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Banach-Stone type theorems for subspaces of continuous functions

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The Banach-Stone theorem

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Theorem (Banach-Stone)

Let K_1, K_2 be compact spaces. The spaces $\mathcal{C}(K_1, \mathbb{F})$ and $\mathcal{C}(K_2, \mathbb{F})$ are isometrically isomorphic if and only if K_1 and K_2 are homeomorphic.

Replacing isometries by Banach space isomorphisms

Theorem (Amir, 1965 and Cambern, 1966)

If there exists an isomorphism $T : \mathcal{C}(K_1, \mathbb{F}) \to \mathcal{C}(K_2, \mathbb{F})$ *such that* $\|T\| \, \|T^{-1}\| < 2$*, then the spaces* K_1 *and* K_2 *are homeomorphic.*

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Theorem (Cohen, 1975)

There exist non-homeomorphic compact spaces K_1 , K_2 and an isomorphism $T : \mathcal{C}(K_1, \mathbb{R}) \to \mathcal{C}(K_2, \mathbb{R})$ with $\|T\| \, \|T^{-1}\| = 2$.

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Theorem (Cengiz, 1978, the "weak Banach-Stone theorem")

If there exists an isomorphism $T : \mathcal{C}(K_1, \mathbb{F}) \to \mathcal{C}(K_2, \mathbb{F})$, then K_1 and K_2 have the same cardinality.

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- Let $\mathcal{A}(X, \mathbb{F})$ stand for the space of affine continuous $\mathbb{F}$-valued functions on X.
- Let $\mathcal{M}^1(X)$ denote the space of Radon probability measures on X.
- If $\mu \in \mathcal{M}^1(X)$, then its barycenter $r(\mu)$ satisfies $f(r(\mu)) = \int_X f d\mu, f \in \mathcal{A}(X, \mathbb{F})$. Also, μ represents $r(\mu)$. The barycenter exists and it is unique.

Definition (Choquet ordering)

Let $\mu, \nu \in \mathcal{M}^1(X)$. Then $\mu \prec \nu$ if $\int_X k d\mu \leq \int_X k d\nu$ for each convex continuous function k on X.

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Theorem (Choquet-Bishop-de-Leeuw)

For each $x \in X$ there exist a $\prec$-maximal measure $\mu \in \mathcal{M}^1(X)$ with $r(\mu) = x$.

Simplices

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Definition (simplex)

The set X is a simplex if for each $x \in X$ there exist a unique $\prec$-maximal measure $\mu \in \mathcal{M}^1(X)$ with $r(\mu) = x$.

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Theorem

If K is a compact, then $\mathcal{C}(K, \mathbb{F}) = \mathcal{A}(\mathcal{M}^1(K), \mathbb{F})$.

Reformulation of isomorphisms theorem

Theorem (Banach-Stone)

If X, Y are Bauer simplices and $\mathcal{A}(X, \mathbb{F})$ is isometric to $\mathcal{A}(Y, \mathbb{F})$, then $\operatorname{ext} X$ is homeomorphic to $\operatorname{ext} Y$.

Theorem (Amir, Cambern)

If X, Y are Bauer simplices and there exists an isomorphism $T : \mathcal{A}(X, \mathbb{F}) \to \mathcal{A}(Y, \mathbb{F})$ with $\|T\| \left\|T^{-1}\right\| < 2$, then $\operatorname{ext} X$ is homeomorphic to $\operatorname{ext} Y$.

Theorem (Cohen)

If X, Y are Bauer simplices and there exists an isomorphism $T : \mathcal{A}(X, \mathbb{F}) \to \mathcal{A}(Y, \mathbb{F})$, then the cardinality of $\operatorname{ext} X$ is equal to the cardinality of $\operatorname{ext} Y$.

Results of Chu and Cohen

Theorem (Chu-Cohen, 1992)

Given compact convex sets X and Y, the sets $\operatorname{ext} X$ and $\operatorname{ext} Y$ are homeomorphic provided there exists an isomorphism $T : \mathcal{A}(X, \mathbb{R}) \to \mathcal{A}(Y, \mathbb{R})$ with $\|T\| \, \|T^{-1}\| < 2$ and one of the following conditions hold:

- *(i) X and Y are simplices such that their extreme points are weak peak points;*
- *(ii) X and Y are metrizable and their extreme points are weak peak points.*

Definition

A point $x \in X$ is a weak peak point if given $\varepsilon \in (0, 1)$ and an open set $U \subset X$ containing x, there exists a in the unit ball $B_{\mathcal{A}(X,\mathbb{F})}$ of $\mathcal{A}(X, \mathbb{F})$ such that $|a| < \varepsilon$ on $\operatorname{ext} X \setminus U$ and $a(x) > 1 - \varepsilon$.

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- If X is a Bauer simplex, then $\mathcal{A}(X,\mathbb{F}) = \mathcal{C}(\operatorname{ext} X,\mathbb{F})$, thus the assumption of weak peak points is always fulfilled in this case.

