On Banach spaces which are weak* sequentially dense in its bidual

José Rodríguez

Universidad de Murcia

Workshop on Banach spaces and Banach lattices

Madrid, September 9, 2019

Research supported by
Agencia Estatal de Investigación/FEDER (MTM2017-86182-P) and Fundación Séneca (20797/PI/18)

Throughout this talk X is a Banach space.

Throughout this talk X is a Banach space.

Goldstine's theorem

B_X is w^*-dense in $B_{X^{**}}$.

Throughout this talk X is a Banach space.

Goldstine's theorem

B_X is w^*-dense in $B_{X^{**}}$. Therefore, X is w^*-dense in X^{**}.

Throughout this talk X is a Banach space.

Goldstine's theorem

B_X is w^*-dense in $B_{X^{**}}$. Therefore, X is w^*-dense in X^{**}.

Notation

$X \in \mathfrak{SD}$ iff X is w^*-**sequentially** dense in X^{**}.

Throughout this talk X is a Banach space.

Goldstine's theorem

B_X is w^*-dense in $B_{X^{**}}$. Therefore, X is w^*-dense in X^{**}.

Notation

$X \in \mathfrak{SD}$ iff X is w^*-**sequentially** dense in X^{**}.

$X \in \mathfrak{SD}$ if

- X is reflexive (obvious)

Throughout this talk X is a Banach space.

Goldstine's theorem

B_X is w^*-dense in $B_{X^{**}}$. Therefore, X is w^*-dense in X^{**}.

Notation

$X \in \mathfrak{SD}$ iff X is w^*-**sequentially** dense in X^{**}.

$X \in \mathfrak{SD}$ if

- X is reflexive (obvious)
- X^* is separable
  [$\iff (B_{X^{**}}, w^*)$ is metrizable]

Throughout this talk X is a Banach space.

Goldstine's theorem

B_X is w^*-dense in $B_{X^{**}}$. Therefore, X is w^*-dense in X^{**}.

Notation

$X \in \mathfrak{S}\mathfrak{D}$ iff X is w^*-**sequentially** dense in X^{**}.

$X \in \mathfrak{S}\mathfrak{D}$ if

- X is reflexive (obvious)
- X^* is separable
  [$\Longleftrightarrow$ $(B_{X^{**}}, w^*)$ is metrizable]

$(B_{X^{**}}, w^*)$ is Fréchet-Urysohn $\Longrightarrow$ $X \in \mathfrak{S}\mathfrak{D}$

Throughout this talk X is a Banach space.

Goldstine's theorem

B_X is w^*-dense in $B_{X^{**}}$. Therefore, X is w^*-dense in X^{**}.

Notation

$X \in \mathfrak{S}\mathfrak{D}$ iff X is w^*-**sequentially** dense in X^{**}.

$X \in \mathfrak{S}\mathfrak{D}$ if

- X is reflexive (obvious)
- X^* is separable [$\iff (B_{X^{**}}, w^*)$ is metrizable]

$X \notin \mathfrak{S}\mathfrak{D}$ if

- X is weakly sequentially complete and non-reflexive (e.g. ℓ_1 and L_1)

$$(B_{X^{**}}, w^*) \text{ is Fréchet-Urysohn } \implies X \in \mathfrak{S}\mathfrak{D}$$

Throughout this talk X is a Banach space.

Goldstine's theorem

B_X is w^*-dense in $B_{X^{**}}$. Therefore, X is w^*-dense in X^{**}.

Notation

$X \in \mathfrak{SD}$ iff X is w^*-**sequentially** dense in X^{**}.

$X \in \mathfrak{SD}$ if

- X is reflexive (obvious)
- X^* is separable
  [$\iff (B_{X^{**}}, w^*)$ is metrizable]

$X \notin \mathfrak{SD}$ if

- X is weakly sequentially complete and non-reflexive (e.g. ℓ_1 and L_1)
- $X = c_0(\Gamma)$ for uncountable Γ

$$(B_{X^{**}}, w^*) \text{ is Fréchet-Urysohn} \implies X \in \mathfrak{SD}$$

Throughout this talk X is a Banach space.

Goldstine's theorem

B_X is w^*-dense in $B_{X^{**}}$. Therefore, X is w^*-dense in X^{**}.

Notation

$X \in \mathfrak{SD}$ iff X is w^*-**sequentially** dense in X^{**}.

