Gerodesics are the curves that (locally) minimize the length.

The flag curvature is a measure of how geodesics get apart.

As in Riemannian Geometry we want a distinguished connection to make computations easier.

We have to deal with a dependence on direction. For example, the fundamental tensor

\[ g_v(u, w) := \frac{1}{2} \frac{\partial^2}{\partial t \partial s} F^2(v + tu + sw) \bigg|_{t=s=0}, \]

gives a scalar product for every \( v \in TM \setminus 0 \).

This dependence can make Finsler computations a sea of tensors.

Because of this, part of the Riemannian community has put Finsler Geometry aside.

We will try to put some order.
Geodesics, Flag curvature and Connections

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\[ g_\nu(u, w) := \frac{1}{2} \frac{\partial^2}{\partial t \partial s} F^2(\nu + tu + sw)|_{t=s=0}, \]

which gives a scalar product for every \( \nu \in TM \setminus \{0\} \).
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Anisotropic tensors

Let us put $A = TM \setminus 0$. An anisotropic tensor $T \in \mathfrak{T}^r_s(M, A)$ in a manifold $M$ is a family of maps of the form

$$T_v : T_{\pi(v)}M^* \times \cdots \times T_{\pi(v)}M^* \times T_{\pi(v)}M \times \cdots \times T_{\pi(v)}M \rightarrow \mathbb{R}$$

for every $v \in A$ such that in a system of coordinates $(\Omega, \varphi = (x^1, \ldots, x^n))$, the functions

$$T_{i_1 i_2 \ldots i_r}^{j_1 j_2 \ldots j_s}(v) = T_v(dx^{i_1}, dx^{i_2}, \ldots, dx^{i_r}, \partial_{j_1}, \partial_{j_2}, \ldots, \partial_{j_s})$$

are smooth.

One can identify $\mathfrak{T}^1_0(M, A)$ with the anisotropic vector fields $\mathcal{X}$, namely, smooth maps

$$A \ni v \rightarrow \mathcal{X}(v) \in TM$$

with $\mathcal{X}(v) \in T_{\pi(v)}M$. 
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for every $v \in A$ such that in a system of coordinates $(\Omega, \varphi = (x^1, \ldots, x^n))$, the functions

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One can identify $S_0^1(M, A)$ with the anisotropic vector fields $\mathcal{X}$, namely, smooth maps

$$A \ni v \to \mathcal{X}(v) \in TM$$

with $\mathcal{X}(v) \in T_{\pi(v)} M$. 

Anisotropic connections

An *anisotropic (linear) connection* is a map

\[ \nabla : \mathcal{A}(M) \times \mathcal{X}(M) \to TM, \quad (v, X, Y) \mapsto \nabla^v_X Y := \nabla(v, X, Y) \in T_{\pi(v)}M, \]

such that

(i) \[ \nabla^v_X(Y + Z) = \nabla^v_X Y + \nabla^v_X Z, \text{ for any } X, Y, Z \in \mathcal{X}(M) \text{ (linear)}, \]

(ii) \[ \nabla^v_X(fY) = X(f)Y|_{\pi(v)} + f(\pi(v))\nabla^v_X Y \text{ for any } f \in \mathcal{F}(M), \]
    \[ X, Y \in \mathcal{X}(M) \text{ Leibniz rule}, \]

(iii) \[ \nabla^v_{fX+hY}Z = f(\pi(v))\nabla^v_X Z + h(\pi(v))\nabla^v_Y Z, \text{ for any } f, h \in \mathcal{F}(M), \]
    \[ X, Y, Z \in \mathcal{X}(M) \text{ (\mathcal{F}(M)-linear)}, \]

(iv) \[ \text{for any } X, Y \in \mathcal{X}(M), \nabla_X Y \in \mathcal{X}_0^1(M, A) \text{ (considered as a map } A \ni v \to \nabla^v_X Y), \]
Anisotropic connections

An *anisotropic (linear) connection* is a map

\[ \nabla : A \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to TM, \quad (\nu, X, Y) \mapsto \nabla^\nu_X Y := \nabla(\nu, X, Y) \in T_{\pi(\nu)} M, \]

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(iii) $\nabla^\nu_{fX + hY} Z = f(\pi(\nu))\nabla^\nu_X Z + h(\pi(\nu))\nabla^\nu_Y Z$, for any $f, h \in \mathcal{F}(M)$,

(iv) for any $X, Y \in \mathfrak{X}(M)$, $\nabla_X Y \in \mathfrak{X}^1_0(M, A)$ (considered as a map $A \ni \nu \to \nabla^\nu_X Y$),
Anisotropic connections

An *anisotropic (linear) connection* is a map

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(iv) for any \( X, Y \in \mathfrak{X}(M), \) \( \nabla_X Y \in \mathcal{X}^1_0(M, A) \) (considered as a map \( A \ni v \to \nabla^v_X Y \)).
The torsion tensor of the connection is defined as

\[ \mathcal{T}_v(X, Y) = \nabla_X^v Y - \nabla_Y^v X - [X, Y] \]

Moreover, we will say that the anistropic linear connection is torsion-free if \( \mathcal{T} = 0 \), namely,

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Christoffel symbols and auto-parallel curves

Given a system of coordinates \((\Omega, \varphi)\), we define the **Christoffel symbols** of \(\nabla\) as the functions

\[
\Gamma^i_{jk} : A \cap T\Omega \rightarrow \mathbb{R}
\]

satisfying that

\[
\nabla^\gamma_{\partial_j} \partial_k = \Gamma^i_{jk}(\nu) \partial_i.
\]

Observe that a connection is torsion-free if and only if its Christoffel symbols are symmetric in \(j\) and \(k\).

