On the disjoint structure of twisted sums

Workshop on Banach spaces and Banach lattices - ICMAT

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Exact sequences of Banach spaces

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$$0 \longrightarrow Y \xrightarrow{\ j\ } Z \xrightarrow{\ q\ } X \longrightarrow 0,$$

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The twisted sum is trivial when $j(Y)$ is complemented in Z $(Z \cong Y \oplus X\ )$

Singular twisted sums

A twisted sum

$$0 \longrightarrow Y \xrightarrow{\;j\;} Z \xrightarrow{\;q\;} X \longrightarrow 0,$$

is said to be singular if for every infinite dimensional closed subspace W of X the exact sequence

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Proposition
The twisted sum is singular $\iff$ q *is strictly singular.*

($q|_M$ is never an isomorphism for inf. dim. subspace M of X)

Quasi-linear maps

Definition
An homogeneous map $\Omega : X \longrightarrow Y$ *is quasi-linear if there exists* $C > 0$ *such that for every* $x_1, x_2 \in X$

$$\|\Omega(x_1 + x_2) - \Omega x_1 - \Omega x_2\| \leq C(\|x_1\| + \|x_2\|).$$

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Kalton - Peck [1979] Every twisted sum can be represented on this way.

Example: Kalton-Peck map

Kalton-Peck map $\Omega_p : \ell_p \curvearrowright \ell_p$, $0 < p < +\infty$, defined by

$$\Omega_p(x) = x \log \frac{|x|}{\|x\|_p}$$

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is singular:

Kalton - Peck [1979] For $1 < p < \infty$.

Castillo-Moreno [2002] For $p = 1$.

Cabello-Castillo-Suárez [2012] For $0 < p < \infty$.

Köthe function spaces

Definition

Let (S, Σ, μ) *be a complete* σ*-finite measure space.*

$L_0 = L_0(S, \Sigma, \mu)$ *locally integrable real valued functions (mod a.e.)*

A Köthe function space $\mathcal{K}$ *is a Banach space of functions in* L_0 *such that*

1. *If* $|f(\omega)| \leq g(\omega)$ *a.e. on* S *and* $g \in \mathcal{K}$*, then* $f \in \mathcal{K}$ *and* $\|f\| \leq \|g\|$*;*
2. $\chi_\sigma \in \mathcal{K}$ *for every* $\sigma \in \Sigma$ *with* $\mu(\sigma) < \infty$*.*

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Examples
Banach spaces with 1-unconditional basis

$L_p[0, 1]$ $(1 \leq p < \infty)$

Complex method of interpolation

Let $\overline{X} = (X_0, X_1)$ be a compatible pair of Köthe function spaces

$\mathcal{F} = \mathcal{F}(X_0, X_1)$ the space of analytic functions on $S = \{z \in \mathbb{C} \,:\, 0 < \Re(z) < 1\}$ Such that

1. $f(j + it) \in X_j$ for every $t \in \mathbb{R}$ and $j = 0, 1$.
2. $t \mapsto f(j + it) \in X_j$ is continuous and bounded $(j = 0, 1)$

$$\|f\| = \max\left\{\sup_{t\in\mathbb{R}} \|f(it)\|_{X_0}\,,\, \sup_{t\in\mathbb{R}} \|f(1+it)\|_{X_1}\right\}$$

For $0 < \theta < 1$, the complex interpolation space X_θ is defined as

$$X_\theta = \{f(\theta) \,:\, f \in \mathcal{F}\}$$

$$\|x\|_{X_\theta} = \inf\{\|f\|_{\mathcal{F}} \,:\, f \in \mathcal{F},\, f(\theta) = x\}$$

X_θ is identified isometrically with the quotient space

$$X_\theta = \mathcal{F}/\{f \in \mathcal{F} \,:\, f(\theta) = 0\}$$

Derived space

Definition. An L_∞-centralizer (resp. an ℓ_∞-centralizer) on a Köthe function (resp. sequence) space $\mathcal{K}$ is a homogeneous map $\Omega : \mathcal{K} \to L_0$ such that there is a constant C such that, for every $f \in L_\infty$ (resp. ℓ_∞) and for every $x \in \mathcal{K}$.

1.) $\Omega(fx) - f\Omega(x) \in \mathcal{K}$,

2.) $\|\Omega(fx) - f\Omega(x)\|_{\mathcal{K}} \leq C\|f\|_\infty \|x\|_{\mathcal{K}}$.