$X \in \mathfrak{SD}$ if

- X is reflexive (obvious)
- X^* is separable
  [$\iff (B_{X^{**}}, w^*)$ is metrizable]

$X \notin \mathfrak{SD}$ if

- X is weakly sequentially complete and non-reflexive (e.g. ℓ_1 and L_1)
- $X = c_0(\Gamma)$ for uncountable Γ

$(B_{X^{**}}, w^*)$ is Fréchet-Urysohn $\implies X \in \mathfrak{SD} \implies X \not\supset \ell_1$.

Throughout this talk X is a Banach space.

Goldstine's theorem

B_X is w^*-dense in $B_{X^{**}}$. Therefore, X is w^*-dense in X^{**}.

Notation

$X \in \mathfrak{SD}$ iff X is w^*-**sequentially** dense in X^{**}.

$X \in \mathfrak{SD}$ if

- X is reflexive (obvious)
- X^* is separable [$\iff (B_{X^{**}}, w^*)$ is metrizable]

$X \notin \mathfrak{SD}$ if

- X is weakly sequentially complete and non-reflexive (e.g. ℓ_1 and L_1)
- $X = c_0(\Gamma)$ for uncountable Γ

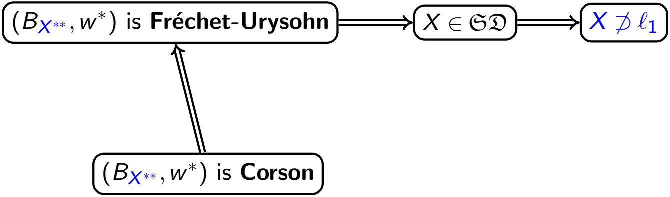
$$(B_{X^{**}}, w^*) \text{ is Fréchet-Urysohn} \implies X \in \mathfrak{SD} \implies X \not\supset \ell_1.$$

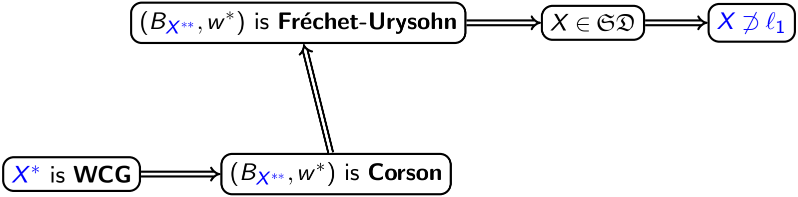
Theorem (Odell-Rosenthal, Bourgain-Fremlin-Talagrand)

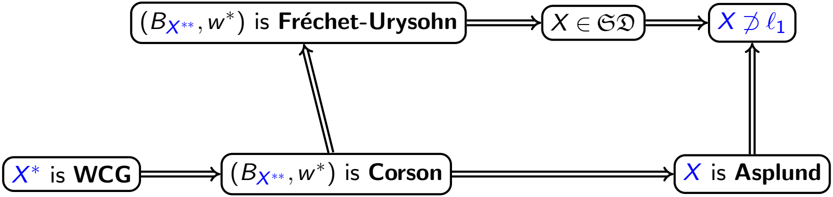
Suppose X is **separable**. Then:

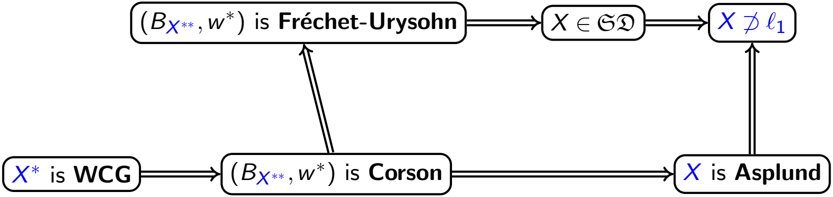
$$(B_{X^{**}}, w^*) \text{ is Fréchet-Urysohn} \iff X \in \mathfrak{SD} \iff X \not\supset \ell_1.$$

$(B_{X^{**}}, w^*)$ is **Fréchet-Urysohn** $\Longrightarrow$ $X \in \mathfrak{S}\mathfrak{D}$ $\Longrightarrow$ $X \not\supset \ell_1$


$(B_{X^{**}}, w^*)$ is **Fréchet-Urysohn**
$X \in \mathfrak{S}\mathfrak{D}$
$X \not\supset \ell_1$
$(B_{X^{**}}, w^*)$ is **Corson**