The **auto-parallel curves** of \(\nabla\) are the curves \(\gamma : [a, b] \rightarrow M\) such that

\[
\dddot{\gamma}^k + \dot{\gamma}^i \dddot{\gamma}^j \Gamma^k_{ij}(\dot{\gamma}) = 0
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We can define a covariant derivative \(D_\gamma\) along any curve \(\gamma\) and then auto-parallel curves satisfy \(D_\gamma \dddot{\gamma} = 0\)
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Semi-sprays and Sprays

**Definition**

Given a manifold $M$, a **semi-spray** on $M$ is a vector field $S$ in a subset $A \subset TM$ which has the following property:

If $\beta : [a, b] \subset \mathbb{R} \to A$ is an integral curve of $S$, then $\beta = \dot{\alpha}$, where

$$\alpha = \pi_M \circ \beta.$$ 

We say that a semi-spray $S$ is a **spray** if, in addition, $A$ is conic, namely, if $v \in A$, then $\lambda v \in A$ for every $\lambda > 0$ and $S$ has the following property:

If $\beta = \dot{\alpha}$ is an integral curve of $S$, then $\tilde{\beta}(t) = \lambda \dot{\alpha}(\lambda t)$ is an integral curve of $S$ for every $\lambda > 0$. 
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Berwald connection of a spray

In natural coordinates for $TM$, a vector field $S$ in $TM$ is a spray if and only if it is expressed as

$$S(v) = \sum_{i=1}^{n} (y^i \frac{\partial}{\partial x^i} - 2G^i(v) \frac{\partial}{\partial y^i})$$

with $G^i : T\Omega \cap A \rightarrow \mathbb{R}$ positive homogeneous functions. Then

$$\Gamma^k_{ij}(v) = \frac{\partial N^k_{ij}}{\partial y^j}(v) = \frac{\partial^2 G^k}{\partial y^i \partial y^j}(v)$$

are the Christoffel symbols of the Berwald connection $\tilde{\nabla}$

Auto-parallel curves of $\tilde{\nabla}$ are the same as integral curves of $S$.

Conclusion: Sprays can be studied as a particular case of anisotropic connection.
Berwald connection of a spray

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Conclusion: Sprays can be studied as a particular case of anisotropic connection.
Berwald connection of a spray

In natural coordinates for $T\mathcal{M}$, a vector field $S$ in $\mathcal{T}\mathcal{M}$ is a spray if and only if it is expressed as

$$S(v) = \sum_{i=1}^{n} (y^i \frac{\partial}{\partial x^i} - 2G^i(v) \frac{\partial}{\partial y^i})$$

with $G^i : T\Omega \cap A \to \mathbb{R}$ positive homogeneous functions. Then

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Conclusion: Sprays can be studied as a particular case of anisotropic connection.
Deriving Anisotropic tensors with $\nabla$

First, we will see how to make the derivative of $f : A \rightarrow \mathbb{R}$ ($A = TM \setminus 0$)

Associated affine connection $\nabla^V$: Given an $A$-admissible vector field $V$ in an open subset $\Omega$, we can define an affine connection $\nabla^V$ using the anisotropic one:

$$(\nabla^V_X Y)_p = \nabla^V_{(p)} X Y.$$ 

$\nabla^V$ is an affine connection in $\Omega \subset M$.

Vertical derivation: given a tensor $T \in \mathcal{S}^r(M, A)$, we define its vertical derivation as the tensor $\partial^\nu T \in \mathcal{S}^r_{s+1}(M, A)$ given by

$$(\partial^\nu T)_v(\theta^1, \theta^2, \ldots, \theta^r, X_1, \ldots, X_s, Z)$$

$$= \frac{\partial}{\partial t} T_{v+tZ(\pi(v))}(\theta^1, \theta^2, \ldots, \theta^r, X_1, \ldots, X_s)|_{t=0}$$

for any

$$(\theta^1, \ldots, \theta^r, X_1, \ldots, X_s, Z) \in T_{\pi(v)}M \times T_{\pi(v)}M \times T_{\pi(v)}M \times T_{\pi(v)}M \times \cdots \times T_{\pi(v)}M$$
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$$ (\partial^\nu T)_v(\theta^1, \theta^2, \ldots, \theta^r, X_1, \ldots, X_s, Z) $$

$$ = \frac{\partial}{\partial t} T_{v+tz}\pi(v)(\theta^1, \theta^2, \ldots, \theta^r, X_1, \ldots, X_s)|_{t=0} $$

for any

$$(\theta^1, \ldots, \theta^r, X_1, \ldots, X_s, Z) \in T_{\pi(v)}M^* \times T_{\pi(v)}M \times T_{\pi(v)}M \times \cdots$$
First, we will see how to make the derivative of \( f : A \rightarrow \mathbb{R} \) \( (A = TM \setminus 0) \)

**Associated affine connection** \( \nabla^V \): Given an \( A \)-admissible vector field \( V \) in an open subset \( \Omega \), we can define an affine connection \( \nabla^V \) using the anisotropic one:

\[
(\nabla^V_X Y)_p = \nabla^V_{(p)} X Y.
\]

\( \nabla^V \) is an affine connection in \( \Omega \subset M \).

