's intersection property of balls*, Canad. Math. Bull., 30 (1987), 455-460.

References

- S. Mazur, *Über Schwache Konvergenz in den Raümen* (L^p), Studia Math. **4** (1933), 128–133.
- R. R. Phelps, *A Representation Theorem for Bounded Convex Sets*, Proc. Amer. Math. Soc. **11** (1960), 976–983.
- J. R. Giles, D. A. Gregory, Brailey Sims, *Characterisation of normed linear spaces with Mazur's intersection property*, Bull. Austral. Math. Soc., 18 (1978), 105-123.
- J. H. M. Whitfield, V. Zizler, *Uniform Mazur's intersection property of balls*, Canad. Math. Bull., 30 (1987), 455-460.
- PB, *The Mazur intersection property for families of closed bounded convex sets in Banach spaces*, Colloq. Math., 63 (1992), 45-56.

References

- Dongjian Chen, Bor-Luh Lin, *On B-convex and Mazur sets of Banach spaces*, Bull. Polish Acad. Sci. Math., 43 (1995), 191-198.

References

- Dongjian Chen, Bor-Luh Lin, *On B-convex and Mazur sets of Banach spaces*, Bull. Polish Acad. Sci. Math., 43 (1995), 191-198.
- Dongjian Chen, Bor-Luh Lin, *Ball separation properties in Banach spaces*, Rocky Mountain J. Math., 28 (1998), 835–873.

References

- Dongjian Chen, Bor-Luh Lin, *On B-convex and Mazur sets of Banach spaces*, Bull. Polish Acad. Sci. Math., 43 (1995), 191-198.
- Dongjian Chen, Bor-Luh Lin, *Ball separation properties in Banach spaces*, Rocky Mountain J. Math., 28 (1998), 835–873.
- M. Jiménez-Sevilla, J. P. Moreno, *Renorming Banach spaces with the Mazur intersection property*, J. Funct. Anal. 144 (1997), no. 2, 486–504.

References

- Dongjian Chen, Bor-Luh Lin, *On B-convex and Mazur sets of Banach spaces*, Bull. Polish Acad. Sci. Math., 43 (1995), 191-198.
- Dongjian Chen, Bor-Luh Lin, *Ball separation properties in Banach spaces*, Rocky Mountain J. Math., 28 (1998), 835–873.
- M. Jiménez-Sevilla, J. P. Moreno, *Renorming Banach spaces with the Mazur intersection property*, J. Funct. Anal. 144 (1997), no. 2, 486–504.
- M. Bačák, P. Hájek, *Mazur intersection property for Asplund spaces*, J. Funct. Anal. 255 (2008), no. 8, 2090–2094.

References

- Dongjian Chen, Bor-Luh Lin, *On B-convex and Mazur sets of Banach spaces*, Bull. Polish Acad. Sci. Math., 43 (1995), 191-198.
- Dongjian Chen, Bor-Luh Lin, *Ball separation properties in Banach spaces*, Rocky Mountain J. Math., 28 (1998), 835–873.
- M. Jiménez-Sevilla, J. P. Moreno, *Renorming Banach spaces with the Mazur intersection property*, J. Funct. Anal. 144 (1997), no. 2, 486–504.
- M. Bačák, P. Hájek, *Mazur intersection property for Asplund spaces*, J. Funct. Anal. 255 (2008), no. 8, 2090–2094.
- PB, J. Ganesh and D. Gothwal, *On Uniform Mazur Intersection Property* Studia Math. 260 (2021), no. 3, 273–283.

THANK YOU!