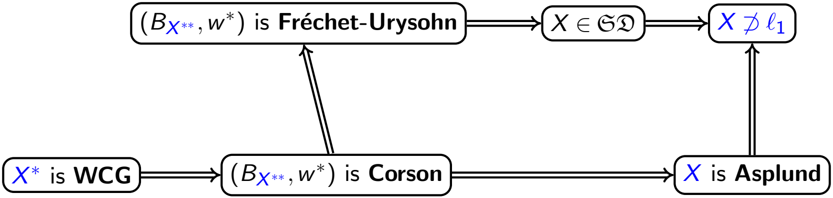

$(B_{X^{**}}, w^*)$ is **Fréchet-Urysohn**
$X \in \mathfrak{S}\mathfrak{D}$
$X \not\supset \ell_1$
X^* is **WCG**
$(B_{X^{**}}, w^*)$ is **Corson**


$(B_{X^{**}}, w^*)$ is **Fréchet-Urysohn**
$X \in \mathfrak{S}\mathfrak{D}$
$X \not\supset \ell_1$
X^* is **WCG**
$(B_{X^{**}}, w^*)$ is **Corson**
X is **Asplund**



Theorem (Deville-Godefroy, Orihuela)

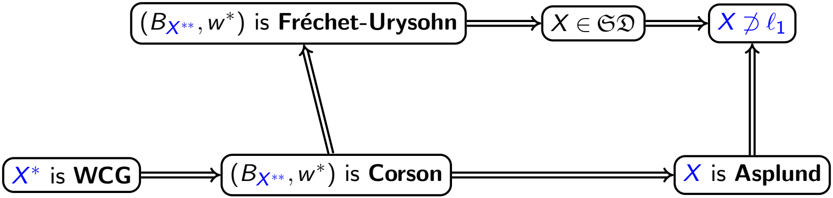
$(B_{X^{**}}, w^*)$ is Corson $\iff$ $X \in \mathfrak{SD}$ and X is Asplund.



Theorem (Deville-Godefroy, Orihuela)

$$(B_{X^{**}}, w^*) \text{ is Corson} \iff X \in \mathfrak{S}\mathfrak{D} \text{ and } X \text{ is Asplund.}$$

A **Banach lattice** X is Asplund if and only if $X \not\supset \ell_1$.



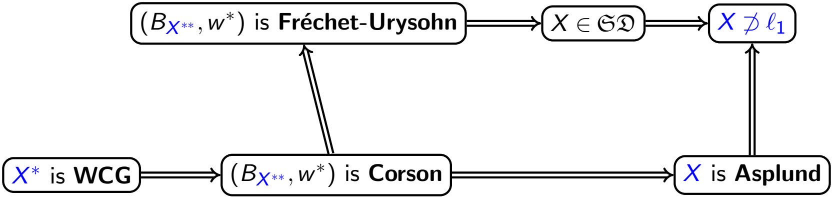
Theorem (Deville-Godefroy, Orihuela)

$(B_{X^{**}}, w^*)$ is Corson $\iff$ $X \in \mathfrak{S}\mathfrak{D}$ and X is Asplund.

A **Banach lattice** X is Asplund if and only if $X \not\supset \ell_1$.

Corollary

Suppose X is a **Banach lattice**. Then $(B_{X^{**}}, w^*)$ is Corson $\iff$ $X \in \mathfrak{S}\mathfrak{D}$.



Theorem (Deville-Godefroy, Orihuela)

$(B_{X^{**}}, w^*)$ is Corson $\iff$ $X \in \mathfrak{SD}$ and X is Asplund.

A **Banach lattice** X is Asplund if and only if $X \not\supset \ell_1$.

Corollary

Suppose X is a **Banach lattice**. Then $(B_{X^{**}}, w^*)$ is Corson $\iff$ $X \in \mathfrak{SD}$.