**Vertical derivation:** given a tensor \( T \in \mathcal{F}^r_s(M, A) \), we define its **vertical derivation** as the tensor \( \partial^\nu T \in \mathcal{F}^r_{s+1}(M, A) \) given by

\[
(\partial^\nu T)_v(\theta^1, \theta^2, \ldots, \theta^r, X_1, \ldots, X_s, Z) = \frac{\partial}{\partial t} T_{v+tZ(\pi(v))}(\theta^1, \theta^2, \ldots, \theta^r, X_1, \ldots, X_s)|_{t=0}
\]

for any

\[
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\]
The subset of anisotropic functions is denoted by

\[ \mathcal{F}(A) = \{ f : A = TM \setminus 0 \to \mathbb{R} \} \]

Lemma

Given an anisotropic connection \( \nabla \) in \( A \), we can define the derivation of a function \( h \in \mathcal{F}(A) \) as \( \nabla_Z(h) \in \mathcal{F}(A) \) defined by

\[ \nabla_Z(h)(v) := Z(h(V))(\pi(v)) - (\partial^\nu h)_v(\nabla_Z^\nu V) \]

with \( V, Z \in \mathcal{X}(\Omega) \), being \( V \) \( A \)-admissible on an open subset \( \Omega \subset M \) such that \( V(\pi(v)) = v \), which satisfies the Leibnitzian property

\[ D(fg) = D(f)g + fD(g) \]

for any \( f, g \in \mathcal{F}(A) \).

Proof: see Lemma 9 of


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In the following Jav19.
A tensor derivation is an \( \mathbb{R} \)-linear map

\[
D : \bigcup_{r \geq 0, s \geq 0} \mathbb{T}^r_s(M, A) \to \bigcup_{r \geq 0, s \geq 0} \mathbb{T}^r_s(M, A)
\]

that preserves the type of the tensor and satisfies the Leibnitz rule for the tensor product

\[
D(T_1 \otimes T_2) = D(T_1) \otimes T_2 + T_1 \otimes D(T_2)
\]

and it commutes with contractions

**Theorem**

Let \( \nabla \) be an anisotropic connection. Then there exists a unique tensor derivation \( \nabla \) such that \((\nabla_X Y)(v) = \nabla^X_Y Y\) for every \( X \in \mathfrak{X}(M) \), and \( \nabla_X(h) \) is defined as before.

Proof: see Theorem 11 of Jav19.
A tensor derivation is an $\mathbb{R}$-linear map

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**Theorem**

Let $\nabla$ be an anisotropic connection. Then there exists a unique tensor derivation $\nabla$ such that $(\nabla_X Y)(\nu) = \nabla^\nu_X Y$ for every $X \in \mathfrak{X}(M)$, and $\nabla_X(h)$ is defined as before.

Proof: see Theorem 11 of Jav19.
Derivative of an anisotropic tensor

By the product rule, if \( T \in \mathcal{X}_2^0(M, A) \), \( X, Y, Z \in \mathcal{X}(M) \),

\[
(\nabla_Z T)_v(X, Y) = \nabla_Z(T(X, Y)) - T_v(\nabla_Z X, Y) - T_v(X, \nabla_Z Y).
\]

By the Lemma, for any \( A \)-admissible extension \( V \) of \( v \),

\[
\nabla_Z(T(X, Y))(v) = Z(T_V(X, Y))(\pi(v)) - (\partial^\nu T)_v(X, Y, \nabla_Z^V V)
\]

Therefore,

\[
(\nabla_Z T)_v(X, Y) = Z(T_V(X, Y))(\pi(v)) - (\partial^\nu T)_v(X, Y, \nabla_Z^V V)
- T_v(\nabla_Z^V X, Y) - T_v(X, \nabla_Z^V Y).
\]

Moral of the story: we can compute \( \nabla_Z T \) using the affine connection \( \nabla^V \).
Derivative of an anisotropic tensor

By the product rule, if $T \in \mathfrak{T}_2^0(M, A)$, $X, Y, Z \in \mathfrak{X}(M)$,

$$(\nabla_Z T)_v(X, Y) = \nabla_Z(T(X, Y)) - T_v(\nabla_Z X, Y) - T_v(X, \nabla_Z Y).$$

By the Lemma, for any $A$-admissible extension $V$ of $v$,

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Moral of the story: we can compute $\nabla_Z T$ using the affine connection $\nabla^V$. 
Derivative of an anisotropic tensor

By the product rule, if $T \in \mathfrak{T}^2_2(M, A)$, $X, Y, Z \in \mathfrak{X}(M)$,

$$(\nabla_Z T)_v(X, Y) = \nabla_Z(T(X, Y)) - T_v(\nabla_Z X, Y) - T_v(X, \nabla_Z Y).$$

By the Lemma, for any $A$-admissible extension $V$ of $v$,

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Moral of the story: we can compute $\nabla_Z T$ using the affine connection $\nabla^V$. 
The curvature tensor of $\nabla$

Using also the Lemma, we can make the derivative of an anisotropic vector field $Y$:

$$\nabla_X(Y)(v) = \nabla_X(Y(V)) - (\partial^\nu Y)_v(\nabla_X V),$$

where $(\partial^\nu Y)_v(z) = \frac{d}{dt} Y(v + tz)|_{t=0}$, for any vector $z \in T_{\pi(v)}M$.

Given $X, Y, Z \in \mathfrak{X}(M)$, define

$$R_v(X, Y)Z := \nabla_X^Y(\nabla_Y Z) - \nabla_Y^Y(\nabla_X Z) - \nabla_{[X,Y]}^Y Z$$

It is easy to prove that it is $\mathcal{F}(M)$-linear in $X, Y, Z$.

Applying above formula, we get

$$\nabla_X^Y(\nabla_Y Z) = \nabla_X^Y(\nabla_Y^Y Z) - \partial^\nu(\nabla_Y Z)_v(\nabla_X^V)$$
The curvature tensor of $\nabla$

Using also the Lemma, we can make the derivative of an anisotropic vector field $\mathcal{Y}$:

$$
\nabla_X(\mathcal{Y})(v) = \nabla_X^\nu(\mathcal{Y}(V)) - (\partial^\nu \mathcal{Y})_v(\nabla_X V),
$$

where $(\partial^\nu \mathcal{Y})_v(z) = \frac{d}{dt} \mathcal{Y}(v + tz)|_{t=0}$, for any vector $z \in T_{\pi(v)}M$.

