New constructions of closed ideals in $L(L_p)$, $1 \leq p \neq 2 < \infty$

Gideon Schechtman

Madrid September 2019

Based on two papers

the first joint with Bill Johnson and Gilles Pisier

the second joint with Bill Johnson

New constructions of closed ideals in $L(L_p)$, $1 \leq p \neq 2 < \infty$

Gideon Schechtman

Madrid September 2019

Based on two papers

the first joint with Bill Johnson and Gilles Pisier

the second joint with Bill Johnson

$L(X)$ is the Banach algebra of bounded linear operators on the Banach space X.

A closed ideal in $L(X)$ is a closed subspace $\mathcal{I}$ of $L(X)$ such that for all $T \in L(X)$ and $S \in \mathcal{I}$, TS and ST are in $\mathcal{I}$.

There are some classical closed ideals in $L(X)$. As long as X has the approximation property, $K(X)$ the set of compact operators is the smallest one. Another is $W(X)$, the set of weakly compact operators; operators T that map the unit ball into a weakly compact set. So $W(X) = L(X)$ iff X is reflexive. An especially important closed ideal is $S(X)$, the space of strictly singular operators on X. An operator T is strictly singular if it is not an into isomorphism when restricted to any infinite dimensional subspace.

Ideals in $L(X)$

$L(X)$ is the Banach algebra of bounded linear operators on the Banach space X.

A closed ideal in $L(X)$ is a closed subspace $\mathcal{I}$ of $L(X)$ such that for all $T \in L(X)$ and $S \in \mathcal{I}$, TS and ST are in $\mathcal{I}$.

There are some classical closed ideals in $L(X)$. As long as X has the approximation property, $K(X)$ the set of compact operators is the smallest one. Another is $W(X)$, the set of weakly compact operators; operators T that map the unit ball into a weakly compact set. So $W(X) = L(X)$ iff X is reflexive. An especially important closed ideal is $S(X)$, the space of strictly singular operators on X. An operator T is strictly singular if it is not an into isomorphism when restricted to any infinite dimensional subspace.

Ideals in $L(X)$

$L(X)$ is the Banach algebra of bounded linear operators on the Banach space X.

A closed ideal in $L(X)$ is a closed subspace $\mathcal{I}$ of $L(X)$ such that for all $T \in L(X)$ and $S \in \mathcal{I}$, TS and ST are in $\mathcal{I}$.

There are some classical closed ideals in $L(X)$. As long as X has the approximation property, $K(X)$ the set of compact operators is the smallest one. Another is $W(X)$, the set of weakly compact operators; operators T that map the unit ball into a weakly compact set. So $W(X) = L(X)$ iff X is reflexive. An especially important closed ideal is $S(X)$, the space of strictly singular operators on X. An operator T is strictly singular if it is not an into isomorphism when restricted to any infinite dimensional subspace.

Ideals in $L(X)$

$L(X)$ is the Banach algebra of bounded linear operators on the Banach space X.

A closed ideal in $L(X)$ is a closed subspace $\mathcal{I}$ of $L(X)$ such that for all $T \in L(X)$ and $S \in \mathcal{I}$, TS and ST are in $\mathcal{I}$.

There are some classical closed ideals in $L(X)$. As long as X has the approximation property, $K(X)$ the set of compact operators is the smallest one. Another is $W(X)$, the set of weakly compact operators; operators T that map the unit ball into a weakly compact set. So $W(X) = L(X)$ iff X is reflexive. An especially important closed ideal is $S(X)$, the space of strictly singular operators on X. An operator T is strictly singular if it is not an into isomorphism when restricted to any infinite dimensional subspace.

Ideals in $L(X)$

$L(X)$ is the Banach algebra of bounded linear operators on the Banach space X.

A closed ideal in $L(X)$ is a closed subspace $\mathcal{I}$ of $L(X)$ such that for all $T \in L(X)$ and $S \in \mathcal{I}$, TS and ST are in $\mathcal{I}$.

There are some classical closed ideals in $L(X)$. As long as X has the approximation property, $K(X)$ the set of compact operators is the smallest one. Another is $W(X)$, the set of weakly compact operators; operators T that map the unit ball into a weakly compact set. So $W(X) = L(X)$ iff X is reflexive. An especially important closed ideal is $S(X)$, the space of strictly singular operators on X. An operator T is strictly singular if it is not an into isomorphism when restricted to any infinite dimensional subspace.

Ideals in $L(X)$

A maximal algebraic ideal is automatically closed since the invertible elements in a Banach algebra form an open set, so every (always proper) closed ideal is contained in a closed maximal ideal. What are the maximal ones? Is there even a largest ideal?

Let $\mathcal{M}(X)$ denote all operators T on X s.t. the identity operator I_X does not factor through T ($I_X \neq BTA$). It is obvious that $\mathcal{M}(X)$ is an ideal in $L(X)$ if it is closed under addition, in which case it clearly is the largest ideal in $L(X)$. It is known, but non trivial, that $\mathcal{M}(L_p)$ is closed under addition, and also that $\mathcal{M}(L_p)$ is the set of L_p-singular operators, that is the set of operators that are not an isomorphism when restricted to any subspace isomorphic to L_p. [Enflo, Starbird '79] for $p = 1$; [Johnson, Maurey, S, Tzafriri '79] for $1 < p \neq 2 < \infty$.

A maximal algebraic ideal is automatically closed since the invertible elements in a Banach algebra form an open set, so every (always proper) closed ideal is contained in a closed maximal ideal. What are the maximal ones? Is there even a largest ideal?

Let $\mathcal{M}(X)$ denote all operators T on X s.t. the identity operator I_X does not factor through T ($I_X \neq BTA$). It is obvious that $\mathcal{M}(X)$ is an ideal in $L(X)$ if it is closed under addition, in which case it clearly is the largest ideal in $L(X)$. It is known, but non trivial, that $\mathcal{M}(L_p)$ is closed under addition, and also that $\mathcal{M}(L_p)$ is the set of L_p-singular operators, that is the set of operators that are not an isomorphism when restricted to any subspace isomorphic to L_p. [Enflo, Starbird '79] for $p = 1$; [Johnson, Maurey, S, Tzafriri '79] for $1 < p \neq 2 < \infty$.

A maximal algebraic ideal is automatically closed since the invertible elements in a Banach algebra form an open set, so every (always proper) closed ideal is contained in a closed maximal ideal. What are the maximal ones? Is there even a largest ideal?

Let $\mathcal{M}(X)$ denote all operators T on X s.t. the identity operator I_X does not factor through T ($I_X \neq BTA$). It is obvious that $\mathcal{M}(X)$ is an ideal in $L(X)$ if it is closed under addition, in which case it clearly is the largest ideal in $L(X)$. It is known, but non trivial, that $\mathcal{M}(L_p)$ is closed under addition, and also that $\mathcal{M}(L_p)$ is the set of L_p-singular operators, that is the set of operators that are not an isomorphism when restricted to any subspace isomorphic to L_p. [Enflo, Starbird '79] for $p = 1$; [Johnson, Maurey, S, Tzafriri '79] for $1 < p \neq 2 < \infty$.

Ideals in $L(X)$

A maximal algebraic ideal is automatically closed since the invertible elements in a Banach algebra form an open set, so every (always proper) closed ideal is contained in a closed maximal ideal. What are the maximal ones? Is there even a largest ideal?

Let $\mathcal{M}(X)$ denote all operators T on X s.t. the identity operator I_X does not factor through T ($I_X \neq BTA$). It is obvious that $\mathcal{M}(X)$ is an ideal in $L(X)$ if it is closed under addition, in which case it clearly is the largest ideal in $L(X)$. It is known, but non trivial, that $\mathcal{M}(L_p)$ is closed under addition, and also that $\mathcal{M}(L_p)$ is the set of L_p-singular operators, that is the set of operators that are not an isomorphism when restricted to any subspace isomorphic to L_p. [Enflo, Starbird '79] for $p = 1$; [Johnson, Maurey, S, Tzafriri '79] for $1 < p \neq 2 < \infty$.

A maximal algebraic ideal is automatically closed since the invertible elements in a Banach algebra form an open set, so every (always proper) closed ideal is contained in a closed maximal ideal. What are the maximal ones? Is there even a largest ideal?

Let $\mathcal{M}(X)$ denote all operators T on X s.t. the identity operator I_X does not factor through T ($I_X \neq BTA$). It is obvious that $\mathcal{M}(X)$ is an ideal in $L(X)$ if it is closed under addition, in which case it clearly is the largest ideal in $L(X)$. It is known, but non trivial, that $\mathcal{M}(L_p)$ is closed under addition, and also that $\mathcal{M}(L_p)$ is the set of L_p-singular operators, that is the set of operators that are not an isomorphism when restricted to any subspace isomorphic to L_p. [Enflo, Starbird '79] for $p = 1$; [Johnson, Maurey, S, Tzafriri '79] for $1 < p \neq 2 < \infty$.

Ideals in $L(X)$

A maximal algebraic ideal is automatically closed since the invertible elements in a Banach algebra form an open set, so every (always proper) closed ideal is contained in a closed maximal ideal. What are the maximal ones? Is there even a largest ideal?

Let $\mathcal{M}(X)$ denote all operators T on X s.t. the identity operator I_X does not factor through T ($I_X \neq BTA$). It is obvious that $\mathcal{M}(X)$ is an ideal in $L(X)$ if it is closed under addition, in which case it clearly is the largest ideal in $L(X)$. It is known, but non trivial, that $\mathcal{M}(L_p)$ is closed under addition, and also that $\mathcal{M}(L_p)$ is the set of L_p-singular operators, that is the set of operators that are not an isomorphism when restricted to any subspace isomorphic to L_p. [Enflo, Starbird '79] for $p = 1$; [Johnson, Maurey, S, Tzafriri '79] for $1 < p \neq 2 < \infty$.

A maximal algebraic ideal is automatically closed since the invertible elements in a Banach algebra form an open set, so every (always proper) closed ideal is contained in a closed maximal ideal. What are the maximal ones? Is there even a largest ideal?

Let $\mathcal{M}(X)$ denote all operators T on X s.t. the identity operator I_X does not factor through T ($I_X \neq BTA$). It is obvious that $\mathcal{M}(X)$ is an ideal in $L(X)$ if it is closed under addition, in which case it clearly is the largest ideal in $L(X)$. It is known, but non trivial, that $\mathcal{M}(L_p)$ is closed under addition, and also that $\mathcal{M}(L_p)$ is the set of L_p-singular operators, that is the set of operators that are not an isomorphism when restricted to any subspace isomorphic to L_p. [Enflo, Starbird '79] for $p = 1$; [Johnson, Maurey, S, Tzafriri '79] for $1 < p \neq 2 < \infty$.

ideals in $L(X)$

A common way of constructing a (not necessarily closed) ideal in $L(X)$ is to take some operator $U : Y \to Z$ between Banach spaces and let $\mathcal{I}_U$ be the collection of all operators on X that factor through U, i.e., all $T \in L(X)$ s.t. there exist $A \in L(X, Y)$ and $B \in L(Z, X)$ s.t. $T = BUA$.

$L(X)\mathcal{I}_U L(X) \subset \mathcal{I}_U$ is clear, so $\mathcal{I}_U$ is an ideal in $L(X)$ if $\mathcal{I}_U$ is closed under addition. One usually guarantees this by using a U s.t. $U \oplus U : Y \oplus Y \to Z \oplus Z$ factors through U, and these are the only U that I will use. Then the closure $\overline{\mathcal{I}}_U$ will be a proper ideal in $L(X)$ as long as I_X does not factor through U.

A common way of constructing a (not necessarily closed) ideal in $L(X)$ is to take some operator $U : Y \to Z$ between Banach spaces and let $\mathcal{I}_U$ be the collection of all operators on X that factor through U, i.e., all $T \in L(X)$ s.t. there exist $A \in L(X, Y)$ and $B \in L(Z, X)$ s.t. $T = BUA$.