Question

$X \in \mathfrak{SD} \implies (B_{X^{**}}, w^*)$ is Fréchet-Urysohn **???**

“Vague” question

Which topological properties does (B_{X^*}, w^*) enjoy whenever $X \in \mathfrak{S}\mathfrak{D}$ **???**

"Vague" question

Which topological properties does (B_{X^*}, w^*) enjoy whenever $X \in \mathfrak{SD}$ **???**

Given a set $A \subset X^*$, we write

$$S_1(A) := \{x^* \in X^* : \exists (x_n^*) \subset A \text{ such that } w^*\text{-}\lim_{n\to\infty} x_n^* = x^*\}.$$

"Vague" question

Which topological properties does (B_{X^*}, w^*) enjoy whenever $X \in \mathfrak{SD}$ **???**

Given a set $A \subset X^*$, we write

$$S_1(A) := \{x^* \in X^* : \exists (x_n^*) \subset A \text{ such that } w^*\text{-}\lim_{n\to\infty} x_n^* = x^*\}.$$

Definition

We say that (B_{X^*}, w^*) is

1. **Efremov** iff $S_1(C) = \overline{C}^{w^*}$ for every convex set $C \subset B_{X^*}$;

"Vague" question

Which topological properties does (B_{X^*}, w^*) enjoy whenever $X \in \mathfrak{SD}$ **???**

Given a set $A \subset X^*$, we write

$$S_1(A) := \left\{x^* \in X^* : \exists (x_n^*) \subset A \text{ such that } w^*\text{-}\lim_{n\to\infty} x_n^* = x^*\right\}.$$

Definition

We say that (B_{X^*}, w^*) is

1. **Efremov** iff $S_1(C) = \overline{C}^{w^*}$ for every convex set $C \subset B_{X^*}$;
2. **convexly sequential** iff for every convex set $C \subset B_{X^*}$ we have

$$S_1(C) = C \quad \Longrightarrow \quad C \text{ is } w^*\text{-closed};$$

"Vague" question

Which topological properties does (B_{X^*}, w^*) enjoy whenever $X \in \mathfrak{S}\mathfrak{D}$ **???**

Given a set $A \subset X^*$, we write

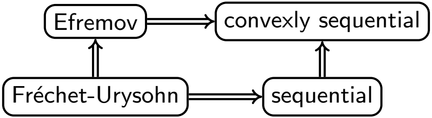
$$S_1(A) := \{x^* \in X^* : \exists (x_n^*) \subset A \text{ such that } w^*\text{-}\lim_{n\to\infty} x_n^* = x^*\}.$$

Definition

We say that (B_{X^*}, w^*) is

1. **Efremov** iff $S_1(C) = \overline{C}^{w^*}$ for every convex set $C \subset B_{X^*}$;
2. **convexly sequential** iff for every convex set $C \subset B_{X^*}$ we have

$$S_1(C) = C \quad \Longrightarrow \quad C \text{ is } w^*\text{-closed};$$



"Vague" question

Which topological properties does (B_{X^*}, w^*) enjoy whenever $X \in \mathfrak{SD}$???

Given a set $A \subset X^*$, we write

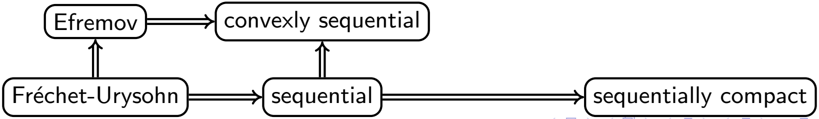
$$S_1(A) := \{x^* \in X^* : \exists (x_n^*) \subset A \text{ such that } w^*\text{-}\lim_{n\to\infty} x_n^* = x^*\}.$$

Definition

We say that (B_{X^*}, w^*) is

1. **Efremov** iff $S_1(C) = \overline{C}^{w^*}$ for every convex set $C \subset B_{X^*}$;
2. **convexly sequential** iff for every convex set $C \subset B_{X^*}$ we have

$$S_1(C) = C \implies C \text{ is } w^*\text{-closed};$$



“Vague” question

Which topological properties does (B_{X^*}, w^*) enjoy whenever $X \in \mathfrak{SD}$ **???**

Given a set $A \subset X^*$, we write

$$S_1(A) := \{x^* \in X^* : \exists (x_n^*) \subset A \text{ such that } w^*\text{-}\lim_{n\to\infty} x_n^* = x^*\}.$$

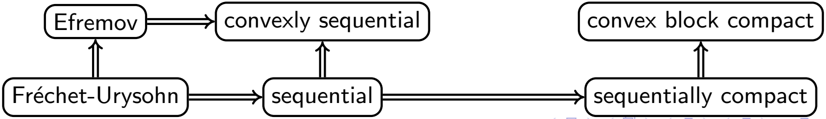
Definition

We say that (B_{X^*}, w^*) is

1. **Efremov** iff $S_1(C) = \overline{C}^{w^*}$ for every convex set $C \subset B_{X^*}$;
2. **convexly sequential** iff for every convex set $C \subset B_{X^*}$ we have

$$S_1(C) = C \quad \Longrightarrow \quad C \text{ is } w^*\text{-closed};$$

3. **convex block compact** iff every sequence in B_{X^*} admits a w^*-convergent convex block subsequence.



