A z^k-invariant subspace without the wandering property

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Spaces over the disc

Definition

Dirichlet-type space, D_α, is:

$$\{f \in Hol(\mathbb{D}) : f(z) = \sum_{k\in\mathbb{N}} a_k z^k, ||f||_\alpha^2 = \sum_{k=0}^{\infty} |a_k|^2 (k+1)^\alpha < \infty\}$$

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- Hilbert spaces with monomials as an orthogonal basis

Invariant subspaces

- The (forward) *shift operator* is bdd:

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Theorem (Aleman, Richter, Sundberg, '96)

For A^2 ($\alpha = -1$), M z-inv. $\Leftrightarrow$
$$[M \ominus SM] = M.$$

Wandering subspaces

- Shimorin ('11) extended ARS'96 to $\alpha \in [-1, 1]$ (different ideas for $\alpha < 0$ and $\alpha > 0$). Hedenmalm-Zhu ('92) and Nowak et al. ('17) showed that the analogous fails for $\alpha \leq -5$ in some sense.

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This is the problem we study (but do not solve) today.

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- Today focus: $\alpha = -16$, z^6 wandering fails.

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- a_4, b_5 different role: $(2z^6 + 5)F_1(z) + (9z^6 + 8)F_2(z) =$

$$= (c_0, ..., c_3, 5a_4, 8b_5, c_6, ..., c_9, 2a_4, 9b_5, c_{12}, ..., c_{15}, 0, 0, c_{18}, ...)$$

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- $\Rightarrow$ unique expression $\Rightarrow$ "Fourier analysis"
- $\Rightarrow M = \{f_1(z^6)F_1(z) + f_2(z^6)F_2(z) : f_1, f_2 \in D_\alpha\}$

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$$f \in M \ominus z^6 M \Leftrightarrow \langle f, z^{6s} F_j \rangle = 0 \ (s \geq 1, j = 1, 2).$$

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Lemma

$f \in M$. Then $f \perp z^6 M \Leftrightarrow \forall s \geq 1$ both

$$0 = \hat{f}_1(s+1)\overline{A_{s+1,1}} + \hat{f}_2(s+1)\overline{A_{s+1,5}} + \hat{f}_1(s) A_{s,3} + \hat{f}_2(s)\overline{A_{s,2}} + \hat{f}_1(s-1) A_{s,1}$$

and

$$0 = \hat{f}_1(s) A_{s,2} + \hat{f}_2(s) A_{s,4} + \hat{f}_1(s-1) A_{s,5}.$$

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- If (*previous*) $+ A_{1,1} = 0$, then $M \perp z^6 M$ spanned by F_2 and F_4 where

$$F_4(z) = F_1(z)(1 - \frac{z^6 A_{1,5}\overline{A_{1,2}}}{|A_{1,2}|^2 - A_{1,3}A_{1,4}}) = (1 + z^6/c)F_1(z).$$

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- Notice then $F_1 \in [M \ominus z^6 M] \Leftrightarrow 1 + z/c$ cyclic $\Leftrightarrow c \geq 1$.
- Optimization problem on 12 variables, with 5 restrictions to show $c < 1$.

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$$N \begin{pmatrix} a_0\overline{a_6} \\ a_1\overline{a_7} \\ a_2\overline{a_8} \\ a_3\overline{a_9} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad N \begin{pmatrix} b_0\overline{a_6} \\ b_1\overline{a_7} \\ b_2\overline{a_8} \\ b_3\overline{a_9} \end{pmatrix} = \begin{pmatrix} \overline{A_{1,5}} \\ 0 \\ 0 \end{pmatrix},$$

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and N is the 3×4 matrix given by

$$N = \begin{pmatrix} \omega_6 & \omega_7 & \omega_8 & \omega_9 \\ \omega_{12} & \omega_{13} & \omega_{14} & \omega_{15} \\ \omega_{18} & \omega_{19} & \omega_{20} & \omega_{21} \end{pmatrix} \quad N_0 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ \omega_6 & \omega_7 & \omega_8 & \omega_9 \\ \omega_{12} & \omega_{13} & \omega_{14} & \omega_{15} \\ \omega_{18} & \omega_{19} & \omega_{20} & \omega_{21} \end{pmatrix}$$

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$$C_2 = 1 + \sum_{i=1}^{3} \frac{E_i^2 \omega_{2k+i}}{d_i}, \quad C_4 = \sum_{i=1}^{3} \frac{G_i^2 \omega_{k+i} d_i}{E_i^2},$$

Further and further reductions...

- Restrictions give $a_6, ..., a_9$ and $b_0, ..., b_3$ in terms of $a_0, ..., a_3$ and 3 mute variables.
- Objective function, homogeneous $\Rightarrow a_0 = 1$.
- Classical calculus techniques and "good luck" reduce from 6 complex to 3 real positive variables: Find $d_1, d_2, d_3 \in \mathbb{R}^+$ such that:

$$4C_2C_4 < 1,$$

where

$$C_2 = 1 + \sum_{i=1}^{3} \frac{E_i^2 \omega_{2k+i}}{d_i}, \quad C_4 = \sum_{i=1}^{3} \frac{G_i^2 \omega_{k+i} d_i}{E_i^2},$$

$$N_0^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ E_1 & G_1 & ... & ... \\ E_2 & G_2 & ... & ... \\ E_3 & G_3 & ... & ... \end{pmatrix}.$$

The solution to the problem

- For us, this becomes something like d_1, d_2, d_3:

$$4 \cdot 10^{-6} \cdot (1 + \frac{82}{d_1} + \frac{440}{d_2} + \frac{194}{d_3}) \cdot (14d_1 + 4d_2 + d_3) < 1.$$

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Remark

Changing those 12 values in the adequate place of the sequence ω will give an equiv. norm in any D_α with the same result for any $k \geq 6$.


END OF TRAIL