The assumption of weak peak points

Theorem (Hess, 1978)

For each $\varepsilon \in (0,1)$ there exist metrizable simplices X, Y and an isomorphism $T : \mathcal{A}(X,\mathbb{R}) \to \mathcal{A}(Y,\mathbb{R})$ with $\|T\| \left\|T^{-1}\right\| < 1+\varepsilon$ such that $\operatorname{ext} X$ is not homeomorphic to $\operatorname{ext} Y$.

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Theorem (Ludvik, Spurny, 2011)

Given compact convex sets X *and* Y *, the sets* $\operatorname{ext} X$ *and* $\operatorname{ext} Y$ *are homeomorphic provided there exists an isomorphism* $T : \mathcal{A}(X,\mathbb{R}) \to \mathcal{A}(Y,\mathbb{R})$ *with* $\|T\| \, \|T^{-1}\| < 2$*, extreme points of* X *and* Y *are weak peak points and both* $\operatorname{ext} X$ *and* $\operatorname{ext} Y$ *are Lindelof.*

Small bound isomorphisms of spaces of affine continuous functions

Theorem (Dostal, Spurny)

Given compact convex sets X and Y, the sets $\operatorname{ext} X$ and $\operatorname{ext} Y$ are homeomorphic provided there exists an isomorphism $T : \mathcal{A}(X, \mathbb{R}) \to \mathcal{A}(Y, \mathbb{R})$ with $\|T\| \, \|T^{-1}\| < 2$, and extreme points of X and Y are weak peak points.

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Theorem (R., Spurny)

Given compact convex sets X and Y, the sets $\operatorname{ext} X$ and $\operatorname{ext} Y$ are homeomorphic provided there exists an isomorphism $T : \mathcal{A}(X,\mathbb{C}) \to \mathcal{A}(Y,\mathbb{C})$ with $\|T\| \, \|T^{-1}\| < 2$ and extreme points of X and Y are weak peak points.

General subspaces of continuous functions

- If $\mathcal{H}$ is a closed subspace of $\mathcal{C}(K, \mathbb{F})$, then the closed dual unit ball $B_{\mathcal{H}^*}$ is a compact convex set with its *weak**-topology.

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- K may be continuously mapped in $B_{\mathcal{H}^*}$ via the evaluation mapping $\phi : x \mapsto \phi_x$, where ϕ_x is a point in $B_{\mathcal{H}^*}$ defined by

$$\phi_x(h) = h(x), \quad h \in \mathcal{H}.$$

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- The Choquet boundary $\operatorname{Ch}_{\mathcal{H}} K$ of $\mathcal{H}$ is the set of points $x \in K$ satisfying that ϕ_x is an extreme point of $B_{\mathcal{H}^*}$.
- A point $x \in \operatorname{Ch}_{\mathcal{H}} K$ is a *weak peak point* (with respect to $\mathcal{H}$), if for each neighbourhood U of x and $\varepsilon \in (0, 1)$ there exists $h \in B_{\mathcal{H}}$ such that $h(x) > 1 - \varepsilon$ and $|h| < \varepsilon$ on $\operatorname{Ch}_{\mathcal{H}} K \setminus U$.

Example

- If $\mathcal{H} = \mathcal{C}(K, \mathbb{F})$, then $\operatorname{Ch}_{\mathcal{H}} K = K$ and by the Urysohn's Lemma, each point of K is a weak peak point.
- If $\mathcal{H} = \mathcal{A}(X, \mathbb{F})$, then $\operatorname{Ch}_{\mathcal{H}} X = \operatorname{ext} X$.

Results on general subspaces of continuous functions

Example

- If $\mathcal{H} = \mathcal{C}(K, \mathbb{F})$, then $\mathrm{Ch}_{\mathcal{H}} K = K$ and by the Urysohn's Lemma, each point of K is a weak peak point.
- If $\mathcal{H} = \mathcal{A}(X, \mathbb{F})$, then $\mathrm{Ch}_{\mathcal{H}} X = \mathrm{ext}\, X$.

Theorem (R., Spurný)

For $i = 1, 2$, let $\mathcal{H}_i$ be a closed subspace of $\mathcal{C}(K_i, \mathbb{F})$ for some compact Hausdorff space K_i. Assume that each point of the Choquet boundary $\mathrm{Ch}_{\mathcal{H}_i} K_i$ is a weak peak point.

- *Let $T\colon \mathcal{H}_1 \to \mathcal{H}_2$ be an isomorphism satisfying $\|T\| \cdot \|T^{-1}\| < 2$. Then $\mathrm{Ch}_{\mathcal{H}_1} K_1$ is homeomorphic to $\mathrm{Ch}_{\mathcal{H}_2} K_2$.*
- *Let $T\colon \mathcal{H}_1 \to \mathcal{H}_2$ be an isomorphism. Then $\mathrm{Ch}_{\mathcal{H}_1} K_1$ and $\mathrm{Ch}_{\mathcal{H}_2} K_2$ have the same cardinality.*

The isomorphic Banach-Stone property

Definition

The Banach space E has the *isomorphic Banach-Stone property* (IBSP), if there exists $\alpha > 1$ such that for all compact spaces K_1, K_2, the existence of an isomorphism $T : \mathcal{C}(K_1, E) \to \mathcal{C}(K_2, E)$ with $\|T\| \|T^{-1}\| < \alpha$ implies that K_1 and K_2 are homeomorphic. The largest possible constant α is called the Banach-Stone constant of E and is denoted by $BS(E)$.

