Norm attaining operators of finite rank

(joint work with Vladimir Kadets, Ginés López, Miguel Martín)

Dirk Werner
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$$\|x\| = \|x^*\| = 1,\ x^*(x) \ge 1 - \varepsilon \quad \Rightarrow$$

$$\exists x_0, x_0^*\colon \quad \|x_0\| = \|x_0^*\| = x_0^*(x_0) = 1,\ \|x - x_0\| \le 2\sqrt{\varepsilon},\ \|x^* - x_0^*\| \le 2\sqrt{\varepsilon}.$$

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- Example: $\mathrm{NA}(c_0) = c_{00} =$ all ℓ_1-sequences of finite support.

On operators which attain their norm

A linear operator $T\colon X \to Y$ is norm attaining ($T \in \mathrm{NA}(X,Y)$):

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Gowers 1990:

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- $X = L_1[0,1]$, $Tx = (\int_0^{1/2} x(t)\,dt, \int_{1/2}^1 x(t)\,dt)$:

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Let $\|f\| = 1$.

(a) If, for some $0 \ne h \in B_{X^*}$,

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(c) There exists some f with a mate.

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Question

Does the Corollary cover all Banach spaces?

The theorems of Read and Rmoutil

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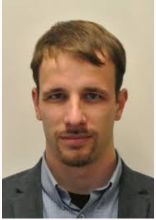

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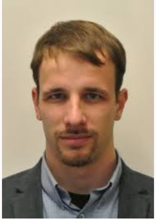
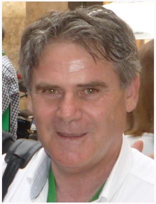

Charles Read (1958–2015), Martin Rmoutil, Bernardo Cascales (1958–2018)

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Main Theorem

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- Is the assumption of the Main Theorem always fulfilled?

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Corollary

In the setting of the Theorem, if X^* has the AP, then every compact operator with domain X can be approximated by finite rank norm attaining operators.