Given $X, Y, Z \in \mathfrak{X}(M)$, define

$$
R_v(X, Y)Z := \nabla_X^\nu(\nabla_Y Z) - \nabla_Y^\nu(\nabla_X Z) - \nabla_{[X, Y]}^\nu Z
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$$
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$$
The curvature tensor of $\nabla$

Using also the Lemma, we can make the derivative of an anisotropic vector field $\mathcal{Y}$:

$$\nabla_X(\mathcal{Y})(\nu) = \nabla^\nu_X(\mathcal{Y}(\mathcal{V})) - (\partial^\nu \mathcal{Y})_\nu(\nabla^\nu_X \mathcal{V}),$$

where $(\partial^\nu \mathcal{Y})_\nu(z) = \frac{d}{dt} \mathcal{Y}(\nu + tz)|_{t=0}$, for any vector $z \in T_{\pi(\nu)}M$.

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$$R^\nu_v(X, Y)Z := \nabla^\nu_X(\nabla_Y Z) - \nabla^\nu_Y(\nabla_X Z) - \nabla^\nu_{[X,Y]}Z$$

It is easy to prove that it is $\mathcal{F}(M)$-linear in $X, Y, Z$.

Applying above formula, we get

$$\nabla^\nu_X(\nabla_Y Z) = \nabla^\nu_X(\nabla^\nu_Y Z) - \partial^\nu(\nabla_Y Z)_\nu(\nabla^\nu_X \mathcal{V})$$
Definition of the curvature tensor

Let us define “the vertical derivative” of the anisotropic connection $\nabla^V$ as

$$P_V(X, Y, Z) = \frac{\partial}{\partial t} \left( \nabla^V_X t^Z Y \right) |_{t=0},$$

Then

$$R_V(X, Y)Z := (\nabla^V_X \nabla^V_Y Z - \nabla^V_Y \nabla^V_X Z - \nabla^V_{[X,Y]} Z - P_V(Y, Z, \nabla^V_X V) + P_V(X, Z, \nabla^V_Y V))(\pi(V)),$$

where $V$ is any vector field with $V(\pi(v)) = v$, is well-defined and $R$ is an anisotropic tensor.
Let us define “the vertical derivative” of the anisotropic connection $\nabla^\nu$ as

$$P^\nu_v(X,Y,Z) = \frac{\partial}{\partial t} \left( \nabla^\nu_X t Z^Y \right) |_{t=0},$$

Then

$$R^\nu_v(X,Y)Z := (\nabla^\nu_X \nabla^\nu_Y Z - \nabla^\nu_Y \nabla^\nu_X Z - \nabla^\nu_{[X,Y]} Z$$

$$- P^\nu_v(Y,Z,\nabla^\nu_X V) + P^\nu_v(X,Z,\nabla^\nu_Y V))(\pi(v)),$$

where $V$ is any vector field with $V(\pi(v)) = v$, is well-defined and $R$ is an anisotropic tensor.
Choosing $V$ to simplify computations

**Proposition**

One can choose and $A$-admissible extension $V$ defined in an open subset $\Omega \subset M$, such that

$$\nabla^V \chi V = 0$$

for any vector field $X \in \mathfrak{X}(\Omega)$. Furthermore, if $T \in \mathfrak{X}_r^s(M, A)$ and $X \in \mathfrak{X}_0^1(M, A)$, then $(\nabla_X T)_V = (\nabla^V_\chi (T_V))_{\pi(v)}$, and the curvature tensor of $\nabla$ can be computed as

$$R_V(X, Y)Z = R^V_{\pi(v)}(X, Y)Z = (\nabla^V_X \nabla^V_Y Z - \nabla^V_Y \nabla^V_X Z)(\pi(v)),$$

for $X, Y, Z \in \mathfrak{X}(\Omega)$ such that $[X, Y] = 0$ (the last condition is not necessary for the first identity), and its derivative as

$$(\nabla_X R)_V(Y, Z)W = (\nabla^V_X R^V)_{\pi(v)}(Y, Z)W - P_V(Z, W, \nabla^V_Y \nabla^V_Y V) + P_V(Y, W, \nabla^V_X \nabla^V_Z V)$$
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$$R_v(X, Y)Z = R_{\pi(v)}^V(X, Y)Z = (\nabla_X^V \nabla_Y^V Z - \nabla_Y^V \nabla_X^V Z)(\pi(v)),$$

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$$(\nabla_X R)_v(Y, Z)W = (\nabla_X^V R^V)_{\pi(v)}(Y, Z)W - P_v(Z, W, \nabla_X^V \nabla_Y^V V) + P_v(Y, W, \nabla_X^V \nabla_Z^V V)$$
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Bianchi identities

As a consequence, we can get Bianchi Identities for anisotropic connections almost for free (using identities for the affine connection $\nabla^V$):

For every $v \in A$ and $u, w, z \in T_{\pi(v)}M$, we have that $R_v(u, w) = - R_v(w, u)$ and $R$ satisfies the 1st Bianchi identity:

$$
\sum_{cyc: u, w, z} R_v(u, w)z = \sum_{cyc: u, w, z} (T_v(T_v(u, w), z) + (\nabla_u T)_v(w, z)),
$$

where $T$ is the torsion of $\nabla$, and the 2nd Bianchi identity:

$$
\sum_{cyc: u, w, z} ((\nabla_u R)_v(w, z)b - P_v(w, b, R_v(u, z)v) + R_v(T_v(u, w), z)b) = 0.
$$

Here $\sum_{cyc: u, w, z}$ denotes the cyclic sum in $u, w, z$. 

