A new weak Hilbert space

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Introduction

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Theorem (Lindenstrauss-Tzafriri)

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The notions of type and cotype

- X has **type 2** if $a_2(X) = \sup_{n\in\mathbb{N}} a_{2,n}(X) < \infty$

$$\mathit{Average}_{\pm} \left\| \sum_{j=1}^{n} \pm x_j \right\| \leq a_{2,n}(X) \cdot \left(\sum_{j=1}^{n} \|x_j\|^2 \right)^{1/2}.$$

- X has **cotype 2** if $c_2(X) = \sup_{n\in\mathbb{N}} c_{2,n}(X) < \infty$

$$\left(\sum_{j=1}^{n} \|x_j\|^2 \right)^{1/2} \leq c_{2,n}(X) \cdot \mathit{Average}_{\pm} \left\| \sum_{j=1}^{n} \pm x_j \right\|.$$

- ℓ_p has type $\min\{p, 2\}$ and cotype $\max\{p, 2\}$.

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- **Weak cotype 2** for X: given $0 < \delta < 1$, every n-dimensional subspace of X contains an $(\delta \cdot n)$-dimensional subspace, say F, that is $C(\delta)$-isomorphic to Hilbert.
- **Weak type 2** for X: There is a projection $P : X \to F$ with $\|P\| \leq C(\delta)$.

Definition (Pisier)

X is a weak Hilbert space if it is both X weak type 2 and weak cotype 2.

A weak Hilbert space that is a twisted Hilbert space $Z(T^2)$

- T^2 is the prototype of a weak Hilbert space.
- A twisted Hilbert space is a Banach space Z containing a copy of ℓ_2 such that $Z/\ell_2 \approx \ell_2$
- Examples of twisted Hilbert spaces: Enflo-Lindenstrauss-Pisier space, Kalton-Peck space.
- $Z(T^2) = \ell_2 \oplus \ell_2$ with a quasi-norm

$$\|(x, y)\| = \|x - \Omega_{T^2}(y)\| + \|y\|.$$

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- How to get such Ω_{T^2}? And why this gives a weak Hilbert space?

Coming back to Kalton once more

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- For $X = T^2$ then Ω_{T^2} is...I have no clue! So then?

A key step: Castillo, Ferenczi and González

- There is information on X and X^* that is reflected into Ω_X even if you do not know the precise form of such Ω_X.
- What information?

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- There is information on X and X^* that is reflected into Ω_X even if you do not know the precise form of such Ω_X.
- What information? For example, the norm of n normalized and disjoint blocks.
- $U_n(X) = \sup\{\| \sum_{j=1}^n u_j\| : u_1 < ... < u_n\}$ and similarly for $U_n(X^*)$.
- $$\left\| \Omega_X(\sum_{j=1}^n u_j) - \sum_{j=1}^n \Omega_X(u_j) - \log \frac{U_n(X)}{U_n(X^*)} \sum_{j=1}^n u_j \right\| \leq 6 \cdot \sqrt{U_n(X) \cdot U_n(X^*)}.$$

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A random view

- We are interested in $a_{2,n}(X), a_{2,n}(X^*)$.
- Our random version of the inequality using an idea of Corrêa.

$$Average_{\pm} \left\| \Omega_X(\sum_{j=1}^{n} \pm x_j) - \sum_{j=1}^{n} \pm\Omega_X(x_j) - \log \frac{a_{2,n}(X)}{a_{2,n}(X^*)} \sum_{j=1}^{n} \pm x_j \right\| \leq \gamma \cdot \left(\sum_{j=1}^{n} \|x_j\|^2 \right)^{1/2}.$$

- γ depends only of $a_{2,n}(X), a_{2,n}(X^*)$.

Conclusion for our twisted Hilbert $Z(T^2)$

- Our random version of the inequality gives that:

$$a_{2,n}(Z(T^2)) \leq C \cdot \max\{a_{2,n}(T^2), a_{2,n}((T^2)^*)\} \to \infty.$$

- In particular, $a_{2,n}(Z(T^2)) \to \infty$ very slowly.
- Also, $c_{2,n}(Z(T^2)) \to \infty$ very slowly (by simple duality).

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- The n-dimensional subspaces of $Z(T^2)$ are $a_{2,n}(Z(T^2)) \cdot c_{2,n}(Z(T^2))$-isomorphic to Hilbert.

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  - The n-dimensional subspaces of $Z(T^2)$ are $a_{2,n}(Z(T^2)) \cdot c_{2,n}(Z(T^2))$-isomorphic to Hilbert.
- We replace $Z(T^2)$ for certain n-codimensional subspaces V_n. The same argument shows that:
  - The $5^{(5^n)}$-dimensional subspaces of V_n ARE HILBERTIAN!!
- Then $Z(T^2)$ is a weak Hilbert space by a result of Johnson.

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- $Z(T^2)$ is no isomorphic to a subspace or a quotient of the Kalton-Peck space or the E-L-P space.
- Neither the Kalton-Peck space nor the E-L-P space is isomorphic to a subspace or a quotient of $Z(T^2)$.

Some answers

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  Does weak type 2 implies Maurey extension property? No.

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- $Z(T^2)$ answers a question of Cuellar: Do there exists nontrivial twisted Hilbert spaces that are weak Hilbert spaces? Yes.
- $Z(T^2)$ solves the "weak type 2 converse" of Milman-Pisier (see Casazza-Shura, Castillo-Plichko,...):
  Does weak type 2 implies Maurey extension property? No.
- $Z(T^2)$ satisfies the J-L lemma and thus it also answers a question of Johnson-Naor:
  It was not known if weak Hilbert spaces with no unconditional basis may satisfy the J-L lemma.

References

1. J. Suárez, A weak Hilbert space that is a twisted Hilbert space. *J. Inst. Math. Jussieu* (to appear).
2. J. Suárez, A space with no unconditional basis that satisfies the J-L lemma. *Results in Mathematics*, 74 (2019), no. 3, Art. 126.