"Vague" question

Which topological properties does (B_{X^*}, w^*) enjoy whenever $X \in \mathfrak{S}\mathfrak{D}$???

Given a set $A \subset X^*$, we write

$$S_1(A) := \{x^* \in X^* : \exists (x_n^*) \subset A \text{ such that } w^*\text{-}\lim_{n\to\infty} x_n^* = x^*\}.$$

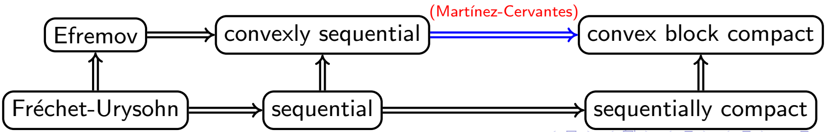
Definition

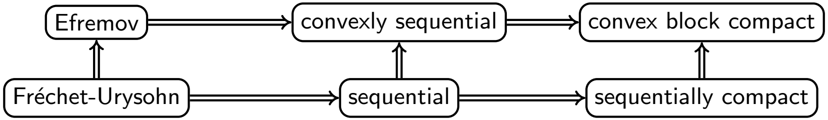
We say that (B_{X^*}, w^*) is

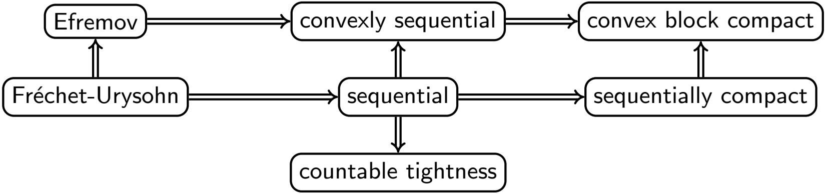
1. **Efremov** iff $S_1(C) = \overline{C}^{w^*}$ for every convex set $C \subset B_{X^*}$;
2. **convexly sequential** iff for every convex set $C \subset B_{X^*}$ we have

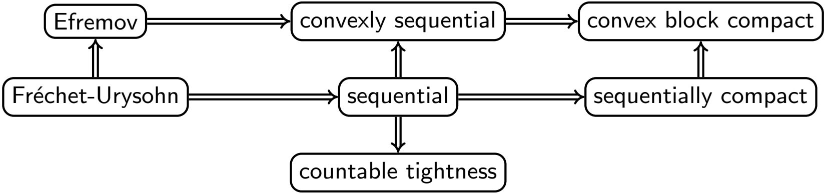
$$S_1(C) = C \quad \Longrightarrow \quad C \text{ is } w^*\text{-closed};$$

3. **convex block compact** iff every sequence in B_{X^*} admits a w^*-convergent convex block subsequence.



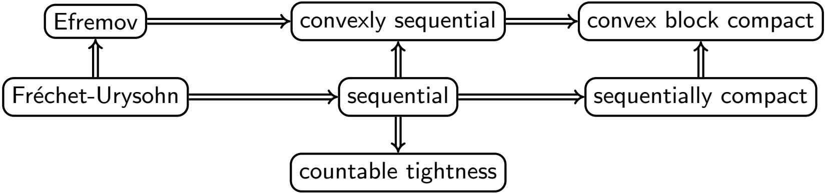

Efremov
convexly sequential
convex block compact
Fréchet-Urysohn
sequential
sequentially compact


Efremov
convexly sequential
convex block compact
Fréchet-Urysohn
sequential
sequentially compact
countable tightness



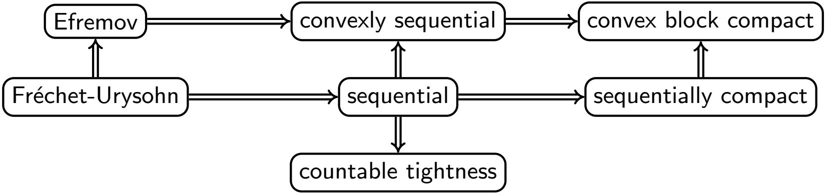
Example

Let $X = JL_2(\mathscr{F})$ be the **Johnson-Lindenstrauss space** associated to a MAD family $\mathscr{F}$. Then $X \in \mathfrak{S}\mathfrak{D}$