Luigi Bianchi (1856-1928)
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Luigi Bianchi (1856-1928)
There is a new “vertical Bianchi identity”:
For every $v \in A$ and $u, w, z, b \in T_{\pi(v)}M$,

$$(\partial^v R)_v(u, w, z, b) = (\nabla_u P)_v(w, z, b) - (\nabla_w P)_v(u, z, b)$$

$$- P_v(w, z, P_v(u, v, b)) + P_v(u, z, P_v(w, v, b))$$

$$+ P_v(T_v(u, w), z, b).$$
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For the last results see Prop. 2.13, 2.14 and 2.15 of


In the following, Jav19b
Back to the Finslerian World
Derivation of the fundamental tensor

Let us recall that the fundamental tensor of a pseudo-Finsler metric $L : A \rightarrow \mathbb{R}$ is defined as

$$g_v(u, w) := \frac{1}{2} \frac{\partial^2}{\partial t \partial s} L(v + tu + sw) \bigg|_{t=s=0},$$

and the Cartan tensor:

$$C_v(w_1, w_2, w_3) := \frac{1}{4} \frac{\partial^3}{\partial s_3 \partial s_2 \partial s_1} L \left( v + \sum_{i=1}^{3} s_i w_i \right) \bigg|_{s_1=s_2=s_3=0}$$

Then it follows that

$$(\partial^\nu g)_v(w_1, w_2, w_3) = 2C_v(w_1, w_2, w_3)$$

and

$$(\nabla_Z g)_v(X, Y) := Z(g_V(X, Y)) - g_V(\nabla^V_Z X, Y) - g_V(X, \nabla^V_Z Y) - 2C_V(\nabla^V_Z V, X, Y),$$
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Chern connection as a Levi-Civita connection

It is well-known that when the Chern connection is interpreted as a family of affine connections is characterized by satisfying:

\[
\begin{align*}
\nabla_X Y - \nabla_Y X &= [X, Y] \quad \text{(torsion-free)} \\
Z(g_V (X, Y)) &= g_V (\nabla_Z X, Y) - g_V (X, \nabla_Z Y) - 2C_V (\nabla_Z V, X, Y)
\end{align*}
\]

But this is the same as $\nabla$ being torsion-free and $\nabla g = 0$ with our definition.

The Chern connection is the Levi-Civita connection of a Finsler metric!!!!
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Some Bibliography about this treatment


The family of affine connections

- The idea of the family of affine connections appears first related to the osculating metric $g_V$
- The osculating metric was considered in 1936 by A. Nazim in his PhD Thesis.
- In 1941, O. Varga proves that when $V$ is a geodesic vector field, there is an affine connection related to the osculating metric having all the integral curves of $V$ as geodesics.
- In 1978, H.-H. Matthias introduces the concept of family of affine connections related to the Chern connection when generalizing the Gromoll-Meyer theorem.
- In 2001, Z. Shen includes this approach to Chern connection in his book about Sprays proving some results of Jacobi operator when $V$ is geodesic.
- In 2004, H.-B. Rademacher uses the family of affine connections to study the energy functional in a sphere theorem for non-reversible Finsler metrics.
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The classical connections

- The classical connections used in Finsler Geometry are connections in the vertical bundle $\pi : \pi^* (TM) \rightarrow TM \setminus 0$, where $\pi^* (TM) = \mathcal{V}(TM \setminus 0)$

- A connection in the vertical bundle is a map

$$\nabla' : \mathfrak{X}(TM \setminus 0) \times \Gamma(\pi^* (TM)) \rightarrow \Gamma(\pi^* (TM))$$

- Consider the natural coordinates of the tangent bundle $\tilde{\varphi} = (x, y)$ associated with a system of coordinates $(\Omega, \varphi)$ in $M$, where $\tilde{\varphi}^{-1}(x^1, \ldots, x^n, y^1, \ldots, y^n) = y^i \left. \frac{\partial}{\partial x^i} \right| \varphi^{-1}(x)$

- The Christoffel symbols are determined by $\nabla' \frac{\partial}{\partial x^i} \left. \frac{\partial}{\partial y^j} \right| = \Gamma^k_{ij}(x, y) \frac{\partial}{\partial y^k}$

- With anisotropic connections you need less information (there are half of Christoffel symbols)
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\[ \nabla' : \mathfrak{X}(TM \setminus 0) \times \Gamma(\pi^*(TM)) \to \Gamma(\pi^*(TM)) \]

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The Christoffel symbols are determined by 
\[ \nabla' \frac{\partial}{\partial x^i} \frac{\partial}{\partial y^j} = \Gamma^k_{ij}(x, y) \frac{\partial}{\partial y^k} \]
and 
\[ \nabla' \frac{\partial}{\partial y^\alpha} \frac{\partial}{\partial y^j} = \Gamma^k_{\alpha j}(x, y) \frac{\partial}{\partial y^k} \]

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- The classical connections used in Finsler Geometry are connections in the vertical bundle $\pi : \pi^*(TM) \rightarrow TM \setminus 0$, where $\pi^*(TM) = V(TM \setminus 0)$.