$L(X)\mathcal{I}_U L(X) \subset \mathcal{I}_U$ is clear, so $\mathcal{I}_U$ is an ideal in $L(X)$ if $\mathcal{I}_U$ is closed under addition. One usually guarantees this by using a U s.t. $U \oplus U : Y \oplus Y \to Z \oplus Z$ factors through U, and these are the only U that I will use. Then the closure $\overline{\mathcal{I}}_U$ will be a proper ideal in $L(X)$ as long as I_X does not factor through U.

ideals in $L(X)$

A common way of constructing a (not necessarily closed) ideal in $L(X)$ is to take some operator $U: Y \to Z$ between Banach spaces and let $\mathcal{I}_U$ be the collection of all operators on X that factor through U, i.e., all $T \in L(X)$ s.t. there exist $A \in L(X, Y)$ and $B \in L(Z, X)$ s.t. $T = BUA$.

$L(X)\mathcal{I}_U L(X) \subset \mathcal{I}_U$ is clear, so $\mathcal{I}_U$ is an ideal in $L(X)$ if $\mathcal{I}_U$ is closed under addition. One usually guarantees this by using a U s.t. $U \oplus U : Y \oplus Y \to Z \oplus Z$ factors through U, and these are the only U that I will use. Then the closure $\overline{\mathcal{I}}_U$ will be a proper ideal in $L(X)$ as long as I_X does not factor through U.

Large and Small Ideals

$\mathcal{I}_U$: All $T \in L(X)$ that factor through U.

$S(X)$: Strictly singular operators on X.

An ideal $\mathcal{I}$ is small if $\mathcal{I} \subset S(X)$; otherwise it is large.

So, for example, $\overline{\mathcal{I}}_U$ is small if U is strictly singular and $U \oplus U$ factors through U.

And, for example, $\overline{\mathcal{I}}_U$ is large if $U = I_Y$ for some complemented subspace Y of X and $Y \oplus Y$ is isomorphic to Y.

To simplify notation, I'll write $\mathcal{I}_Y$ instead of $\mathcal{I}_{I_Y}$.

Large and Small Ideals

$\mathcal{I}_U$: All $T \in L(X)$ that factor through U.

$S(X)$: Strictly singular operators on X.

An ideal $\mathcal{I}$ is small if $\mathcal{I} \subset S(X)$; otherwise it is large.

So, for example, $\overline{\mathcal{I}}_U$ is small if U is strictly singular and $U \oplus U$ factors through U.

And, for example, $\overline{\mathcal{I}}_U$ is large if $U = I_Y$ for some complemented subspace Y of X and $Y \oplus Y$ is isomorphic to Y.

To simplify notation, I'll write $\mathcal{I}_Y$ instead of $\mathcal{I}_{I_Y}$.

Large and Small Ideals

$\mathcal{I}_U$: All $T \in L(X)$ that factor through U.

$S(X)$: Strictly singular operators on X.

An ideal $\mathcal{I}$ is small if $\mathcal{I} \subset S(X)$; otherwise it is large.

So, for example, $\overline{\mathcal{I}}_U$ is small if U is strictly singular and $U \oplus U$ factors through U.

And, for example, $\overline{\mathcal{I}}_U$ is large if $U = I_Y$ for some complemented subspace Y of X and $Y \oplus Y$ is isomorphic to Y.

To simplify notation, I'll write $\mathcal{I}_Y$ instead of $\mathcal{I}_{I_Y}$.

Large and Small Ideals

$\mathcal{I}_U$: All $T \in L(X)$ that factor through U.

$S(X)$: Strictly singular operators on X.

An ideal $\mathcal{I}$ is small if $\mathcal{I} \subset S(X)$; otherwise it is large.

So, for example, $\overline{\mathcal{I}}_U$ is small if U is strictly singular and $U \oplus U$ factors through U.

And, for example, $\overline{\mathcal{I}}_U$ is large if $U = I_Y$ for some complemented subspace Y of X and $Y \oplus Y$ is isomorphic to Y.

To simplify notation, I'll write $\mathcal{I}_Y$ instead of $\mathcal{I}_{I_Y}$.

Large and Small Ideals

$\mathcal{I}_U$: All $T \in L(X)$ that factor through U.

$S(X)$: Strictly singular operators on X.

An ideal $\mathcal{I}$ is small if $\mathcal{I} \subset S(X)$; otherwise it is large.

So, for example, $\overline{\mathcal{I}}_U$ is small if U is strictly singular and $U \oplus U$ factors through U.

And, for example, $\overline{\mathcal{I}}_U$ is large if $U = I_Y$ for some complemented subspace Y of X and $Y \oplus Y$ is isomorphic to Y.

To simplify notation, I'll write $\mathcal{I}_Y$ instead of $\mathcal{I}_{I_Y}$.

Large and Small Ideals

$\mathcal{I}_U$: All $T \in L(X)$ that factor through U.

$S(X)$: Strictly singular operators on X.

An ideal $\mathcal{I}$ is small if $\mathcal{I} \subset S(X)$; otherwise it is large.

So, for example, $\overline{\mathcal{I}}_U$ is small if U is strictly singular and $U \oplus U$ factors through U.

And, for example, $\overline{\mathcal{I}}_U$ is large if $U = I_Y$ for some complemented subspace Y of X and $Y \oplus Y$ is isomorphic to Y.

To simplify notation, I'll write $\mathcal{I}_Y$ instead of $\mathcal{I}_{I_Y}$.

Ideals in $L(L_1)$

An ideal $\mathcal{I}$ is small if $\mathcal{I} \subset S(X)$; otherwise it is large.

Small closed ideals in $L(L_1)$ include $K(L_1)$, $S(L_1)$, and $W(L_1)$.
But $W(L_1) = S(L_1)$ Dunford-Pettis property of L_1.

Large closed ideals in $L(L_1)$ include $\overline{\mathcal{I}}_{\ell_1}$ and the largest ideal $\mathcal{M}(L_1)$ (and also the Dunford–Pettis opertors).

Incidently, Every large ideal in $L(L_1)$ contains $\overline{\mathcal{I}}_{\ell_1}$ and $\overline{\mathcal{I}}_{\ell_1}$ contains any small ideal in $L(L_1)$.

Until recently this is all that were known. This led Pietsch to ask in his 1979 book “Operator Ideals" whether there are infinitely many closed ideals in $L(L_1)$.

Ideals in $L(L_1)$

An ideal $\mathcal{I}$ is small if $\mathcal{I} \subset S(X)$; otherwise it is large.

Small closed ideals in $L(L_1)$ include $K(L_1)$, $S(L_1)$, and $W(L_1)$. But $W(L_1) = S(L_1)$ Dunford-Pettis property of L_1.

Large closed ideals in $L(L_1)$ include $\overline{\mathcal{I}}_{\ell_1}$ and the largest ideal $\mathcal{M}(L_1)$ (and also the Dunford–Pettis opertors).

Incidently, Every large ideal in $L(L_1)$ contains $\overline{\mathcal{I}}_{\ell_1}$ and $\overline{\mathcal{I}}_{\ell_1}$ contains any small ideal in $L(L_1)$.

Until recently this is all that were known. This led Pietsch to ask in his 1979 book "Operator Ideals" whether there are infinitely many closed ideals in $L(L_1)$.

Ideals in $L(L_1)$

An ideal $\mathcal{I}$ is small if $\mathcal{I} \subset S(X)$; otherwise it is large.

Small closed ideals in $L(L_1)$ include $K(L_1)$, $S(L_1)$, and $W(L_1)$. But $W(L_1) = S(L_1)$ Dunford-Pettis property of L_1.

Large closed ideals in $L(L_1)$ include $\overline{\mathcal{I}}_{\ell_1}$ and the largest ideal $\mathcal{M}(L_1)$ (and also the Dunford–Pettis opertors).

Incidently, Every large ideal in $L(L_1)$ contains $\overline{\mathcal{I}}_{\ell_1}$ and $\overline{\mathcal{I}}_{\ell_1}$ contains any small ideal in $L(L_1)$.

Until recently this is all that were known. This led Pietsch to ask in his 1979 book "Operator Ideals" whether there are infinitely many closed ideals in $L(L_1)$.

Ideals in $L(L_1)$

An ideal $\mathcal{I}$ is small if $\mathcal{I} \subset S(X)$; otherwise it is large.

Small closed ideals in $L(L_1)$ include $K(L_1)$, $S(L_1)$, and $W(L_1)$. But $W(L_1) = S(L_1)$ Dunford-Pettis property of L_1.

Large closed ideals in $L(L_1)$ include $\overline{\mathcal{I}}_{\ell_1}$ and the largest ideal $\mathcal{M}(L_1)$ (and also the Dunford–Pettis opertors).

Incidently, Every large ideal in $L(L_1)$ contains $\overline{\mathcal{I}}_{\ell_1}$ and $\overline{\mathcal{I}}_{\ell_1}$ contains any small ideal in $L(L_1)$.

Until recently this is all that were known. This led Pietsch to ask in his 1979 book “Operator Ideals" whether there are infinitely many closed ideals in $L(L_1)$.

Ideals in $L(L_1)$

An ideal $\mathcal{I}$ is small if $\mathcal{I} \subset S(X)$; otherwise it is large.

Small closed ideals in $L(L_1)$ include $K(L_1)$, $S(L_1)$, and $W(L_1)$. But $W(L_1) = S(L_1)$ Dunford-Pettis property of L_1.

Large closed ideals in $L(L_1)$ include $\overline{\mathcal{I}}_{\ell_1}$ and the largest ideal $\mathcal{M}(L_1)$ (and also the Dunford–Pettis opertors).

Incidently, Every large ideal in $L(L_1)$ contains $\overline{\mathcal{I}}_{\ell_1}$ and $\overline{\mathcal{I}}_{\ell_1}$ contains any small ideal in $L(L_1)$.

Until recently this is all that were known. This led Pietsch to ask in his 1979 book “Operator Ideals" whether there are infinitely many closed ideals in $L(L_1)$.

Ideals in $L(L_1)$ - the difficulty

It is easy to build closed ideals in $L(X)$; in particular, in $L(L_1)$; but difficult to prove that ideals are different. For example, for $1 < p < \infty$, let $\mathcal{I}_{L_p}$ be the (non closed) ideal of operators on L_1 that factor through L_p. These are all different, but their closures $\overline{\mathcal{I}}_{L_p}$ are all equal to the weakly compact operators on L_1.

One would guess that the key to solving Pietsch's problem was to find just one new closed ideal in $L(L_1)$. A couple of years ago Bill and I did that. The ideal is the closure of $\mathcal{I}_{J_2}$, where $J_2 : \ell_1 \to L_1$ maps the unit vector basis of ℓ_1 onto the Rademacher functions IID Bernoulli random variables that take on the values 1 and -1, each with probability $1/2$. We were excited when we were able to prove that $\overline{\mathcal{I}}_{J_2}$ is different from the previously known ideals. We then looked at $\overline{\mathcal{I}}_{J_p}$, $1 < p < 2$, where $J_p : \ell_1 \to L_1$ maps the unit vector basis of ℓ_1 onto IID p-stable random variables. The ideals $\mathcal{I}_{J_p}$ are all different, but it turns out that all the $\overline{\mathcal{I}}_{J_p}$ are equal to $\overline{\mathcal{I}}_{J_2}$!

Ideals in $L(L_1)$ - the difficulty

It is easy to build closed ideals in $L(X)$; in particular, in $L(L_1)$; but difficult to prove that ideals are different. For example, for $1 < p < \infty$, let $\mathcal{I}_{L_p}$ be the (non closed) ideal of operators on L_1 that factor through L_p. These are all different, but their closures $\overline{\mathcal{I}}_{L_p}$ are all equal to the weakly compact operators on L_1.

One would guess that the key to solving Pietsch's problem was to find just one new closed ideal in $L(L_1)$. A couple of years ago Bill and I did that. The ideal is the closure of $\mathcal{I}_{J_2}$, where $J_2 : \ell_1 \to L_1$ maps the unit vector basis of ℓ_1 onto the Rademacher functions IID Bernoulli random variables that take on the values 1 and -1, each with probability $1/2$. We were excited when we were able to prove that $\overline{\mathcal{I}}_{J_2}$ is different from the previously known ideals. We then looked at $\overline{\mathcal{I}}_{J_p}$, $1 < p < 2$, where $J_p : \ell_1 \to L_1$ maps the unit vector basis of ℓ_1 onto IID p-stable random variables. The ideals $\mathcal{I}_{J_p}$ are all different, but it turns out that all the $\overline{\mathcal{I}}_{J_p}$ are equal to $\overline{\mathcal{I}}_{J_2}$!