Example

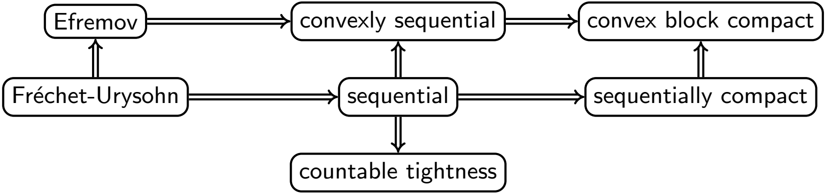
Let $X = JL_2(\mathscr{F})$ be the **Johnson-Lindenstrauss space** associated to a MAD family $\mathscr{F}$. Then $X \in \mathfrak{S}\mathfrak{D}$ and (B_{X^*}, w^*) is **not** Fréchet-Urysohn.



Example

Let $X = JL_2(\mathscr{F})$ be the **Johnson-Lindenstrauss space** associated to a MAD family $\mathscr{F}$. Then $X \in \mathfrak{S}\mathfrak{D}$ and (B_{X^*}, w^*) is **not** Fréchet-Urysohn. Moreover:

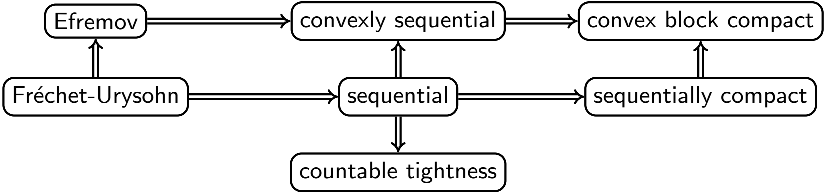
- (B_{X^*}, w^*) is sequential (Martínez-Cervantes);



Example

Let $X = JL_2(\mathscr{F})$ be the **Johnson-Lindenstrauss space** associated to a MAD family $\mathscr{F}$. Then $X \in \mathfrak{S}\mathfrak{D}$ and (B_{X^*}, w^*) is **not** Fréchet-Urysohn. Moreover:

- (B_{X^*}, w^*) is sequential (Martínez-Cervantes);
- Under CH, there exist MAD families $\mathscr{F}$ for which (B_{X^*}, w^*) is/isn't Efremov (Avilés, Martínez-Cervantes, R.).



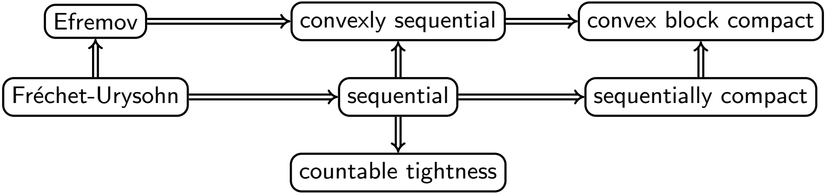
Example

Let $X = JL_2(\mathscr{F})$ be the **Johnson-Lindenstrauss space** associated to a MAD family $\mathscr{F}$. Then $X \in \mathfrak{S}\mathfrak{D}$ and (B_{X^*}, w^*) is **not** Fréchet-Urysohn. Moreover:

- (B_{X^*}, w^*) is sequential (Martínez-Cervantes);
- Under CH, there exist MAD families $\mathscr{F}$ for which (B_{X^*}, w^*) is/isn't Efremov (Avilés, Martínez-Cervantes, R.).

Theorem

If $X \in \mathfrak{S}\mathfrak{D}$, then (B_{X^*}, w^*) has countable tightness [Hernández-Rubio]



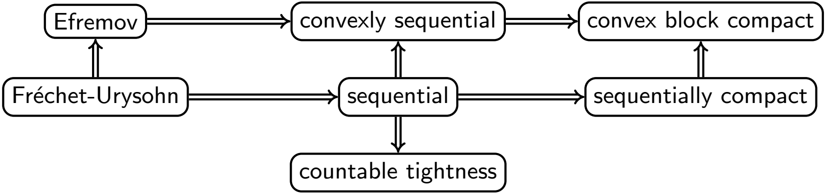
Example

Let $X = JL_2(\mathscr{F})$ be the **Johnson-Lindenstrauss space** associated to a MAD family $\mathscr{F}$. Then $X \in \mathfrak{SD}$ and (B_{X^*}, w^*) is **not** Fréchet-Urysohn. Moreover:

- (B_{X^*}, w^*) is sequential (Martínez-Cervantes);
- Under CH, there exist MAD families $\mathscr{F}$ for which (B_{X^*}, w^*) is/isn't Efremov (Avilés, Martínez-Cervantes, R.).