- A connection in the vertical bundle is a map

\[ \nabla' : \mathfrak{X}(TM \setminus 0) \times \Gamma(\pi^*(TM)) \rightarrow \Gamma(\pi^*(TM)) \]

- Consider the natural coordinates of the tangent bundle $\bar{\varphi} = (x, y)$ associated with a system of coordinates $(\Omega, \varphi)$ in $M$, where $\bar{\varphi}^{-1}(x^1, \ldots, x^n, y^1, \ldots, y^n) = y^i \frac{\partial}{\partial x^i} \bigr|_{\varphi^{-1}(x)}$

- The Christoffel symbols are determined by

\[ \nabla'_i \frac{\partial}{\partial y^j} = \Gamma^k_{ij}(x, y) \frac{\partial}{\partial y^k} \]

\[ \nabla'_\alpha \frac{\partial}{\partial y^j} = \Gamma^k_{\alpha j}(x, y) \frac{\partial}{\partial y^k} \]

- With anisotropic connections you need less information (there are half of Christoffel symbols).
The classical connections

- The classical connections used in Finsler Geometry are connections in the vertical bundle $\pi : \pi^*(TM) \to TM \setminus 0$, where $\pi^*(TM) = \mathcal{V}(TM \setminus 0)$

- A connection in the vertical bundle is a map

  $$\nabla' : \mathfrak{X}(TM \setminus 0) \times \Gamma(\pi^*(TM)) \to \Gamma(\pi^*(TM))$$

- Consider the natural coordinates of the tangent bundle $\bar{\varphi} = (x, y)$ associated with a system of coordinates $(\Omega, \varphi)$ in $M$, where $\bar{\varphi}^{-1}(x^1, \ldots, x^n, y^1, \ldots, y^n) = y^i \left. \frac{\partial}{\partial x^i} \right|_{\varphi^{-1}(x)}$

- The Christoffel symbols are determined by $\nabla' \frac{\partial}{\partial y^j} = \Gamma^k_{ij}(x, y) \frac{\partial}{\partial y^k}$
  
  and $\nabla' \frac{\partial}{\partial y^\alpha} \frac{\partial}{\partial y^j} = \Gamma^k_{\alpha j}(x, y) \frac{\partial}{\partial y^k}$

- With anisotropic connections you need less information (there are half of Christoffel symbols)
Anisotropic connections associated with the classical connections

A Finsler metric has always associated a spray (constructed from its geodesics), and then an Ehresmann connection, which allows one to lift a vector field $X \in \mathfrak{X}(M)$ to a vector field $X^H \in \mathfrak{X}(TM)$. Moreover, using

$$i_v : T_{\pi(v)}M \to \mathcal{V}_v TM$$

given by $i_v(u) = \frac{d}{dt}(v + tu)|_{t=0}$, we also can lift a vector field $X \in \mathfrak{X}(M)$ to a section of $\pi : \pi^*(TM) \to TM \setminus 0$. 
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Recall that, we can define an anisotropic connection

\[ \nabla^\nu_X Y = (\nabla_X\nu Y^\nu)(\nu), \]

and an anisotropic tensor

\[ \mathbb{C}_\nu(X, Y) = (\nabla_X\nu Y^\nu)(\nu), \]

Then for classical connections it turns out that

- **Chern-Rund connection** projects into the Levi-Civita and \( \mathbb{C} = 0 \)
- **Cartan connection** projects into Levi-Civita and \( \mathbb{C} = \mathbb{C}^b \)
- **Berwald connection** projects into Berwald and \( \mathbb{C} = 0 \)
- **Hashiguchi connection** projects into Berwald and \( \mathbb{C} = \mathbb{C}^b \)

For details see §4.4 of Jav19b
Recall that, we can define an anisotropic connection

\[ \nabla^v_X Y = (\nabla_X^\mu Y^\nu)(v), \]

and an anisotropic tensor

\[ C_v(X, Y) = (\nabla_X^v Y^\nu)(v), \]

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Let $\tilde{\nabla}$ be the Berwald connection. The Berwald tensor $B$ is the vertical derivative of $\tilde{\nabla}$ and the Landsberg curvature

$$\mathcal{L}_\nu(u, w, z) = 2g_\nu(B_\nu(u, w, z), v)$$

If we define $\mathcal{L}^b$ as $g_\nu(\mathcal{L}^b_\nu(u, w), z) = \mathcal{L}_\nu(u, w, z)$, we can write down the difference tensor between the Chern and Berwald connections as

$$\nabla^\nu_X Y - \tilde{\nabla}^\nu_X Y = \mathcal{L}^b_\nu(X, Y),$$

see (7.17) in


As a consequence if $P$ is the vertical derivative of the Chern connection: $P_\nu(v, v, u) = 0$, for every $v \in A$ and $u, w \in T_{\pi(v)}M$ (see Prop. 3.6 of Jav19b)
Berwald and Chern connections

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$$\mathcal{L}_v(u, w, z) = 2g_v(B_v(u, w, z), v)$$

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Observe that $\mathcal{L}_v^b(v, u) = 0$. It turns out that all the torsion-free connections such that
\[
\nabla_v^Y X - \hat{\nabla}_X^Y Y = Q_v(X, Y),
\]
with $Q_v(v, u) = 0$, are good connections to study Finsler Geometry.
Properties of the curvature tensor

**Proposition**

Let \((M, L)\) be a pseudo-Finsler manifold and \(\nabla\), its Levi-Civita-Chern connection. Then the curvature tensor \(R\) associated with \(\nabla\) satisfies the symmetries:

\[
g_v(R_v(u, w)z, b) + g_v(R_v(u, w)b, z) = 2C_v(R_v(w, u)v, z, b)
\]

and

\[
g_v(R_v(u, w)z, b) - g_v(R_v(z, b)u, w) = \\
C_v(R_v(w, z)v, u, b) + C_v(R_v(z, u)v, w, b) + C_v(R_v(u, b)v, z, w) \\
+ C_v(R_v(b, w)v, z, u) + C_v(R_v(z, b)v, u, w) + C_v(R_v(w, u)v, z, b).
\]

For a proof see Prop. 3.1 of Jav19b.
Let $(M, F)$ be a Finsler metric. Then there exists $r > 0$ such that

$$\exp_p : B_0^+(r) \to \exp_p(B_0^+(r))$$

is a diffeomorphism, and in this case, for any $q \in B_p^+(r)$ the radial geodesic from $p$ to $q$ is, up to reparametrizations, the unique minimizer of the Finslerian distance.
Proof:

- Given any other curve $c$ from $p$ to $q$, $c(u) = \exp_p(s(u)v(u))$, with $v: [0, 1] \to \Sigma_p$, $s(0) = 0$, $s(1) = \tilde{r}$

- $\sigma(t, u) = \exp_p(tv(u))$, $T(t, u) = \frac{\partial \sigma}{\partial t}(t, u)$,

$$U(t, u) = \frac{\partial \sigma}{\partial u}(t, u) = d\exp_p(t\dot{v}(u))$$

- $c(u) = \sigma(s(u), u)$, $\dot{c}(u) = \dot{s}(u) T(s(u), u) + U(s(u), u)$

- By the Fund. Ineq. $g_T(T, \dot{c}) \leq F(T)F(\dot{c})$, and by the Gauss Lemma $g_T(T, U) = 0$

- $F(T)F(\dot{c}) \geq g_T(T, \dot{c}) = \dot{s}(u)g_T(T, T)$ and then $\dot{s}(u) \leq F(\dot{c})$ ($F(T) = 1$)

- $\ell_F(c) = \int_0^1 F(\dot{c})du \geq \int_0^1 \dot{s}(u)du = s(1) - s(0) = \tilde{r}$. 
Proof:

- given any other curve \( c \) from \( p \) to \( q \), \( c(u) = \exp_p(s(u)v(u)) \), with \( v : [0, 1] \to \Sigma_p, s(0) = 0, s(1) = \tilde{r} \)
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\[31/45\]
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31/45
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Variation of the energy

For any piecewise smooth curve $\gamma : [a, b] \subset \mathbb{R} \to M$, let us define the energy functional as

$$E(\gamma) = \frac{1}{2} \int_{a}^{b} F(\dot{\gamma})^2 ds.$$  \hfill (1)

The second fundamental form of $\mathcal{P}$ in the direction of $N$ as the tensor

$$S^\mathcal{P}_N : \mathfrak{X}(\mathcal{P}) \times \mathfrak{X}(\mathcal{P}) \to \mathfrak{X}(\mathcal{P})^\perp_N$$

given by $S^\mathcal{P}_N(U, W) = \text{nor}_N \nabla^N_U W$, where $\text{nor}_N$ is computed with the metric $g_N$, and $\mathfrak{X}(\mathcal{P})^\perp_N$ is the space of $g_N$-orthogonal vector fields to $\mathcal{P}$.
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Second variation and Flag curvature

\[E'(0) := \frac{d(E(\gamma_s))}{ds} \bigg|_{s=0}\]

\[= - \int_a^b g_{\dot{\gamma}}(W, D_{\dot{\gamma}} \dot{\gamma}) \ dt + g_{\dot{\gamma}}(W, \dot{\gamma}) \bigg|_a^b \]

\[+ \sum_{i=1}^h \left( \mathcal{L}_L(\dot{\gamma}(t_i^+))(W(t_i)) - \mathcal{L}_L(\dot{\gamma}(t_i^-))(W(t_i)) \right),\]

where \(\mathcal{L}_L(v)(w) = g_v(v, w)\) is the Legendre transform and \(D\) the Levi-Civita connection. Moreover, if \(\gamma\) is a geodesic which is \(g_{\dot{\gamma}}\)-orthogonal to two submanifolds \(\mathcal{P}\) and \(\mathcal{Q}\) at the endpoints, then

\[E''(0) = \int_a^b \left( -g_{\dot{\gamma}}(R_{\dot{\gamma}}(\dot{\gamma}, W)W, \dot{\gamma}) + g_{\dot{\gamma}}(D_{\dot{\gamma}} W, D_{\dot{\gamma}} W) \right) \ dt \]

\[+ g_{\dot{\gamma}(b)}(S^\mathcal{P}_{\dot{\gamma}(b)}(W, W), \dot{\gamma}(b)) - g_{\dot{\gamma}(a)}(S^\mathcal{Q}_{\dot{\gamma}(a)}(W, W), \dot{\gamma}(a)),\]
Second variation and Flag curvature

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\[ = - \int_{a}^{b} g_{\dot{\gamma}}(W, D_{\dot{\gamma}} \dot{\gamma}) \ dt + g_{\dot{\gamma}}(W, \dot{\gamma})|_{a}^{b} \]

\[ + \sum_{i=1}^{h} (\mathcal{L}_L(\dot{\gamma}(t_i^+))(W(t_i)) - \mathcal{L}_L(\dot{\gamma}(t_i^-))(W(t_i))), \]

where \( \mathcal{L}_L(v)(w) = g_v(v, w) \) is the Legendre transform and \( D \) the Levi-Civita connection. Moreover, if \( \gamma \) is a geodesic which is \( g_{\dot{\gamma}} \)-orthogonal to two submanifolds \( P \) and \( Q \) at the endpoints, then

\[ E''(0) = \int_{a}^{b} \left( -g_{\dot{\gamma}}(R_{\dot{\gamma}}(\dot{\gamma}, W)W, \dot{\gamma}) + g_{\dot{\gamma}}(D_{\dot{\gamma}}^2 W, D_{\dot{\gamma}} \dot{\gamma} W) \right) \ dt \]

\[ + g_{\dot{\gamma}}(b)(S_{\dot{\gamma}}^P(W, W), \dot{\gamma}(b)) - g_{\dot{\gamma}}(a)(S_{\dot{\gamma}}^Q(W, W), \dot{\gamma}(a)), \]
The main difficulty is that the square of a Finsler metric $F^2$ is smooth in the zero section if and only if it is a Riemannian metric.

As a consequence, the energy functional is not $C^2$ in the space of curves with regularity $H^1$.

There are two possibilities: to consider the space of broken geodesics or to proceed as in the reference, providing a splitting lemma for the Finslerian energy functional.