Ideals in $L(L_1)$ - the difficulty

It is easy to build closed ideals in $L(X)$; in particular, in $L(L_1)$; but difficult to prove that ideals are different. For example, for $1 < p < \infty$, let $\mathcal{I}_{L_p}$ be the (non closed) ideal of operators on L_1 that factor through L_p. These are all different, but their closures $\overline{\mathcal{I}}_{L_p}$ are all equal to the weakly compact operators on L_1.

One would guess that the key to solving Pietsch's problem was to find just one new closed ideal in $L(L_1)$. A couple of years ago Bill and I did that. The ideal is the closure of $\mathcal{I}_{J_2}$, where $J_2 : \ell_1 \to L_1$ maps the unit vector basis of ℓ_1 onto the Rademacher functions IID Bernoulli random variables that take on the values 1 and -1, each with probability $1/2$. We were excited when we were able to prove that $\overline{\mathcal{I}}_{J_2}$ is different from the previously known ideals. We then looked at $\overline{\mathcal{I}}_{J_p}$, $1 < p < 2$, where $J_p : \ell_1 \to L_1$ maps the unit vector basis of ℓ_1 onto IID p-stable random variables. The ideals $\mathcal{I}_{J_p}$ are all different, but it turns out that all the $\overline{\mathcal{I}}_{J_p}$ are equal to $\overline{\mathcal{I}}_{J_2}$!

Ideals in $L(L_1)$ - the difficulty

It is easy to build closed ideals in $L(X)$; in particular, in $L(L_1)$; but difficult to prove that ideals are different. For example, for $1 < p < \infty$, let $\mathcal{I}_{L_p}$ be the (non closed) ideal of operators on L_1 that factor through L_p. These are all different, but their closures $\overline{\mathcal{I}}_{L_p}$ are all equal to the weakly compact operators on L_1.

One would guess that the key to solving Pietsch's problem was to find just one new closed ideal in $L(L_1)$. A couple of years ago Bill and I did that. The ideal is the closure of $\mathcal{I}_{J_2}$, where $J_2 : \ell_1 \to L_1$ maps the unit vector basis of ℓ_1 onto the Rademacher functions IID Bernoulli random variables that take on the values 1 and -1, each with probability $1/2$. We were excited when we were able to prove that $\overline{\mathcal{I}}_{J_2}$ is different from the previously known ideals. We then looked at $\overline{\mathcal{I}}_{J_p}$, $1 < p < 2$, where $J_p : \ell_1 \to L_1$ maps the unit vector basis of ℓ_1 onto IID p-stable random variables. The ideals $\mathcal{I}_{J_p}$ are all different, but it turns out that all the $\overline{\mathcal{I}}_{J_p}$ are equal to $\overline{\mathcal{I}}_{J_2}$!

Ideals in $L(L_1)$ - the difficulty

It is easy to build closed ideals in $L(X)$; in particular, in $L(L_1)$; but difficult to prove that ideals are different. For example, for $1 < p < \infty$, let $\mathcal{I}_{L_p}$ be the (non closed) ideal of operators on L_1 that factor through L_p. These are all different, but their closures $\overline{\mathcal{I}}_{L_p}$ are all equal to the weakly compact operators on L_1.

One would guess that the key to solving Pietsch's problem was to find just one new closed ideal in $L(L_1)$. A couple of years ago Bill and I did that. The ideal is the closure of $\mathcal{I}_{J_2}$, where $J_2 : \ell_1 \to L_1$ maps the unit vector basis of ℓ_1 onto the Rademacher functions IID Bernoulli random variables that take on the values 1 and -1, each with probability $1/2$. We were excited when we were able to prove that $\overline{\mathcal{I}}_{J_2}$ is different from the previously known ideals. We then looked at $\overline{\mathcal{I}}_{J_p}$, $1 < p < 2$, where $J_p : \ell_1 \to L_1$ maps the unit vector basis of ℓ_1 onto IID p-stable random variables. The ideals $\mathcal{I}_{J_p}$ are all different, but it turns out that all the $\overline{\mathcal{I}}_{J_p}$ are equal to $\overline{\mathcal{I}}_{J_2}$!

Ideals in $L(L_1)$ - the difficulty

It is easy to build closed ideals in $L(X)$; in particular, in $L(L_1)$; but difficult to prove that ideals are different. For example, for $1 < p < \infty$, let $\mathcal{I}_{L_p}$ be the (non closed) ideal of operators on L_1 that factor through L_p. These are all different, but their closures $\overline{\mathcal{I}}_{L_p}$ are all equal to the weakly compact operators on L_1.

One would guess that the key to solving Pietsch's problem was to find just one new closed ideal in $L(L_1)$. A couple of years ago Bill and I did that. The ideal is the closure of $\mathcal{I}_{J_2}$, where $J_2 : \ell_1 \to L_1$ maps the unit vector basis of ℓ_1 onto the Rademacher functions IID Bernoulli random variables that take on the values 1 and -1, each with probability $1/2$. We were excited when we were able to prove that $\overline{\mathcal{I}}_{J_2}$ is different from the previously known ideals. We then looked at $\overline{\mathcal{I}}_{J_p}$, $1 < p < 2$, where $J_p : \ell_1 \to L_1$ maps the unit vector basis of ℓ_1 onto IID p-stable random variables. The ideals $\mathcal{I}_{J_p}$ are all different, but it turns out that all the $\overline{\mathcal{I}}_{J_p}$ are equal to $\overline{\mathcal{I}}_{J_2}$!

Theorem.

[JPS] There are at least $2^{\aleph_0}$ (small) closed ideals in $L(L_1)$.

It remains open whether there are infinitely many large closed ideals in $L(L_1)$. This is connected to the unsolved problem whether every infinite dimensional complemented subspace of L_1 is isomorphic either to ℓ_1 or to L_1. Also open is whether there are more than $2^{\aleph_0}$ closed ideals in $L(L_1)$.

The new ideals are a familty $(\overline{\mathcal{I}}_{U_q})_{2<q<\infty}$, where $U_q : \ell_1 \to L_1\{-1,1\}^{\mathbb{N}}$ maps the unit vector basis of ℓ_1 to a carefully chosen $\Lambda(q)$-set of characters. (A set of characters is $\Lambda(q)$ if the L_1 norm is equivalent to the L_q norm on their linear span.) Bourgain's solution to Rudin's $\Lambda(q)$-set problem is used (could be avoided by using B-space theory results from the 1970s).

The problem is to show that these ideals are all different.

Ideals in $L(L_1)$

Theorem.

[JPS] There are at least $2^{\aleph_0}$ (small) closed ideals in $L(L_1)$.

It remains open whether there are infinitely many large closed ideals in $L(L_1)$. This is connected to the unsolved problem whether every infinite dimensional complemented subspace of L_1 is isomorphic either to ℓ_1 or to L_1. Also open is whether there are more than $2^{\aleph_0}$ closed ideals in $L(L_1)$.

The new ideals are a familty $(\overline{\mathcal{I}}_{U_q})_{2<q<\infty}$, where $U_q : \ell_1 \to L_1\{-1,1\}^{\mathbb{N}}$ maps the unit vector basis of ℓ_1 to a carefully chosen $\Lambda(q)$-set of characters. (A set of characters is $\Lambda(q)$ if the L_1 norm is equivalent to the L_q norm on their linear span.) Bourgain's solution to Rudin's $\Lambda(q)$-set problem is used (could be avoided by using B-space theory results from the 1970s).

The problem is to show that these ideals are all different.

Theorem.

[JPS] There are at least $2^{\aleph_0}$ (small) closed ideals in $L(L_1)$.

It remains open whether there are infinitely many large closed ideals in $L(L_1)$. This is connected to the unsolved problem whether every infinite dimensional complemented subspace of L_1 is isomorphic either to ℓ_1 or to L_1. Also open is whether there are more than $2^{\aleph_0}$ closed ideals in $L(L_1)$.

The new ideals are a familty $(\overline{\mathcal{I}}_{U_q})_{2<q<\infty}$, where $U_q : \ell_1 \to L_1\{-1,1\}^{\mathbb{N}}$ maps the unit vector basis of ℓ_1 to a carefully chosen $\Lambda(q)$-set of characters. (A set of characters is $\Lambda(q)$ if the L_1 norm is equivalent to the L_q norm on their linear span.) Bourgain's solution to Rudin's $\Lambda(q)$-set problem is used

(could be avoided by using B-space theory results from the 1970s).

The problem is to show that these ideals are all different.

Theorem.

[JPS] There are at least $2^{\aleph_0}$ (small) closed ideals in $L(L_1)$.

It remains open whether there are infinitely many large closed ideals in $L(L_1)$. This is connected to the unsolved problem whether every infinite dimensional complemented subspace of L_1 is isomorphic either to ℓ_1 or to L_1. Also open is whether there are more than $2^{\aleph_0}$ closed ideals in $L(L_1)$.

The new ideals are a familty $(\overline{\mathcal{I}}_{U_q})_{2<q<\infty}$, where $U_q : \ell_1 \to L_1\{-1,1\}^{\mathbb{N}}$ maps the unit vector basis of ℓ_1 to a carefully chosen $\Lambda(q)$-set of characters. (A set of characters is $\Lambda(q)$ if the L_1 norm is equivalent to the L_q norm on their linear span.) Bourgain's solution to Rudin's $\Lambda(q)$-set problem is used (could be avoided by using B-space theory results from the 1970s).

The problem is to show that these ideals are all different.

Ideals in $L(L_1)$

Theorem.

[JPS] There are at least $2^{\aleph_0}$ (small) closed ideals in $L(L_1)$.

It remains open whether there are infinitely many large closed ideals in $L(L_1)$. This is connected to the unsolved problem whether every infinite dimensional complemented subspace of L_1 is isomorphic either to ℓ_1 or to L_1. Also open is whether there are more than $2^{\aleph_0}$ closed ideals in $L(L_1)$.

The new ideals are a familty $(\overline{\mathcal{I}}_{U_q})_{2<q<\infty}$, where $U_q : \ell_1 \to L_1\{-1,1\}^{\mathbb{N}}$ maps the unit vector basis of ℓ_1 to a carefully chosen $\Lambda(q)$-set of characters. (A set of characters is $\Lambda(q)$ if the L_1 norm is equivalent to the L_q norm on their linear span.) Bourgain's solution to Rudin's $\Lambda(q)$-set problem is used (could be avoided by using B-space theory results from the 1970s).

The problem is to show that these ideals are all different.

Large Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$

An ideal $\mathcal{I}$ is small if $\mathcal{I} \subset S(X)$; otherwise it is large.

[S '75] There are infinitely many isomorphically different complemented subspaces of L_p, each isomorphic to its square, hence there are infinitely many (large) closed ideals in $L(L_p)$.

[Bourgain, Rosenthal, S '81] There are $\aleph_1$ isomorphically different complemented subspaces of L_p, each isomorphic to its square, hence there are $\aleph_1$ (large) closed ideals in $L(L_p)$.

This leaves open whether there are there more than $\aleph_1$ (large?/small?) closed ideals in $L(L_p)$? Maybe there are even $2^{2^{\aleph_0}}$ (large?/small?) closed ideals.

Large Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$

An ideal $\mathcal{I}$ is small if $\mathcal{I} \subset S(X)$; otherwise it is large.

[S '75] There are infinitely many isomorphically different complemented subspaces of L_p, each isomorphic to its square, hence there are infinitely many (large) closed ideals in $L(L_p)$.

[Bourgain, Rosenthal, S '81] There are $\aleph_1$ isomorphically different complemented subspaces of L_p, each isomorphic to its square, hence there are $\aleph_1$ (large) closed ideals in $L(L_p)$.