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Kalton [1992] Every centralizer induce an exact sequence

$$0 \longrightarrow \mathcal{K} \stackrel{\jmath}{\longrightarrow} d_\Omega\mathcal{K} \stackrel{q}{\longrightarrow} \mathcal{K} \longrightarrow 0$$

where $d_\Omega\mathcal{K} = \{(w, x) : w \in L_0, x \in \mathcal{K} : w - \Omega x \in \mathcal{K}\}$ endowed with the quasi-norm

$$\|(w, x)\|_{d_\Omega\mathcal{K}} = \|x\|_{\mathcal{K}} + \|w - \Omega x\|_{\mathcal{K}}$$

Derived space

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Examples

- *The Kalton-Peck spaces can be obtained as derived spaces:* $\ell_p = (\ell_\infty, \ell_1)_\theta$, *with* $p = 1/\theta$
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  $\ell_p = (\ell_\infty, \ell_1)_\theta$, *with* $p = 1/\theta$
  $\Omega_\theta = \alpha \Omega_p$
- *Kalton-Peck functions version*
  $L_p = (L_\infty, L_1)_\theta$, *with* $p = 1/\theta$
  $\Omega_\theta(f) = f \log \left(\frac{|f|}{\|f\|_p} \right)$

Singularity and centralizers

Examples

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Proposition (CCFM)
There is no singular centralizer on admissible superreflexive Köthe space.

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Proposition (CCFM)
There is no singular centralizer on admissible superreflexive Köthe space.

Definition. A Köthe space $\mathcal{K}$ is admissible when for some strictly positive functions $h, k \in L_0$ one has

$$\|hk\|_1 \leq \|x\|_{\mathcal{K}} \leq \|kx\|_\infty \quad \textit{for every} \quad x \in \mathcal{K}$$

Disjoint singularity

Let $\mathcal{K}$ be a Köthe space and $\Omega : \mathcal{K} \to Y$ be a quasi-linear map

$$0 \longrightarrow Y \xrightarrow{\ i\ } Y \oplus_\Omega \mathcal{K} \xrightarrow{\ q\ } \mathcal{K} \longrightarrow 0$$

Ω is called disjointly singular if for every infinite dimensional subspace generated by a disjointly supported sequence W of $\mathcal{K}$ the exact sequence

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is nontrivial.

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- singular $\implies$ disjointly singular.
- Castillo, Ferenczi, González [2017] Let X be a Banach space with unconditional basis. A quasi-linear map $\Omega : X \to Y$ is singular $\iff$ is disjointly singular (with respect to the induced lattice structure.)

Criteria for disjoint singularity

In Castillo, Ferenczi, González [2017], criteria for disjointly singular centralizers are introduced.

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Definition. Let $\mathcal{K}$ be a Köthe function space. For each $n \in \mathbb{N}$ let $M_{\mathcal{K}}(n) = \sup\{\|x_1 + \ldots x_n\| : \|x_i\| \leq 1,\ (x_i) \text{ disjoint in } \mathcal{K}\}$

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Definition. $f \sim g$ if $\liminf \frac{f(n)}{g(n)} \leq \limsup \frac{f(n)}{g(n)} < \infty$.

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Theorem (Castillo, Ferenczi, González - 2017)
Let(X_0, X_1) be an admissible pair of Köthe function spaces and $0 < \theta < 1$. Suppose that X_θ is reflexive and

1) $M_{X_0} \not\sim M_{X_1}$;

2) $M_{X_0}^{1-\theta} M_{X_1}^{\theta} \sim M_X$:

3) $M_W \sim M_{X_\theta}$ for every $W \subset X_\theta$ generated by a disjoint sequence

Then Ω_θ is disjointly singular.