Example

It is known that the following Banach spaces E have the IBSP:

- finite-dimensional Hilbert spaces, and $BS(E) \geq \sqrt{2}$ (Cambern, 1976),
- uniformly convex spaces, and $BS(E) \geq (1 - \delta_E(1))^{-1}$, where $\delta_E : [0,2] \to [0,1]$ is the modulus of convexity of E (Cambern, 1985),
- uniformly non-square spaces (Behrends, Cambern, 1988),
- reflexive spaces with $\lambda(E) > 1$, and $BS(E) \geq \lambda(E)$ (Cidral, Galego, R.-Villamizar, 2015), where

$$\lambda(E) = \inf\{\max\{\|e_1 + \lambda e_2\| : \lambda \in S_{\mathbb{F}}\} : e_1, e_2 \in S_E\}.$$

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- For each uniformly convex space E with dimension at least two, $(1 - \delta_E(1))^{-1} < \lambda(E)$ (Cidral, Galego, R.-Villamizar, 2015).

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- For each uniformly convex space E with dimension at least two, $(1 - \delta_E(1))^{-1} < \lambda(E)$ (Cidral, Galego, R.-Villamizar, 2015).
- It holds that $2^{\frac{1}{p}} = \lambda(l_p) = BS(l_p)$ for $2 \leq p < \infty$ (Cidral, Galego, R.-Villamizar, 2015).

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- It holds that $2^{\frac{1}{p}} = \lambda(l_p) = BS(l_p)$ for $2 \leq p < \infty$ (Cidral, Galego, R.-Villamizar, 2015).
- For real Banach spaces E, the fact that $\lambda(E) > 1$ implies that E is reflexive.

Subspaces of vector-valued functions

- (Al-Halees, Fleming, 2015) Several results in the spirit of the isomorphic Banach-stone theorem for subspaces $\mathcal{H} \subseteq \mathcal{C}(K, E)$, that are so called $\mathcal{C}(K, \mathbb{F})$-modules, that is, closed with respect to multiplication by functions from $\mathcal{C}(K, \mathbb{F})$.

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- The authors posed a question if this module condition could be weakened or removed.

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- If $\mathcal{H} = \mathcal{A}(X, E)$, the space of affine continuous E-valued functions on a compact convex set X, then $\mathrm{Ch}_{\mathcal{H}}\, X = \mathrm{ext}\, X$ and a the definition of weak peak points of $\mathcal{H}$ coincides with the one for $\mathcal{A}(X, \mathbb{F})$.

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Theorem (R., Spurný)

Let for $i = 1, 2$, $\mathcal{H}_i$ be a closed subpace of $\mathcal{C}(K_i, E_i)$ for some compact space K_i and a reflexive Banach space E_i over $\mathbb{F}$ with $\lambda(E_i) > 1$. Let each point of $\operatorname{Ch}_{\mathcal{H}_i} K_i$ be a weak peak point. If there exists an isomorphism $T : \mathcal{H}_1 \to \mathcal{H}_2$ with $\|T\| \|T^{-1}\| < \min\{\lambda(E_1), \lambda(E_2)\}$, then $\operatorname{Ch}_{\mathcal{H}_1} K_1$ and $\operatorname{Ch}_{\mathcal{H}_2} K_2$ are homeomorphic.

The weak Banach-Stone property

Definition

A Banach space E has the *weak Banach-Stone property* (WBSP) if for all compact spaces K_1, K_2, the existence of an isomorphism $T : \mathcal{C}(K_1, E) \to \mathcal{C}(K_2, E)$ implies that K_1 and K_2 are either both finite or they have the same cardinality.

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Example

It is known that the following Banach spaces have the (WBSP):

- spaces having nontrivial Rademacher cotype, such that either E is separable or E^* has the Radon-Nikodym property (Candido, Galego, 2013),
- spaces not containing an isomorphic copy of c_0 (Galego, Rincón-Villamizar, 2015).

Thw weak Banach-Stone property

Theorem (R., Spurný)

Let for $i = 1, 2$, $\mathcal{H}_i$ be a closed subpace of $\mathcal{C}(K_i, E_i)$ for some compact space K_i and a Banach space E_i over $\mathbb{F}$ not containing an isomorphic copy of c_0. Let each point of $\mathrm{Ch}_{\mathcal{H}_i} K_i$ be a weak peak point. If there exists an isomorphism $T : \mathcal{H}_1 \to \mathcal{H}_2$, then either both the spaces $\mathrm{Ch}_{\mathcal{H}_1} K_1$ and $\mathrm{Ch}_{\mathcal{H}_2} K_2$ are finite or they have the same cardinality.

Acknowledgement

Thank you.