This leaves open whether there are there more than $\aleph_1$ (large?/small?) closed ideals in $L(L_p)$? Maybe there are even $2^{2^{\aleph_0}}$ (large?/small?) closed ideals.

Large Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$

An ideal $\mathcal{I}$ is small if $\mathcal{I} \subset S(X)$; otherwise it is large.

[S '75] There are infinitely many isomorphically different complemented subspaces of L_p, each isomorphic to its square, hence there are infinitely many (large) closed ideals in $L(L_p)$.

[Bourgain, Rosenthal, S '81] There are $\aleph_1$ isomorphically different complemented subspaces of L_p, each isomorphic to its square, hence there are $\aleph_1$ (large) closed ideals in $L(L_p)$.

This leaves open whether there are there are more than $\aleph_1$ (large?/small?) closed ideals in $L(L_p)$? Maybe there are even $2^{2^{\aleph_0}}$ (large?/small?) closed ideals.

Large Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$

An ideal $\mathcal{I}$ is small if $\mathcal{I} \subset S(X)$; otherwise it is large.

[S '75] There are infinitely many isomorphically different complemented subspaces of L_p, each isomorphic to its square, hence there are infinitely many (large) closed ideals in $L(L_p)$.

[Bourgain, Rosenthal, S '81] There are $\aleph_1$ isomorphically different complemented subspaces of L_p, each isomorphic to its square, hence there are $\aleph_1$ (large) closed ideals in $L(L_p)$.

This leaves open whether there are there more than $\aleph_1$ (large?/small?) closed ideals in $L(L_p)$? Maybe there are even $2^{2^{\aleph_0}}$ (large?/small?) closed ideals.

Small Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$

The following solved the first problem for small ideals

Theorem. (Schlumprecht,Zsak '18)

There are infinitely many; in fact, at least $2^{\aleph_0}$; (small) closed ideals in $L(L_p)$, $1 < p \neq 2 < \infty$.

The ideals constructed in [SZ '18] are all of the form $\overline{\mathcal{I}}_U$ with U a basis to basis mapping from ℓ_r to ℓ_s but the bases for ℓ_r, ℓ_s are not the standard unit vector basis.

Whether there are more than $2^{\aleph_0}$ small closed ideals in $L(L_p)$ remains open.

But,

Small Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$

The following solved the first problem for small ideals

Theorem. (Schlumprecht,Zsak '18)

There are infinitely many; in fact, at least $2^{\aleph_0}$; (small) closed ideals in $L(L_p)$, $1 < p \neq 2 < \infty$.

The ideals constructed in [SZ '18] are all of the form $\overline{\mathcal{I}}_U$ with U a basis to basis mapping from ℓ_r to ℓ_s but the bases for ℓ_r, ℓ_s are not the standard unit vector basis.

Whether there are more than $2^{\aleph_0}$ small closed ideals in $L(L_p)$ remains open.

But,

Small Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$

The following solved the first problem for small ideals

Theorem. (Schlumprecht,Zsak '18)

There are infinitely many; in fact, at least $2^{\aleph_0}$; (small) closed ideals in $L(L_p)$, $1 < p \neq 2 < \infty$.

The ideals constructed in [SZ '18] are all of the form $\overline{\mathcal{I}}_U$ with U a basis to basis mapping from ℓ_r to ℓ_s but the bases for ℓ_r, ℓ_s are not the standard unit vector basis.

Whether there are more than $2^{\aleph_0}$ small closed ideals in $L(L_p)$ remains open.

But,

The following solved the first problem for small ideals

Theorem. (Schlumprecht,Zsak '18)

There are infinitely many; in fact, at least $2^{\aleph_0}$; (small) closed ideals in $L(L_p)$, $1 < p \neq 2 < \infty$.

The ideals constructed in [SZ '18] are all of the form $\overline{\mathcal{I}}_U$ with U a basis to basis mapping from ℓ_r to ℓ_s but the bases for ℓ_r, ℓ_s are not the standard unit vector basis.

Whether there are more than $2^{\aleph_0}$ small closed ideals in $L(L_p)$ remains open.

But,

Small Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$

The following solved the first problem for small ideals

Theorem. (Schlumprecht,Zsak '18)

There are infinitely many; in fact, at least $2^{\aleph_0}$; (small) closed ideals in $L(L_p)$, $1 < p \neq 2 < \infty$.

The ideals constructed in [SZ '18] are all of the form $\overline{\mathcal{I}}_U$ with U a basis to basis mapping from ℓ_r to ℓ_s but the bases for ℓ_r, ℓ_s are not the standard unit vector basis.

Whether there are more than $2^{\aleph_0}$ small closed ideals in $L(L_p)$ remains open.

But,

We recently proved,

Theorem. (JS '19)

There are $2^{2^{\aleph_0}}$; (large) closed ideals in $L(L_p)$, $1 < p \neq 2 < \infty$.

The proof relays on fine properties of spaces spanned by independent random variables in L_p, $2 < p < \infty$, a topic investigated mostly by Rosenthal in the 1970-s.

More Large Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$

We recently proved,

Theorem. (JS '19)

There are $2^{2^{\aleph_0}}$ (large) closed ideals in $L(L_p)$, $1 < p \neq 2 < \infty$.

The proof relays on fine properties of spaces spanned by independent random variables in L_p, $2 < p < \infty$, a topic investigated mostly by Rosenthal in the 1970-s.

Recall that for a sequence $u = \{u_j\}_{j=1}^{\infty}$ of positive real numbers and for $p > 2$, the Banach space $X_{p,u}$ is the real sequence space with norm

$$\|\{a_j\}_{j=1}^{\infty}\| = \max\{(\sum_{j=1}^{\infty} |a_j|^p)^{1/p}, (\sum_{j=1}^{\infty} |a_j u_j|^2)^{1/2}\}.$$

Rosenthal proved that $X_{p,u}$ is isomorphic to a complemented subspace of L_p with the isomorphism constant and the complementation constant depending only on p.

If u is such that $\lim_{j\to 0} u_j = 0$ but $\sum_{j=1}^{\infty} |u_j|^{\frac{2p}{p-2}} = \infty$ then one gets a space isomorphically different from ℓ_p, ℓ_2 and $\ell_p \oplus \ell_2$.

Recall that for a sequence $u = \{u_j\}_{j=1}^{\infty}$ of positive real numbers and for $p > 2$, the Banach space $X_{p,u}$ is the real sequence space with norm

$$\|\{a_j\}_{j=1}^{\infty}\| = \max\{(\sum_{j=1}^{\infty} |a_j|^p)^{1/p}, (\sum_{j=1}^{\infty} |a_j u_j|^2)^{1/2}\}.$$

Rosenthal proved that $X_{p,u}$ is isomorphic to a complemented subspace of L_p with the isomorphism constant and the complementation constant depending only on p.

If u is such that $\lim_{j\to 0} u_j = 0$ but $\sum_{j=1}^{\infty} |u_j|^{\frac{2p}{p-2}} = \infty$ then one gets a space isomorphically different from ℓ_p, ℓ_2 and $\ell_p \oplus \ell_2$.

More Large Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$

Recall that for a sequence $u = \{u_j\}_{j=1}^{\infty}$ of positive real numbers and for $p > 2$, the Banach space $X_{p,u}$ is the real sequence space with norm

$$\|\{a_j\}_{j=1}^{\infty}\| = \max\{(\sum_{j=1}^{\infty} |a_j|^p)^{1/p}, (\sum_{j=1}^{\infty} |a_j u_j|^2)^{1/2}\}.$$

Rosenthal proved that $X_{p,u}$ is isomorphic to a complemented subspace of L_p with the isomorphism constant and the complementation constant depending only on p.

If u is such that $\lim_{j\to 0} u_j = 0$ but $\sum_{j=1}^{\infty} |u_j|^{\frac{2p}{p-2}} = \infty$ then one gets a space isomorphically different from ℓ_p, ℓ_2 and $\ell_p \oplus \ell_2$.

More Large Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$

Recall that for a sequence $u = \{u_j\}_{j=1}^{\infty}$ of positive real numbers and for $p > 2$, the Banach space $X_{p,u}$ is the real sequence space with norm

$$\|\{a_j\}_{j=1}^{\infty}\| = \max\{(\sum_{j=1}^{\infty} |a_j|^p)^{1/p}, (\sum_{j=1}^{\infty} |a_j u_j|^2)^{1/2}\}.$$

Rosenthal proved that $X_{p,u}$ is isomorphic to a complemented subspace of L_p with the isomorphism constant and the complementation constant depending only on p.

If u is such that $\lim_{j\to 0} u_j = 0$ but $\sum_{j=1}^{\infty} |u_j|^{\frac{2p}{p-2}} = \infty$ then one gets a space isomorphically different from ℓ_p, ℓ_2 and $\ell_p \oplus \ell_2$.

More Large Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$

$$\|\{a_j\}_{j=1}^\infty\|_{X_{p,u}} = \max\{(\sum_{j=1}^\infty |a_j|^p)^{1/p}, (\sum_{j=1}^\infty |a_j u_j|^2)^{1/2}\}.$$

However, for different u satisfying the two conditions above the different $X_{p,u}$ spaces are mutually isomorphic. We denote by X_p any of these spaces. We'll need more properties of the spaces $X_{p,u}$ but right now we only need the representation above and we think of $X_{p,u}$ as a subspace of $\ell_p \oplus_\infty \ell_2$.

Let $\{e_j\}_{j=1}^\infty$ be the unit vector basis of ℓ_p and $\{f_j\}_{j=1}^\infty$ be the unit vector basis of ℓ_2. Let $v = \{v_j\}_{j=1}^\infty$ and $w = \{w_j\}_{j=1}^\infty$ be two positive real sequences such that $\delta_j = w_j/v_j \to 0$ as $j \to \infty$. Set

$$g_j^v = e_j + v_j f_j \in \ell_p \oplus_\infty \ell_2 \quad \text{and} \quad g_j^w = e_j + w_j f_j \in \ell_p \oplus_\infty \ell_2.$$

Then $\{g_j^v\}_{j=1}^\infty$ is the unit vector basis of $X_{p,v}$ and $\{g_j^w\}_{j=1}^\infty$ is the unit vector basis of $X_{p,w}$.

More Large Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$

$$\|\{a_j\}_{j=1}^\infty\|_{X_{p,u}} = \max\{(\sum_{j=1}^\infty |a_j|^p)^{1/p}, (\sum_{j=1}^\infty |a_j u_j|^2)^{1/2}\}.$$

However, for different u satisfying the two conditions above the different $X_{p,u}$ spaces are mutually isomorphic. We denote by X_p any of these spaces. We'll need more properties of the spaces $X_{p,u}$ but right now we only need the representation above and we think of $X_{p,u}$ as a subspace of $\ell_p \oplus_\infty \ell_2$.

Let $\{e_j\}_{j=1}^\infty$ be the unit vector basis of ℓ_p and $\{f_j\}_{j=1}^\infty$ be the unit vector basis of ℓ_2. Let $v = \{v_j\}_{j=1}^\infty$ and $w = \{w_j\}_{j=1}^\infty$ be two positive real sequences such that $\delta_j = w_j/v_j \to 0$ as $j \to \infty$. Set

$$g_j^v = e_j + v_j f_j \in \ell_p \oplus_\infty \ell_2 \quad \text{and} \quad g_j^w = e_j + w_j f_j \in \ell_p \oplus_\infty \ell_2.$$

Then $\{g_j^v\}_{j=1}^\infty$ is the unit vector basis of $X_{p,v}$ and $\{g_j^w\}_{j=1}^\infty$ is the unit vector basis of $X_{p,w}$.

More Large Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$

$$\|\{a_j\}_{j=1}^{\infty}\|_{X_{p,u}} = \max\{(\sum_{j=1}^{\infty} |a_j|^p)^{1/p}, (\sum_{j=1}^{\infty} |a_j u_j|^2)^{1/2}\}.$$

However, for different u satisfying the two conditions above the different $X_{p,u}$ spaces are mutually isomorphic. We denote by X_p any of these spaces. We'll need more properties of the spaces $X_{p,u}$ but right now we only need the representation above and we think of $X_{p,u}$ as a subspace of $\ell_p \oplus_\infty \ell_2$.