Examples

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2. F. Cabello [2017] Lorentz spaces $(L_{p_0,q_0}, L_{p_1,q_1})_\theta = L_{p,q}$ with associated derivation

$$\Omega(x) = \alpha\mathcal{K}(x) + \beta\kappa(x)$$

Here $\mathcal{K}(\cdot)$ is the Kalton-Peck map and $\kappa(x) = x\ r_x$ where $r_x(t) = m\{s : |x(s)| > |x(t)|$ or $|x(s)| = |x(t)|$ and $s \leq t\}$

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3. (CCFM) Disjointly singular quasi-linear maps on $C[0,1]$ and ℓ_∞

Characterizations of disjoint singularity

Definition
Let $\Omega : \mathcal{K} \curvearrowright \mathcal{K}$ *be a centralizer. A pair of nonzero elements* $f = (w_0, x_0), g = (w_1, x_1)$ *of* $d_\Omega\mathcal{K}$ *are said to be* *disjoint* *if the functions* $f, g : S \to \mathbb{C} \times \mathbb{C}$ *are disjointly supported.*

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An operator $\tau : d_\Omega\mathcal{K} \to \mathcal{K}$ *is said to be disjointly singular if the restriction of* τ *to any infinite dimensional subspace generated by a disjoint sequence of vectors is not an isomorphism.*

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Theorem (CCFM)
A centralizer Ω on a reflexive Köthe space $\mathcal{K}$ is disjointly singular $\iff$ q_Ω is disjointly singular.

Characterization of disjoint singularity for L_p

Proposition (CCFM)
A centralizer Ω *defined on* L_p *is not disjointly singular if and only if there is a disjointly supported normalized sequence* $u = (u_n)_n$ *and a constant* $C > 0$ *such that for every* $\lambda = (\lambda_k)_k \in c_{00}$ *and every* $n \in \mathbb{N}$ *one has*

$$\mathrm{Ave}_{\epsilon} \left\| \Omega \left(\sum_{k=1}^{n} \epsilon_k \lambda_k u_k \right) - \sum_{k=1}^{n} \epsilon_k \Omega(\lambda_k u_k) \right\| \leq C \|\lambda\|_p.$$

Super singularity

Definition. An operator $\tau : X \to Y$ between two Banach spaces is said to be super-SS if there does not exists $c > 0$ and a sequence of subspaces E_n of X, with $\dim E_n = n$, such that

$$\|Tx\| \geq c\|x\| \quad \text{for all } x \in \bigcup_n E_n$$

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Ω is super-singular if and only the quotient map q_Ω of the exact sequence it defines is super strictly singular.

Proposition (CCFM)
No super singular quasi-linear maps between B-convex Banach spaces exist.

Disjoint super singularity

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Proposition (CCFM)
Let $\Omega : \mathcal{K} \curvearrowright \mathcal{K}$ be a centralizer on a Köthe space. The following are equivalent

1. *All ultrapowers $\Omega_{\mathcal{U}}$ of Ω are disjointly singular.*
2. *The quotient map q_Ω is super-disjointly singular.*

When 1. and 2. hold Ω is said to be super-disjointly singular.

Disjoint super singularity

Proposition (CCFM)
Let (X_0, X_1) be an interpolation couple of Köthe function spaces and let $0 < \theta < 1$ so that X_θ is an $L_p(\mu)$-space. If

1) $M_{X_0} \not\sim M_{X_1}$;

2) $M_{X_0}^{1-\theta} M_{X_1}^{\theta} \sim n^{1/p}$.

Then the induced centralizer Ω_θ on X_θ is super disjointly singular

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Examples

- *The Kalton-Peck centralizer on L_p is super disjointly singular for $1 < p < \infty$.*
- *If $\mathcal{S}$ denotes the Schreier space then $(\mathcal{S}, \mathcal{S}^*)_{1/2} = \ell_2$ then the associated centralizer is super disjointly singular*

Examples

- Lorentz function spaces. $(L_{p_0,p_1}, L_{p_1,q_1})_\theta = L_{p,q}$ the associated centralizer super disjointly singular when $\min\{p_0, p_1\} \neq \min\{p_1, q_1\}$.

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Remark
super-disjointly singular $\implies$ *disjointly singular*

Let $1 \leq p_1 < p_0 < 2$ *and* $0 < \theta < 1$. *For* $p^{-1} = (1-\theta)p_1^{-1} + \theta p_0^{-1}$ *one has* $(\ell_{p_0}(\bigoplus \ell_2^n), \ell_{p_1}(\bigoplus \ell_2^n))_\theta = \ell_p(\bigoplus \ell_2^k)$ *with associated centralizer*

$$\Omega_\theta(x) = \left(\left(\frac{p}{p_1} - \frac{p}{p_0}\right)\log\left(\frac{\|x^k\|_2}{\|x\|}\right)x^k\right)_k.$$

Then Ω_θ *is disjointly singular but is not super- disjointly singular.*

The end

Thank you for your attention!