E. Caponio, M. Á. Javaloyes, and A. Masiello.
Morse theory of causal geodesics in a stationary spacetime via Morse theory of geodesics of a Finsler metric.
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Flag Curvature

the flag curvature of $F$ is given by:

$$K(v, w) = \frac{g_v(R_v(v, w)w, v)}{F(v)^2 g_v(w, w) - g_v(v, w)^2}.$$

It measures how geodesics taking initial values in the plane $\pi = \text{span}\{v, w\}$ get apart from the geodesic $\gamma$ such that $\dot{\gamma}(0) = v$. 
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It measures how geodesics taking initial values in the plane $\pi = \text{span}\{v, w\}$ get apart from the geodesic $\gamma$ such that $\dot{\gamma}(0) = v$.
Interpretation of the Flag Curvature

On $T_x M$

$\|W\| = 1 = F(T)$

$\|tW\| = t$

On $M$, for $t \approx 0^+$

$K > 0$

$U(t)$

$\|U(t)\| < t$

$x$

$\sigma(t, 0)$

$K < 0$

$U(t)$

$\|U(t)\| > t$

$x$

$\sigma(t, 0)$

(5.5.3)

$\|U(t)\|^2 = t^2 - \frac{1}{3} K(T, W) t^4 + O(t^5)$.

See §5.5 of:

Constant flag curvature

- There is no classification of Finsler manifolds with constant flag curvature
- The only known case is the family of Randers metrics
- In this case, a Randers metrics has constant flag curvature if and only if its Zermelo data \((h, W)\) satisfies that \(h\) has constant curvature and \(W\) is a homothety. This was proved in the reference:

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- In this case, a Randers metrics has constant flag curvature if and only if its Zermelo data \((h, W)\) satisfies that \(h\) has constant curvature and \(W\) is a homothety. This was proved in the reference:

Submanifolds and orthogonal geodesics

- Geodesics $\gamma : [a, b] \to M$ (Critical points of $E$) are auto-parallel curves of the Chern connection orthogonal to $\mathcal{P}$ and $\mathcal{Q}$, namely, $g_{\gamma(a)}(\dot{\gamma}(a), u) = 0$ for all $u \in T_{\gamma(a)}\mathcal{P}$ and $g_{\gamma(b)}(\dot{\gamma}(b), w) = 0$ for all $w \in T_{\gamma(b)}\mathcal{Q}$.

- They are also critical points of the length functional.

- Do they realize locally the distance to a submanifold $\mathcal{P}$?

$$d_F(p, \mathcal{P}) = \inf_{\gamma \in C(p, \mathcal{P})} \int_a^b F(\dot{\gamma}) \, ds$$

where $C(p, \mathcal{P}) = \{ \gamma : [a, b] \to M : \gamma(a) = p, \gamma(b) \in \mathcal{P} \}$.

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- The main difficulty to generalize this result to Finsler Geometry is that the orthogonal space to \( P, \nu(P) \), is not a vector bundle (it is not linear)

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\nu(P) = \{ v \in TM : \pi(v) \in P, g(v, u) = 0 \quad \forall u \in T_{\pi(v)}P \}
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- When $\mathcal{P}$ is a hypersurface, the orthogonal space provides two vector bundles
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Existence of Tubular neighbourhoods

**Theorem**

For any precompact open subset $Q$ of $\mathcal{P}$, there exists $\varepsilon > 0$ such that the exponential map is defined in the open subset

$$\mathcal{V}_\varepsilon = \{ v \in \nu(Q) : F(v) < \varepsilon \},$$

$\exp : \mathcal{V}_\varepsilon \to \mathcal{V}_\varepsilon \setminus Q$ is a diffeomorphism for a certain open subset $\mathcal{V}_\varepsilon \subset M$

and the geodesic $\gamma_v : [0, 1] \to M$ minimizes the distance from $\mathcal{P}$ to $\gamma_v(1)$ for all $v \in \mathcal{V}_\varepsilon$.

In particular, $\exp(\mathcal{V}_\varepsilon)$ is a tubular neighbourhood of $Q$.

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For every $X \in \mathfrak{X}(M)$, we define

$$\mathcal{L}_X(L) = X(L(V)) - (\partial^\nu L)_v([X, V]),$$

where $V$ is any vector field that extends $v \in A$. Now observe that

$$(\partial^\nu L)_v(w) = 2g_v(v, w).$$

Then using the Chern connection and that $L(v) = g_v(v, v)$ we get

$$\mathcal{L}_X(L) = 2g_v(\nabla_X V, V) - 2g_v(V, [X, V]) = 2g_v(\nabla_v X, v).$$

It follows that $X$ is a Killing field of $L$ if and only if $\mathcal{L}_X L = 0$ and conformal if and only if $\mathcal{L}_X L = fL$ for some function $f : M \to \mathbb{R}$. 
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\[(L_X g)_v(Y, Z) = X(g_v(Y, Z)) - g_v([X, Y], Z) - g_v(Y, [X, Z]) - 2C_v([X, V], Y, Z)\]

since \(\partial^\nu g = 2C\). Using the Chern connection we get the tensorial expression

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which gives another characterization of Killing fields:

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Let us observe that given a diffeomorphism $\psi : M \to M$, we can define the pullback $\psi^*(T)$ of an anisotropic tensor $T$ as the anisotropic tensor given by $\psi^*(T)_v = T_{\psi^*(v)}$, where $\psi^*$ is the differential of $\psi$.

### Proposition

If $X \in \mathfrak{X}(M)$ and $T \in \mathfrak{T}^0_s(M, A)$, then

$$\mathcal{L}_X T = \lim_{t \to 0} \frac{1}{t} (\psi_t^*(T) - T),$$

where $\psi_t$ is the (possibly local) flow of $X$.

For the proof see Prop. 28 of Jav19.
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