Let $\{e_j\}_{j=1}^{\infty}$ be the unit vector basis of ℓ_p and $\{f_j\}_{j=1}^{\infty}$ be the unit vector basis of ℓ_2. Let $v = \{v_j\}_{j=1}^{\infty}$ and $w = \{w_j\}_{j=1}^{\infty}$ be two positive real sequences such that $\delta_j = w_j/v_j \to 0$ as $j \to \infty$. Set

$$g_j^v = e_j + v_j f_j \in \ell_p \oplus_\infty \ell_2 \quad \text{and} \quad g_j^w = e_j + w_j f_j \in \ell_p \oplus_\infty \ell_2.$$

Then $\{g_j^v\}_{j=1}^{\infty}$ is the unit vector basis of $X_{p,v}$ and $\{g_j^w\}_{j=1}^{\infty}$ is the unit vector basis of $X_{p,w}$.

$$\|\{a_j\}_{j=1}^{\infty}\|_{X_{p,u}} = \max\{(\sum_{j=1}^{\infty} |a_j|^p)^{1/p}, (\sum_{j=1}^{\infty} |a_j u_j|^2)^{1/2}\}.$$

However, for different u satisfying the two conditions above the different $X_{p,u}$ spaces are mutually isomorphic. We denote by X_p any of these spaces. We'll need more properties of the spaces $X_{p,u}$ but right now we only need the representation above and we think of $X_{p,u}$ as a subspace of $\ell_p \oplus_\infty \ell_2$.

Let $\{e_j\}_{j=1}^{\infty}$ be the unit vector basis of ℓ_p and $\{f_j\}_{j=1}^{\infty}$ be the unit vector basis of ℓ_2. Let $v = \{v_j\}_{j=1}^{\infty}$ and $w = \{w_j\}_{j=1}^{\infty}$ be two positive real sequences such that $\delta_j = w_j / v_j \to 0$ as $j \to \infty$. Set

$$g_j^v = e_j + v_j f_j \in \ell_p \oplus_\infty \ell_2 \quad \text{and} \quad g_j^w = e_j + w_j f_j \in \ell_p \oplus_\infty \ell_2.$$

Then $\{g_j^v\}_{j=1}^{\infty}$ is the unit vector basis of $X_{p,v}$ and $\{g_j^w\}_{j=1}^{\infty}$ is the unit vector basis of $X_{p,w}$.

$$\|\{a_j\}_{j=1}^{\infty}\|_{X_{p,u}} = \max\{(\sum_{j=1}^{\infty} |a_j|^p)^{1/p}, (\sum_{j=1}^{\infty} |a_j u_j|^2)^{1/2}\}.$$

However, for different u satisfying the two conditions above the different $X_{p,u}$ spaces are mutually isomorphic. We denote by X_p any of these spaces. We'll need more properties of the spaces $X_{p,u}$ but right now we only need the representation above and we think of $X_{p,u}$ as a subspace of $\ell_p \oplus_\infty \ell_2$.

Let $\{e_j\}_{j=1}^{\infty}$ be the unit vector basis of ℓ_p and $\{f_j\}_{j=1}^{\infty}$ be the unit vector basis of ℓ_2. Let $v = \{v_j\}_{j=1}^{\infty}$ and $w = \{w_j\}_{j=1}^{\infty}$ be two positive real sequences such that $\delta_j = w_j/v_j \to 0$ as $j \to \infty$. Set

$$g_j^v = e_j + v_j f_j \in \ell_p \oplus_\infty \ell_2 \quad \text{and} \quad g_j^w = e_j + w_j f_j \in \ell_p \oplus_\infty \ell_2.$$

Then $\{g_j^v\}_{j=1}^{\infty}$ is the unit vector basis of $X_{p,v}$ and $\{g_j^w\}_{j=1}^{\infty}$ is the unit vector basis of $X_{p,w}$.

$$\|\{a_j\}_{j=1}^{\infty}\|_{X_{p,u}} = \max\{(\sum_{j=1}^{\infty} |a_j|^p)^{1/p}, (\sum_{j=1}^{\infty} |a_j u_j|^2)^{1/2}\}.$$

However, for different u satisfying the two conditions above the different $X_{p,u}$ spaces are mutually isomorphic. We denote by X_p any of these spaces. We'll need more properties of the spaces $X_{p,u}$ but right now we only need the representation above and we think of $X_{p,u}$ as a subspace of $\ell_p \oplus_\infty \ell_2$.

Let $\{e_j\}_{j=1}^{\infty}$ be the unit vector basis of ℓ_p and $\{f_j\}_{j=1}^{\infty}$ be the unit vector basis of ℓ_2. Let $v = \{v_j\}_{j=1}^{\infty}$ and $w = \{w_j\}_{j=1}^{\infty}$ be two positive real sequences such that $\delta_j = w_j/v_j \to 0$ as $j \to \infty$. Set

$$g_j^v = e_j + v_j f_j \in \ell_p \oplus_\infty \ell_2 \quad \text{and} \quad g_j^w = e_j + w_j f_j \in \ell_p \oplus_\infty \ell_2.$$

Then $\{g_j^v\}_{j=1}^{\infty}$ is the unit vector basis of $X_{p,v}$ and $\{g_j^w\}_{j=1}^{\infty}$ is the unit vector basis of $X_{p,w}$.

$$\|\{a_j\}_{j=1}^{\infty}\|_{X_{p,u}} = \max\{(\sum_{j=1}^{\infty} |a_j|^p)^{1/p}, (\sum_{j=1}^{\infty} |a_j u_j|^2)^{1/2}\}.$$

However, for different u satisfying the two conditions above the different $X_{p,u}$ spaces are mutually isomorphic. We denote by X_p any of these spaces. We'll need more properties of the spaces $X_{p,u}$ but right now we only need the representation above and we think of $X_{p,u}$ as a subspace of $\ell_p \oplus_\infty \ell_2$.

Let $\{e_j\}_{j=1}^{\infty}$ be the unit vector basis of ℓ_p and $\{f_j\}_{j=1}^{\infty}$ be the unit vector basis of ℓ_2. Let $v = \{v_j\}_{j=1}^{\infty}$ and $w = \{w_j\}_{j=1}^{\infty}$ be two positive real sequences such that $\delta_j = w_j/v_j \to 0$ as $j \to \infty$. Set

$$g_j^v = e_j + v_j f_j \in \ell_p \oplus_\infty \ell_2 \quad \text{and} \quad g_j^w = e_j + w_j f_j \in \ell_p \oplus_\infty \ell_2.$$

Then $\{g_j^v\}_{j=1}^{\infty}$ is the unit vector basis of $X_{p,v}$ and $\{g_j^w\}_{j=1}^{\infty}$ is the unit vector basis of $X_{p,w}$.

More Large Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$

$$\|\{a_j\}_{j=1}^{\infty}\|_{X_{p,u}} = \max\{(\sum_{j=1}^{\infty} |a_j|^p)^{1/p}, (\sum_{j=1}^{\infty} |a_j u_j|^2)^{1/2}\}.$$

However, for different u satisfying the two conditions above the different $X_{p,u}$ spaces are mutually isomorphic. We denote by X_p any of these spaces. We'll need more properties of the spaces $X_{p,u}$ but right now we only need the representation above and we think of $X_{p,u}$ as a subspace of $\ell_p \oplus_\infty \ell_2$.

Let $\{e_j\}_{j=1}^{\infty}$ be the unit vector basis of ℓ_p and $\{f_j\}_{j=1}^{\infty}$ be the unit vector basis of ℓ_2. Let $v = \{v_j\}_{j=1}^{\infty}$ and $w = \{w_j\}_{j=1}^{\infty}$ be two positive real sequences such that $\delta_j = w_j/v_j \to 0$ as $j \to \infty$. Set

$$g_j^v = e_j + v_j f_j \in \ell_p \oplus_\infty \ell_2 \quad \text{and} \quad g_j^w = e_j + w_j f_j \in \ell_p \oplus_\infty \ell_2.$$

Then $\{g_j^v\}_{j=1}^{\infty}$ is the unit vector basis of $X_{p,v}$ and $\{g_j^w\}_{j=1}^{\infty}$ is the unit vector basis of $X_{p,w}$.

More Large Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$

$$g_j^v = e_j + v_j f_j \in \ell_p \oplus_\infty \ell_2 \text{ and } g_j^w = e_j + w_j f_j \in \ell_p \oplus_\infty \ell_2.$$

Define also $\Delta = \Delta(w, v)$

$$\Delta : X_{p,w} \to X_{p,v}$$

by

$$\Delta g_j^w = \delta_j g_j^v.$$

Note that Δ is the restriction to $X_{p,w}$ of

$$K : \ell_p \oplus_\infty \ell_2 \to \ell_p \oplus_\infty \ell_2$$

defined by

$$K(e_j) = \delta_j e_j \text{ and } K(f_j) = f_j$$

Consequently, $\|\Delta\| \leq \|K\| = \max\{1, \max_{1 \leq j < \infty} \delta_j\}$.

More Large Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$

$$g_j^v = e_j + v_j f_j \in \ell_p \oplus_\infty \ell_2 \text{ and } g_j^w = e_j + w_j f_j \in \ell_p \oplus_\infty \ell_2.$$

Define also $\Delta = \Delta(w, v)$

$$\Delta : X_{p,w} \to X_{p,v}$$

by

$$\Delta g_j^w = \delta_j g_j^v.$$

Note that Δ is the restriction to $X_{p,w}$ of

$$K : \ell_p \oplus_\infty \ell_2 \to \ell_p \oplus_\infty \ell_2$$

defined by

$$K(e_j) = \delta_j e_j \text{ and } K(f_j) = f_j$$

Consequently, $\|\Delta\| \leq \|K\| = \max\{1, \max_{1 \leq j < \infty} \delta_j\}$.

$$g_j^v = e_j + v_j f_j \in \ell_p \oplus_\infty \ell_2 \text{ and } g_j^w = e_j + w_j f_j \in \ell_p \oplus_\infty \ell_2.$$

Define also $\Delta = \Delta(w, v)$

$$\Delta : X_{p,w} \to X_{p,v}$$

by

$$\Delta g_j^w = \delta_j g_j^v.$$

Note that Δ is the restriction to $X_{p,w}$ of

$$K : \ell_p \oplus_\infty \ell_2 \to \ell_p \oplus_\infty \ell_2$$

defined by

$$K(e_j) = \delta_j e_j \text{ and } K(f_j) = f_j$$

Consequently, $\|\Delta\| \leq \|K\| = \max\{1, \max_{1 \leq j < \infty} \delta_j\}$.

More Large Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$

$$g_j^v = e_j + v_j f_j \in \ell_p \oplus_\infty \ell_2 \text{ and } g_j^w = e_j + w_j f_j \in \ell_p \oplus_\infty \ell_2.$$

Define also $\Delta = \Delta(w, v)$

$$\Delta : X_{p,w} \to X_{p,v}$$

by

$$\Delta g_j^w = \delta_j g_j^v.$$

Note that Δ is the restriction to $X_{p,w}$ of

$$K : \ell_p \oplus_\infty \ell_2 \to \ell_p \oplus_\infty \ell_2$$

defined by

$$K(e_j) = \delta_j e_j \text{ and } K(f_j) = f_j$$

Consequently, $\|\Delta\| \leq \|K\| = \max\{1, \max_{1 \leq j < \infty} \delta_j\}$.

Denote by $\{h_j^w\}$ the dual basis to $\{g_j^w\}$ (and by $\{h_j^v\}$ the dual basis to $\{g_j^v\}$,

It was proved by Rosenthal that $[h_j^w]$ and $[h_j^v]$ contain copies of ℓ_r for all $q = p/(p-1) \leq r \leq 2$

A major part in our proof is the fact that for any sequence $r_i \nearrow 2$ and n_i such that $n_i^{\frac{1}{r_i} - \frac{1}{2}} \nearrow \infty$ (i.e. $d(\ell_{r_i}^{n_i}, \ell_2^{n_i}) \to \infty$) there are sequences $v = \{v_j\}_{j=1}^{\infty}$ and $w = \{w_j\}_{j=1}^{\infty}$ such that $\delta_j = w_j/v_j \to 0$ and

Δ^* isomorphically uniformly preserves these copies of $\ell_{r_i}^{n_i}$.

