On minimal and unbounded topologies

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Motivation

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Theorem

Let X be a Banach lattice. Then the norm topology is the finest topology on X that respects the lattice structure.

What do I mean by "respect the lattice structure"

A subset V of a vector lattice X is called **solid** if $v \in V$ and $|x| \leq |v|$ implies $x \in V$.

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A linear topology is locally solid iff lattice operations are uniformly continuous.

Or equivalently if lattice operations are continuous and from $0 \leq x_\alpha \leq y_\alpha \xrightarrow{\tau} 0$ we can conclude that $x_\alpha \xrightarrow{\tau} 0$.

How to "weaken a topology"

For each solid set $V \subseteq X$ and each $a \geq 0$ define $V_a := \{x \in X : |x| \wedge a \in V\}$. It is easy to see that V_a is also solid and $V \subseteq V_a$.

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Let A be an ideal of a vector lattice X, and τ a locally solid topology on X.

Since τ is a locally solid topology, it has a base, $\mathcal{N}_0$, at zero consisting of solid sets. The collection $\{V_a : V \in \mathcal{N}_0, a \in A_+\}$ defines a locally solid topology, $u_A\tau$, on X.

Basic properties

- $u_A\tau \subseteq \tau$, and if B is another ideal then $u_A(u_B\tau) = u_B(u_A\tau) = u_{A\cap B}\tau$.

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- $x_\alpha \xrightarrow{u_A\tau} x$ iff $|x_\alpha - x| \wedge a \xrightarrow{\tau} 0$ for all $a \in A_+$.
- $u_A\tau$ is Hausdorff iff τ is Hausdorff and A is order dense in X.
- We define $u\tau := u_X\tau$. The map $\tau \mapsto u\tau$ from the set of locally solid topologies on X to itself is idempotent.

Definition

A locally solid topology is **unbounded** if $\tau = u\tau$ or, equivalently, if $\tau = u\sigma$ for some locally solid topology σ.

Examples of unbounded topologies include $u_A\tau$ for every ideal A and locally solid topology τ.

Unbounded order convergence

Since order convergence is topological iff X is finite dimensional, we must separately define unbounded order convergence.

We say that $x_\alpha \xrightarrow{uo} x$ iff $|x_\alpha - x| \wedge a \xrightarrow{o} 0$ for each $a \in X_+$.

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uo-convergence, although not part of the theory of unbounded topologies, plays a key role.

Why don't we test *uo* against ideals?

Define $x_\alpha \xrightarrow{u_A o} x$ if $|x_\alpha - x| \wedge a \xrightarrow{o} 0$ for each $a \in A_+$.

Theorem (Li, Chen)

- *If A is order dense then $u_A o = uo$;*
- *If A is not order dense then $u_A o$ fails to have unique limits.*

The situation is very different for topologies since there is a "gap" between an ideal being order dense and topologically dense.

Minimal topologies

Definition

A Hausdorff locally solid topology τ is said to be **minimal** if it follows from $\sigma \subseteq \tau$ and σ Hausdorff and locally solid that $\sigma = \tau$.

Minimal topologies have been studied by Aliprantis, Burkinshaw, Labuda, and Conradie ('78-'05).

The connection with unbounded convergences

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4. $\tau = u_A\tau$ *for every order dense ideal* $A \subseteq X$*.*

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Moreover, minimal topologies are **unique***, and in spaces of measurable functions the minimal topology is convergence in measure.*

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$C[0, 1]$ admits no minimal topology.

Properties of minimal topologies

Theorem (Conradie, Taylor, Kandic)

Let τ be a minimal topology:

- *τ is locally convex iff X is atomic. In general, $0 \neq f \in (X, \tau)^*$ implies f is a linear combination of coordinate functionals of atoms;*

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- *τ is complete iff τ has MCP iff τ is monotonically complete iff X is universally complete;*

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- τ *is locally bounded iff* X *is finite dimensional.*
- *etc...*

Another look at the characterization of minimal topologies

Theorem

Let (X, τ) be a Hausdorff locally solid vector lattice. TFAE:

1. τ *is minimal;*
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5. τ *extends to a locally solid topology on* X^u.

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5. *τ extends to a locally solid topology on X^u.*

The universal completion satisfies two universal properties: It is the **smallest** universally complete space containing X, but it is also the **largest** order dense extension of X. It is therefore rare that topologies extend to X^u, but minimal topologies satisfy $\widehat{(X, \tau)} = X^u$.

Sequential aspects

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The universal σ-completion X^s is unique, and exists iff X is almost σ-order complete. In this case it is the largest sequentially order dense extension of X.

Theorem

Let X be an almost σ-order complete vector lattice and τ a Hausdorff locally solid topology on X. TFAE:

1. τ *extends to a (Hausdorff) locally solid topology on* X^s;
2. τ *is* σ*-universal.*

Moreover, the extension specified in (i) is unique.

There is no problem with measurable cardinals here (but there will be later).

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σ-universal topologies are not unique! The space $\mathcal{L}_0[0,1]$ is universally σ-complete and admits infinitely many Hausdorff locally solid topologies.

The coarsest topology on $\mathcal{L}_0[0,1]$ is pointwise convergence, but one can also consider the topology of simultaneous pointwise convergence and convergence in measure.

Further results

Theorem

Let τ be a Hausdorff locally solid topology on an almost σ-order complete vector lattice X. TFAE:

1. *τ is σ-universal;*
2. *τ is σ-Fatou and disjoint sequences are τ-null;*
3. *τ is σ-Fatou and disjoint sequences are τ-bounded.*

Theorem

The restriction of a minimal topology to a sublattice is minimal iff the sublattice is regular.

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It is open whether every σ-universal topology is unbounded.

Roughly speaking, given a property P, the characterization of when σ-universal topologies have P is either the same as the characterization of when minimal topologies have P, or measurable cardinals cause some wild behaviour to occur.

Thank you