(Δ^* also preserves the modular space $\ell_{\{r_i\}}$.)

Denote by $\{h_j^w\}$ the dual basis to $\{g_j^w\}$ (and by $\{h_j^v\}$ the dual basis to $\{g_j^v\}$,

It was proved by Rosenthal that $[h_j^w]$ and $[h_j^v]$ contain copies of ℓ_r for all $q = p/(p-1) \leq r \leq 2$

A major part in our proof is the fact that for any sequence $r_i \nearrow 2$ and n_i such that $n_i^{\frac{1}{r_i}-\frac{1}{2}} \nearrow \infty$ (i.e. $d(\ell_{r_i}^{n_i}, \ell_2^{n_i}) \to \infty$) there are sequences $v = \{v_j\}_{j=1}^{\infty}$ and $w = \{w_j\}_{j=1}^{\infty}$ such that $\delta_j = w_j/v_j \to 0$ and

Δ^* isomorphically uniformly preserves these copies of $\ell_{r_i}^{n_i}$.

(Δ^* also preserves the modular space $\ell_{\{r_i\}}$.)

More Large Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$

Denote by $\{h_j^w\}$ the dual basis to $\{g_j^w\}$ (and by $\{h_j^v\}$ the dual basis to $\{g_j^v\}$,

It was proved by Rosenthal that $[h_j^w]$ and $[h_j^v]$ contain copies of ℓ_r for all $q = p/(p-1) \leq r \leq 2$

A major part in our proof is the fact that for any sequence $r_i \nearrow 2$ and n_i such that $n_i^{\frac{1}{r_i}-\frac{1}{2}} \nearrow \infty$ (i.e. $d(\ell_{r_i}^{n_i}, \ell_2^{n_i}) \to \infty$) there are sequences $v = \{v_j\}_{j=1}^{\infty}$ and $w = \{w_j\}_{j=1}^{\infty}$ such that $\delta_j = w_j/v_j \to 0$ and

Δ^* isomorphically uniformly preserves these copies of $\ell_{r_i}^{n_i}$.

(Δ^* also preserves the modular space $\ell_{\{r_i\}}$.)

More Large Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$

Denote by $\{h_j^w\}$ the dual basis to $\{g_j^w\}$ (and by $\{h_j^v\}$ the dual basis to $\{g_j^v\}$,

It was proved by Rosenthal that $[h_j^w]$ and $[h_j^v]$ contain copies of ℓ_r for all $q = p/(p-1) \leq r \leq 2$

A major part in our proof is the fact that for any sequence $r_i \nearrow 2$ and n_i such that $n_i^{\frac{1}{r_i}-\frac{1}{2}} \nearrow \infty$ (i.e. $d(\ell_{r_i}^{n_i}, \ell_2^{n_i}) \to \infty$) there are sequences $v = \{v_j\}_{j=1}^{\infty}$ and $w = \{w_j\}_{j=1}^{\infty}$ such that $\delta_j = w_j/v_j \to 0$ and

Δ^* isomorphically uniformly preserves these copies of $\ell_{r_i}^{n_i}$.

(Δ^* also preserves the modular space $\ell_{\{r_i\}}$.)

More Large Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$

Denote by $\{h_j^w\}$ the dual basis to $\{g_j^w\}$ (and by $\{h_j^v\}$ the dual basis to $\{g_j^v\}$,

It was proved by Rosenthal that $[h_j^w]$ and $[h_j^v]$ contain copies of ℓ_r for all $q = p/(p-1) \le r \le 2$

A major part in our proof is the fact that for any sequence $r_i \nearrow 2$ and n_i such that $n_i^{\frac{1}{r_i}-\frac{1}{2}} \nearrow \infty$ (i.e. $d(\ell_{r_i}^{n_i}, \ell_2^{n_i}) \to \infty$) there are sequences $v = \{v_j\}_{j=1}^{\infty}$ and $w = \{w_j\}_{j=1}^{\infty}$ such that $\delta_j = w_j/v_j \to 0$ and

Δ^* isomorphically uniformly preserves these copies of $\ell_{r_i}^{n_i}$.

(Δ^* also preserves the modular space $\ell_{\{r_i\}}$.)

More Large Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$

Denote by $\{h_j^w\}$ the dual basis to $\{g_j^w\}$ (and by $\{h_j^v\}$ the dual basis to $\{g_j^v\}$,

It was proved by Rosenthal that $[h_j^w]$ and $[h_j^v]$ contain copies of ℓ_r for all $q = p/(p-1) \leq r \leq 2$

A major part in our proof is the fact that for any sequence $r_i \nearrow 2$ and n_i such that $n_i^{\frac{1}{r_i}-\frac{1}{2}} \nearrow \infty$ (i.e. $d(\ell_{r_i}^{n_i}, \ell_2^{n_i}) \to \infty$) there are sequences $v = \{v_j\}_{j=1}^{\infty}$ and $w = \{w_j\}_{j=1}^{\infty}$ such that $\delta_j = w_j/v_j \to 0$ and

Δ^* isomorphically uniformly preserves these copies of $\ell_{r_i}^{n_i}$.

(Δ^* also preserves the modular space $\ell_{\{r_i\}}$.)

Denote by $\{h_j^w\}$ the dual basis to $\{g_j^w\}$ (and by $\{h_j^v\}$ the dual basis to $\{g_j^v\}$,

It was proved by Rosenthal that $[h_j^w]$ and $[h_j^v]$ contain copies of ℓ_r for all $q = p/(p-1) \leq r \leq 2$

A major part in our proof is the fact that for any sequence $r_i \nearrow 2$ and n_i such that $n_i^{\frac{1}{r_i}-\frac{1}{2}} \nearrow \infty$ (i.e. $d(\ell_{r_i}^{n_i}, \ell_2^{n_i}) \to \infty$) there are sequences $v = \{v_j\}_{j=1}^{\infty}$ and $w = \{w_j\}_{j=1}^{\infty}$ such that $\delta_j = w_j/v_j \to 0$ and

Δ^* isomorphically uniformly preserves these copies of $\ell_{r_i}^{n_i}$.

(Δ^* also preserves the modular space $\ell_{\{r_i\}}$.)

More Large Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$

Denote by $\{h_j^w\}$ the dual basis to $\{g_j^w\}$ (and by $\{h_j^v\}$ the dual basis to $\{g_j^v\}$,

It was proved by Rosenthal that $[h_j^w]$ and $[h_j^v]$ contain copies of ℓ_r for all $q = p/(p-1) \leq r \leq 2$

A major part in our proof is the fact that for any sequence $r_i \nearrow 2$ and n_i such that $n_i^{\frac{1}{r_i}-\frac{1}{2}} \nearrow \infty$ (i.e. $d(\ell_{r_i}^{n_i}, \ell_2^{n_i}) \to \infty$) there are sequences $v = \{v_j\}_{j=1}^{\infty}$ and $w = \{w_j\}_{j=1}^{\infty}$ such that $\delta_j = w_j/v_j \to 0$ and

Δ^* isomorphically uniformly preserves these copies of $\ell_{r_i}^{n_i}$.

(Δ^* also preserves the modular space $\ell_{\{r_i\}}$.)

More Large Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$

For $1 < p < 2$, we construct new ideals of the form

$$\overline{\mathcal{I}}_{\Delta^*(w,v)},$$

that is the ideal of all operators factoring through $\Delta^*(w, v)$, for different sequences $(w, v) = \{w_i, v_i\}$.

More precisely, we build a continuum $\mathcal{C}$ of different sequences (w, v) such that $\overline{\mathcal{I}}_{\Delta^*(w,v)}$ are all different. This already produces a continuum of different ideals.

If $\mathcal{A} \subset \mathcal{C}$ one can look at the closed ideal generated by $\{\Delta^*(w, v)\}_{(w,v)\in\mathcal{A}}$ We show moreover that (with the right choice of $\mathcal{C}$) if $\mathcal{A} \neq \mathcal{B}$ then the two closed ideal generated by $\mathcal{A}$ and $\mathcal{B}$ are different.

This produces the required $2^{2^{\aleph_0}}$ ideals.

More Large Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$

For $1 < p < 2$, we construct new ideals of the form

$$\overline{\mathcal{I}}_{\Delta^*(w,v)},$$

that is the ideal of all operators factoring through $\Delta^*(w, v)$, for different sequences $(w, v) = \{w_i, v_i\}$.

More precisely, we build a continuum $\mathcal{C}$ of different sequences (w, v) such that $\overline{\mathcal{I}}_{\Delta^*(w,v)}$ are all different. This already produces a continuum of different ideals.

If $\mathcal{A} \subset \mathcal{C}$ one can look at the closed ideal generated by $\{\Delta^*(w, v)\}_{(w,v)\in\mathcal{A}}$ We show moreover that (with the right choice of $\mathcal{C}$) if $\mathcal{A} \neq \mathcal{B}$ then the two closed ideal generated by $\mathcal{A}$ and $\mathcal{B}$ are different.

This produces the required $2^{2^{\aleph_0}}$ ideals.

For $1 < p < 2$, we construct new ideals of the form

$$\overline{\mathcal{I}}_{\Delta^*(w,v)},$$

that is the ideal of all operators factoring through $\Delta^*(w, v)$, for different sequences $(w, v) = \{w_i, v_i\}$.

More precisely, we build a continuum $\mathcal{C}$ of different sequences (w, v) such that $\overline{\mathcal{I}}_{\Delta^*(w,v)}$ are all different. This already produces a continuum of different ideals.

If $\mathcal{A} \subset \mathcal{C}$ one can look at the closed ideal generated by $\{\Delta^*(w, v)\}_{(w,v)\in\mathcal{A}}$ We show moreover that (with the right choice of $\mathcal{C}$) if $\mathcal{A} \neq \mathcal{B}$ then the two closed ideal generated by $\mathcal{A}$ and $\mathcal{B}$ are different.

This produces the required $2^{2^{\aleph_0}}$ ideals.

For $1 < p < 2$, we construct new ideals of the form

$$\overline{\mathcal{I}}_{\Delta^*(w,v)},$$

that is the ideal of all operators factoring through $\Delta^*(w, v)$, for different sequences $(w, v) = \{w_i, v_i\}$.

More precisely, we build a continuum $\mathcal{C}$ of different sequences (w, v) such that $\overline{\mathcal{I}}_{\Delta^*(w,v)}$ are all different. This already produces a continuum of different ideals.

If $\mathcal{A} \subset \mathcal{C}$ one can look at the closed ideal generated by $\{\Delta^*(w, v)\}_{(w,v)\in\mathcal{A}}$ We show moreover that (with the right choice of $\mathcal{C}$) if $\mathcal{A} \neq \mathcal{B}$ then the two closed ideal generated by $\mathcal{A}$ and $\mathcal{B}$ are different.

This produces the required $2^{2^{\aleph_0}}$ ideals.

More Large Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$, main proposition

For appropriate (w, v) the operator $T = \Delta^*(w, v)$ has the following properties:

X (in our case $X^*_{p,v}$) is a Banach space with a 1-unconditional basis $\{e_i\}$ (in our case $\{h_i^v\}$). $T : X \to X$ is a norm one operator satisfying:

(a) For every M there is a finite dimensional subspace E of X such that $d(E) > M$ and $\|Tx\| \geq 1/2$ for all $x \in E$.

and

(b) For every m there is an n such that every m-dimensional subspace E of $[e_i]_{i \geq n}$ satisfies $\gamma_2(T_{|E}) \leq 2$.

We proved the following Proposition.

More Large Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$, main proposition

For appropriate (w, v) the operator $T = \Delta^*(w, v)$ has the following properties:

X (in our case $X^*_{p,v}$) is a Banach space with a 1-unconditional basis $\{e_i\}$ (in our case $\{h_i^v\}$). $T : X \to X$ is a norm one operator satisfying:

(a) For every M there is a finite dimensional subspace E of X such that $d(E) > M$ and $\|Tx\| \geq 1/2$ for all $x \in E$.

and

(b) For every m there is an n such that every m-dimensional subspace E of $[e_i]_{i \geq n}$ satisfies $\gamma_2(T_{|E}) \leq 2$.

We proved the following Proposition.

More Large Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$, main proposition

For appropriate (w, v) the operator $T = \Delta^*(w, v)$ has the following properties:

X (in our case $X^*_{p,v}$) is a Banach space with a 1-unconditional basis $\{e_i\}$ (in our case $\{h^v_i\}$). $T : X \to X$ is a norm one operator satisfying:

(a) For every M there is a finite dimensional subspace E of X such that $d(E) > M$ and $\|Tx\| \geq 1/2$ for all $x \in E$.

and
(b) For every m there is an n such that every m-dimensional subspace E of $[e_i]_{i \geq n}$ satisfies $\gamma_2(T_{|E}) \leq 2$.

We proved the following Proposition.

More Large Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$, main proposition

For appropriate (w, v) the operator $T = \Delta^*(w, v)$ has the following properties:

X (in our case $X^*_{p,v}$) is a Banach space with a 1-unconditional basis $\{e_i\}$ (in our case $\{h^v_i\}$). $T : X \to X$ is a norm one operator satisfying:

(a) For every M there is a finite dimensional subspace E of X such that $d(E) > M$ and $\|Tx\| \geq 1/2$ for all $x \in E$.

and

(b) For every m there is an n such that every m-dimensional subspace E of $[e_i]_{i \geq n}$ satisfies $\gamma_2(T_{|E}) \leq 2$.

We proved the following Proposition.

More Large Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$, main proposition

For appropriate (w, v) the operator $T = \Delta^*(w, v)$ has the following properties:

X (in our case $X^*_{p,v}$) is a Banach space with a 1-unconditional basis $\{e_i\}$ (in our case $\{h^v_i\}$). $T : X \to X$ is a norm one operator satisfying:

(a) For every M there is a finite dimensional subspace E of X such that $d(E) > M$ and $\|Tx\| \geq 1/2$ for all $x \in E$.

and
(b) For every m there is an n such that every m-dimensional subspace E of $[e_i]_{i \geq n}$ satisfies $\gamma_2(T_{|E}) \leq 2$.

We proved the following Proposition.

More Large Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$, main proposition

Proposition

Let $T : X = [e_i] \to X$ satisfy (a) and (b). Then there exist a subsequence of $\mathbb{N}$, $1 = p_1 < q_1 < p_2 < q_2 < \dots$ with the following properties:

Denoting for each k, $G_k = [e_i]_{i=p_k}^{q_k}$. Let $\mathcal{C}$ be a continuum of subsequences of $\mathbb{N}$ each two of which has a finite intersection. For each $\alpha \in \mathcal{C}$, $P_\alpha : X \to [G_k]_{k\in\alpha}$ denotes the natural basis projection and $T_\alpha = TP_\alpha$.

If $\alpha_1, \dots, \alpha_s \in \mathcal{C}$ (possibly with repetitions) and $\alpha \in \mathcal{C} \setminus \{\alpha_1, \dots, \alpha_s\}$ then for all $A_1, \dots, A_s \in L(X)$ and all $B_1, \dots, B_s \in L(X)$

$$\|T_\alpha - \sum^{s} A_i T_{\alpha_i} B_i\| \geq 1/4.$$

More Large Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$, main proposition

Proposition

Let $T : X = [e_i] \to X$ satisfy (a) and (b). Then there exist a subsequence of $\mathbb{N}$, $1 = p_1 < q_1 < p_2 < q_2 < \dots$ with the following properties:

Denoting for each k, $G_k = [e_i]_{i=p_k}^{q_k}$. Let $\mathcal{C}$ be a continuum of subsequences of $\mathbb{N}$ each two of which has a finite intersection. For each $\alpha \in \mathcal{C}$, $P_\alpha : X \to [G_k]_{k\in\alpha}$ denotes the natural basis projection and $T_\alpha = TP_\alpha$.

If $\alpha_1, \dots, \alpha_s \in \mathcal{C}$ (possibly with repetitions) and $\alpha \in \mathcal{C} \setminus \{\alpha_1, \dots, \alpha_s\}$ then for all $A_1, \dots, A_s \in L(X)$ and all $B_1, \dots, B_s \in L(X)$

$$\|T_\alpha - \sum^{s} A_i T_{\alpha_i} B_i\| \geq 1/4.$$

More Large Ideals in $L(L_p)$, $1 < p \neq 2 < \infty$, main proposition

Proposition

Let $T : X = [e_i] \to X$ satisfy (a) and (b). Then there exist a subsequence of $\mathbb{N}$, $1 = p_1 < q_1 < p_2 < q_2 < \dots$ with the following properties:

Denoting for each k, $G_k = [e_i]_{i=p_k}^{q_k}$. Let $\mathcal{C}$ be a continuum of subsequences of $\mathbb{N}$ each two of which has a finite intersection. For each $\alpha \in \mathcal{C}$, $P_\alpha : X \to [G_k]_{k \in \alpha}$ denotes the natural basis projection and $T_\alpha = TP_\alpha$.

If $\alpha_1, \dots, \alpha_s \in \mathcal{C}$ (possibly with repetitions) and $\alpha \in \mathcal{C} \setminus \{\alpha_1, \dots, \alpha_s\}$ then for all $A_1, \dots, A_s \in L(X)$ and all $B_1, \dots, B_s \in L(X)$

$$\|T_\alpha - \sum^{s} A_i T_{\alpha_i} B_i\| \geq 1/4.$$

If I have time left

back to small ideals in $L(L_1)$

Theorem.

[JPS] There are at least $2^{\aleph_0}$ small closed ideals in $L(L_1)$.

The new ideals are a familty $(\overline{\mathcal{I}}_{U_q})_{2<q<\infty}$, where $U_q : \ell_1 \to L_1\{-1,1\}^{\mathbb{N}}$ maps the unit vector basis of ℓ_1 to a carefully chosen $\Lambda(q)$-set of characters.

The following lemma is the heart of the proof.

Lemma

Let $1 \le p < q < \infty$, $\{v_1, \dots, v_N\} \subset L_q$, and let $T : L_1 \to L_1^{N^{\frac{p}{2}}}$ be an operator. Suppose that C and ϵ satisfy

1. $\max_{\epsilon_i=\pm1} \|\sum_{i=1}^N \epsilon_i v_i\|_q \le CN^{1/2}$, *and*
2. $\min_{1\le i\le N} \|Tv_i\|_1 \ge \epsilon$.

Then $\|T\| \ge (\epsilon/C)N^{\frac{q-p}{2q}}$.

back to small ideals in $L(L_1)$

Theorem.

[JPS] There are at least $2^{\aleph_0}$ small closed ideals in $L(L_1)$.

The new ideals are a familty $(\overline{\mathcal{I}}_{U_q})_{2<q<\infty}$, where $U_q : \ell_1 \to L_1\{-1,1\}^{\mathbb{N}}$ maps the unit vector basis of ℓ_1 to a carefully chosen $\Lambda(q)$-set of characters.

The following lemma is the heart of the proof.

Lemma

Let $1 \le p < q < \infty$, $\{v_1, \dots, v_N\} \subset L_q$, and let $T : L_1 \to L_1^{N^{\frac{p}{2}}}$ be an operator. Suppose that C and ϵ satisfy

1. $\max_{\epsilon_i=\pm 1} \|\sum_{i=1}^N \epsilon_i v_i\|_q \le CN^{1/2}$, *and*
2. $\min_{1\le i\le N} \|Tv_i\|_1 \ge \epsilon$.

Then $\|T\| \ge (\epsilon/C)N^{\frac{q-p}{2q}}$.

back to small ideals in $L(L_1)$

Theorem.

[JPS] There are at least $2^{\aleph_0}$ small closed ideals in $L(L_1)$.

The new ideals are a familty $(\overline{\mathcal{I}}_{U_q})_{2<q<\infty}$, where $U_q : \ell_1 \to L_1\{-1,1\}^{\mathbb{N}}$ maps the unit vector basis of ℓ_1 to a carefully chosen $\Lambda(q)$-set of characters.

The following lemma is the heart of the proof.

Lemma

Let $1 \le p < q < \infty$, $\{v_1, \dots, v_N\} \subset L_q$, and let $T : L_1 \to L_1^{N^{\frac{p}{2}}}$ be an operator. Suppose that C and ϵ satisfy

1. $\max_{\epsilon_i=\pm 1} \|\sum_{i=1}^N \epsilon_i v_i\|_q \le CN^{1/2}$, *and*
2. $\min_{1\le i\le N} \|Tv_i\|_1 \ge \epsilon$.

Then $\|T\| \ge (\epsilon/C)N^{\frac{q-p}{2q}}$.

Proof: Take u_i^* in $L_\infty^{N^{p/2}} = (L_1^{N^{\frac{p}{2}}})^*$ with $|u_i^*| \equiv 1$ so that $\langle u_i^*, Tv_i\rangle = \|Tv_i\|_1 \geq \epsilon$. Then

$$\epsilon N \leq \sum_{i=1}^{N} \langle T^* u_i^*, v_i\rangle := \int_0^1 \sum_{i=1}^{N} (T^* u_i^*)(b) v_i(b)\, db$$

$$\leq \int_0^1 \sup_{a\in[0,1]} |\sum_{i=1}^{N} (T^* u_i^*)(a) v_i(b)|\, db$$

$$=: \int_0^1 \|\sum_{i=1}^{N} v_i(b) T^* u_i^*\|_{L_\infty[0,1]} \quad db$$

$$\leq \|T\| \int_0^1 \|\sum_{i=1}^{N} v_i(b) u_i^*\|_{L_\infty^{N^{p/2}}}\, db$$

$$\leq \|T\| N^{\frac{p}{2q}} \int_0^1 \Big(\int_{[N^{\frac{p}{2}}]} |\sum_{i=1}^{N} u_i^*(c) v_i(b)|^q\, dc\Big)^{\frac{1}{q}}\, db$$

back to small ideals in $L(L_1)$

Proof: Take u_i^* in $L_\infty^{N^{p/2}} = (L_1^{N^{\frac{p}{2}}})^*$ with $|u_i^*| \equiv 1$ so that $\langle u_i^*, Tv_i \rangle = \|Tv_i\|_1 \geq \epsilon$. Then

$$\epsilon N \leq \sum_{i=1}^N \langle T^* u_i^*, v_i \rangle := \int_0^1 \sum_{i=1}^N (T^* u_i^*)(b) v_i(b)\, db$$

$$\leq \int_0^1 \sup_{a \in [0,1]} |\sum_{i=1}^N (T^* u_i^*)(a) v_i(b)|\, db$$

$$=: \int_0^1 \|\sum_{i=1}^N v_i(b) T^* u_i^* \|_{L_\infty[0,1]}\quad db$$

$$\leq \|T\| \int_0^1 \|\sum_{i=1}^N v_i(b) u_i^* \|_{L_\infty^{N^{p/2}}} db$$

$$\leq \|T\| N^{\frac{p}{2q}} \int_0^1 \Big(\int_{[N^{\frac{p}{2}}]} |\sum_{i=1}^N u_i^*(c) v_i(b)|^q\, dc \Big)^{\frac{1}{q}} db$$

back to small ideals in $L(L_1)$

Proof: Take u_i^* in $L_\infty^{N^{p/2}} = (L_1^{N^{\frac{p}{2}}})^*$ with $|u_i^*| \equiv 1$ so that $\langle u_i^*, Tv_i \rangle = \|Tv_i\|_1 \geq \epsilon$. Then

$$\epsilon N \leq \sum_{i=1}^{N} \langle T^* u_i^*, v_i \rangle := \int_0^1 \sum_{i=1}^{N} (T^* u_i^*)(b) v_i(b)\, db$$

$$\leq \int_0^1 \sup_{a \in [0,1]} |\sum_{i=1}^{N} (T^* u_i^*)(a) v_i(b)|\, db$$

$$=: \int_0^1 \|\sum_{i=1}^{N} v_i(b) T^* u_i^* \|_{L_\infty[0,1]} \quad db$$

$$\leq \|T\| \int_0^1 \|\sum_{i=1}^{N} v_i(b) u_i^* \|_{L_\infty^{N^{p/2}}}\, db$$

$$\leq \|T\| N^{\frac{p}{2q}} \int_0^1 \Big(\int_{[N^{\frac{p}{2}}]} |\sum_{i=1}^{N} u_i^*(c) v_i(b)|^q\, dc \Big)^{\frac{1}{q}}\, db$$

Proof: Take u_i^* in $L_\infty^{N^{p/2}} = (L_1^{N^{\frac{p}{2}}})^*$ with $|u_i^*| \equiv 1$ so that $\langle u_i^*, Tv_i\rangle = \|Tv_i\|_1 \ge \epsilon$. Then

$$\epsilon N \le \sum_{i=1}^{N} \langle T^* u_i^*, v_i\rangle := \int_0^1 \sum_{i=1}^{N} (T^* u_i^*)(b) v_i(b)\, db$$

$$\le \int_0^1 \sup_{a\in[0,1]} |\sum_{i=1}^{N} (T^* u_i^*)(a) v_i(b)|\, db$$

$$=: \int_0^1 \|\sum_{i=1}^{N} v_i(b) T^* u_i^*\|_{L_\infty[0,1]} \quad db$$

$$\le \|T\| \int_0^1 \|\sum_{i=1}^{N} v_i(b) u_i^*\|_{L_\infty^{N^{p/2}}}\, db$$

$$\le \|T\| N^{\frac{p}{2q}} \int_0^1 \Big(\int_{[N^{\frac{p}{2}}]} |\sum_{i=1}^{N} u_i^*(c) v_i(b)|^q\, dc\Big)^{\frac{1}{q}}\, db$$

Proof: Take u_i^* in $L_\infty^{N^{p/2}} = (L_1^{N^{\frac{p}{2}}})^*$ with $|u_i^*| \equiv 1$ so that $\langle u_i^*, Tv_i\rangle = \|Tv_i\|_1 \geq \epsilon$. Then

$$
\begin{aligned}
\epsilon N &\leq \sum_{i=1}^N \langle T^* u_i^*, v_i\rangle := \int_0^1 \sum_{i=1}^N (T^* u_i^*)(b) v_i(b)\, db \\
&\leq \int_0^1 \sup_{a\in[0,1]} |\sum_{i=1}^N (T^* u_i^*)(a) v_i(b)|\, db \\
&=: \int_0^1 \|\sum_{i=1}^N v_i(b) T^* u_i^*\|_{L_\infty[0,1]} \quad db \\
&\leq \|T\| \int_0^1 \|\sum_{i=1}^N v_i(b) u_i^*\|_{L_\infty^{N^{p/2}}}\, db \\
&\leq \|T\| N^{\frac{p}{2q}} \int_0^1 \Big(\int_{[N^{\frac{p}{2}}]} |\sum_{i=1}^N u_i^*(c) v_i(b)|^q\, dc\Big)^{\frac{1}{q}}\, db
\end{aligned}
$$

$$\epsilon N \le \sum_{i=1}^{N} \langle T^* u_i^*, v_i \rangle := \int_0^1 \sum_{i=1}^{N} (T^* u_i^*)(b) v_i(b)\, db$$

$$\le \ldots..$$

$$\le \|T\| N^{\frac{p}{2q}} \int_0^1 \Big(\int_{[N^{\frac{p}{2}}]} |\sum_{i=1}^{N} u_i^*(c) v_i(b)|^q \, dc \Big)^{\frac{1}{q}} \, db$$

$$\le \|T\| N^{\frac{p}{2q}} \Big(\int_{[N^{\frac{p}{2}}]} \int_0^1 |\sum_{i=1}^{N} u_i^*(c) v_i(b)|^q \, db\, dc \Big)^{\frac{1}{q}}$$

$$\le C\|T\| N^{\frac{p+q}{2q}}.$$

So,

$$\|T\| \ge (\epsilon/C) N^{1-\frac{p+q}{2q}} = (\epsilon/C) N^{\frac{q-p}{2q}}.$$

back to small ideals in $L(L_1)$

$$
\begin{aligned}
\epsilon N &\le \sum_{i=1}^{N} \langle T^* u_i^*, v_i \rangle := \int_0^1 \sum_{i=1}^{N} (T^* u_i^*)(b) v_i(b)\, db \\
&\le ..... \\
&\le \|T\| N^{\frac{p}{2q}} \int_0^1 \Big(\int_{[N^{\frac{p}{2}}]} |\sum_{i=1}^{N} u_i^*(c) v_i(b)|^q \, dc \Big)^{\frac{1}{q}} \, db \\
&\le \|T\| N^{\frac{p}{2q}} \Big(\int_{[N^{\frac{p}{2}}]} \int_0^1 |\sum_{i=1}^{N} u_i^*(c) v_i(b)|^q \, db\, dc \Big)^{\frac{1}{q}} \\
&\le C \|T\| N^{\frac{p+q}{2q}}.
\end{aligned}
$$

So,

$$
\|T\| \ge (\epsilon / C) N^{1 - \frac{p+q}{2q}} = (\epsilon / C) N^{\frac{q-p}{2q}}.
$$

back to small ideals in $L(L_1)$

$$\epsilon N \le \sum_{i=1}^{N} \langle T^* u_i^*, v_i \rangle := \int_0^1 \sum_{i=1}^{N} (T^* u_i^*)(b) v_i(b)\, db$$

$$\le \ldots..$$

$$\le \|T\| N^{\frac{p}{2q}} \int_0^1 \Big(\int_{[N^{\frac{p}{2}}]} \Big|\sum_{i=1}^{N} u_i^*(c) v_i(b)\Big|^q \, dc\Big)^{\frac{1}{q}} \, db$$

$$\le \|T\| N^{\frac{p}{2q}} \Big(\int_{[N^{\frac{p}{2}}]} \int_0^1 \Big|\sum_{i=1}^{N} u_i^*(c) v_i(b)\Big|^q \, db\, dc\Big)^{\frac{1}{q}}$$

$$\le C\|T\| N^{\frac{p+q}{2q}}.$$

So,

$$\|T\| \ge (\epsilon/C) N^{1-\frac{p+q}{2q}} = (\epsilon/C) N^{\frac{q-p}{2q}}.$$

back to small ideals in $L(L_1)$

$$
\begin{aligned}
\epsilon N &\le \sum_{i=1}^{N} \langle T^* u_i^*, v_i\rangle := \int_0^1 \sum_{i=1}^{N} (T^* u_i^*)(b) v_i(b)\, db \\
&\le \ldots.. \\
&\le \|T\| N^{\frac{p}{2q}} \int_0^1 \Big(\int_{[N^{\frac{p}{2}}]} \Big| \sum_{i=1}^{N} u_i^*(c) v_i(b) \Big|^q dc \Big)^{\frac{1}{q}} db \\
&\le \|T\| N^{\frac{p}{2q}} \Big(\int_{[N^{\frac{p}{2}}]} \int_0^1 \Big| \sum_{i=1}^{N} u_i^*(c) v_i(b) \Big|^q db\, dc \Big)^{\frac{1}{q}} \\
&\le C \|T\| N^{\frac{p+q}{2q}}.
\end{aligned}
$$

So,

$$
\|T\| \ge (\epsilon/C) N^{1-\frac{p+q}{2q}} = (\epsilon/C) N^{\frac{q-p}{2q}}.
$$

back to small ideals in $L(L_1)$

$$\begin{aligned}\epsilon N &\le \sum_{i=1}^{N} \langle T^* u_i^*, v_i \rangle := \int_0^1 \sum_{i=1}^{N} (T^* u_i^*)(b) v_i(b)\, db \\ &\le ..... \\ &\le \|T\| N^{\frac{p}{2q}} \int_0^1 \Big(\int_{[N^{\frac{p}{2}}]} \Big| \sum_{i=1}^{N} u_i^*(c) v_i(b) \Big|^q dc \Big)^{\frac{1}{q}} db \\ &\le \|T\| N^{\frac{p}{2q}} \Big(\int_{[N^{\frac{p}{2}}]} \int_0^1 \Big| \sum_{i=1}^{N} u_i^*(c) v_i(b) \Big|^q db\, dc \Big)^{\frac{1}{q}} \\ &\le C \|T\| N^{\frac{p+q}{2q}}.\end{aligned}$$

So,

$$\|T\| \ge (\epsilon/C) N^{1-\frac{p+q}{2q}} = (\epsilon/C) N^{\frac{q-p}{2q}}.$$

back to small ideals in $L(L_1)$

$$\begin{aligned}
\epsilon N &\le \sum_{i=1}^{N} \langle T^* u_i^*, v_i \rangle := \int_0^1 \sum_{i=1}^{N} (T^* u_i^*)(b) v_i(b)\, db \\
&\le ..... \\
&\le \|T\| N^{\frac{p}{2q}} \int_0^1 \Big(\int_{[N^{\frac{p}{2}}]} \Big| \sum_{i=1}^{N} u_i^*(c) v_i(b) \Big|^q \, dc \Big)^{\frac{1}{q}} \, db \\
&\le \|T\| N^{\frac{p}{2q}} \Big(\int_{[N^{\frac{p}{2}}]} \int_0^1 \Big| \sum_{i=1}^{N} u_i^*(c) v_i(b) \Big|^q \, db\, dc \Big)^{\frac{1}{q}} \\
&\le C \|T\| N^{\frac{p+q}{2q}}.
\end{aligned}$$

So,

$$\|T\| \ge (\epsilon/C) N^{1-\frac{p+q}{2q}} = (\epsilon/C) N^{\frac{q-p}{2q}}.$$

Small ideals in $L(L_\infty)$

When X is non reflexive, distinct closed ideals in $L(X)$ do not naturally generate distinct closed ideals in $L(X^*)$.

For example, $L(L_1)$ has at least two closed large ideals; the ideal of operators that factor through ℓ_1 and the unique maximal ideal, but $L(L_\infty)$ has no large ideals.

However, distinct small ideals in $L(L_1)$ do dualize to produce distinct small ideals in $L(L_\infty)$. Consequently, $L(L_\infty)$ contains a continuum of small ideals.

The proof uses special properties of L_1.

Small ideals in $L(L_\infty)$

When X is non reflexive, distinct closed ideals in $L(X)$ do not naturally generate distinct closed ideals in $L(X^*)$.

For example, $L(L_1)$ has at least two closed large ideals; the ideal of operators that factor through ℓ_1 and the unique maximal ideal, but $L(L_\infty)$ has no large ideals.

However, distinct small ideals in $L(L_1)$ do dualize to produce distinct small ideals in $L(L_\infty)$. Consequently, $L(L_\infty)$ contains a continuum of small ideals.

The proof uses special properties of L_1.

Small ideals in $L(L_\infty)$

When X is non reflexive, distinct closed ideals in $L(X)$ do not naturally generate distinct closed ideals in $L(X^*)$.

For example, $L(L_1)$ has at least two closed large ideals; the ideal of operators that factor through ℓ_1 and the unique maximal ideal, but $L(L_\infty)$ has no large ideals.

However, distinct small ideals in $L(L_1)$ do dualize to produce distinct small ideals in $L(L_\infty)$. Consequently, $L(L_\infty)$ contains a continuum of small ideals.

The proof uses special properties of L_1.

Small ideals in $L(L_\infty)$

When X is non reflexive, distinct closed ideals in $L(X)$ do not naturally generate distinct closed ideals in $L(X^*)$.

For example, $L(L_1)$ has at least two closed large ideals; the ideal of operators that factor through ℓ_1 and the unique maximal ideal, but $L(L_\infty)$ has no large ideals.

However, distinct small ideals in $L(L_1)$ do dualize to produce distinct small ideals in $L(L_\infty)$. Consequently, $L(L_\infty)$ contains a continuum of small ideals.

The proof uses special properties of L_1.

Thank you!