Theorem

If $X \in \mathfrak{SD}$, then (B_{X^*}, w^*) has countable tightness [Hernández-Rubio] and is convexly sequential [Avilés, Martínez-Cervantes, R.].



Example

Let $X = JL_2(\mathscr{F})$ be the **Johnson-Lindenstrauss space** associated to a MAD family $\mathscr{F}$. Then $X \in \mathfrak{S}\mathfrak{D}$ and (B_{X^*}, w^*) is **not** Fréchet-Urysohn. Moreover:

- (B_{X^*}, w^*) is sequential (Martínez-Cervantes);
- Under CH, there exist MAD families $\mathscr{F}$ for which (B_{X^*}, w^*) is/isn't Efremov (Avilés, Martínez-Cervantes, R.).

Theorem

If $X \in \mathfrak{S}\mathfrak{D}$, then (B_{X^*}, w^*) has countable tightness [Hernández-Rubio]
and is convexly sequential [Avilés, Martínez-Cervantes, R.].

Question

$$X \in \mathfrak{S}\mathfrak{D} \implies (B_{X^*}, w^*) \text{ is sequential or sequentially compact } \textbf{???}$$

$$X \in \mathfrak{S}\mathfrak{D} \quad \Longrightarrow \quad (B_{X^*}, w^*) \text{ is convexly sequential.}$$

Ingredients:

$X \in \mathfrak{S}\mathfrak{D} \quad \Longrightarrow \quad (B_{X^*}, w^*)$ is convexly sequential.

Ingredients:

Theorem (Bourgain)

$X \not\supset \ell_1 \Longrightarrow (B_{X^*}, w^*)$ is convex block compact.

$X \in \mathfrak{S}\mathfrak{D} \implies (B_{X^*}, w^*)$ is convexly sequential.

Ingredients:

Theorem (Bourgain)

$X \not\supset \ell_1 \implies (B_{X^*}, w^*)$ is convex block compact.

Definition

Let $K \subset X^*$ be convex w^*-compact.
A set $B \subset K$ is a **boundary** of K iff

$$\forall x \in X \quad \exists x_0^* \in B \quad \text{such that}$$
$$x_0^*(x) = \sup\{x^*(x) : x^* \in K\}.$$

$X \in \mathfrak{SD} \implies (B_{X^*}, w^*)$ is convexly sequential.

Ingredients:

Theorem (Bourgain)

$X \not\supset \ell_1 \implies (B_{X^*}, w^*)$ is convex block compact.

Definition

Let $K \subset X^*$ be convex w^*-compact.
A set $B \subset K$ is a **boundary** of K iff

$$\forall x \in X \quad \exists x_0^* \in B \quad \text{such that} \quad x_0^*(x) = \sup\{x^*(x) : x^* \in K\}.$$

Theorem (Efremov, Godefroy)

$X \in \mathfrak{SD} \implies K = \overline{\mathrm{conv}(B)}^{\|\cdot\|}$ for all K and B as above.

$X \in \mathfrak{S}\mathfrak{D} \implies (B_{X^*}, w^*)$ is convexly sequential.

Ingredients:

Theorem (Bourgain)

$X \not\supset \ell_1 \implies (B_{X^*}, w^*)$ is convex block compact.

Definition

Let $K \subset X^*$ be convex w^*-compact.
A set $B \subset K$ is a **boundary** of K iff

$$\forall x \in X \quad \exists x_0^* \in B \quad \text{such that} \quad x_0^*(x) = \sup\{x^*(x) : x^* \in K\}.$$

Theorem (Efremov, Godefroy)

$X \in \mathfrak{S}\mathfrak{D} \implies K = \overline{\mathrm{conv}(B)}^{\|\cdot\|}$ for all K and B as above.

Sketch of proof of the implication:

Take $C \subset B_{X^*}$ convex.

CLAIM: $\overline{C}^{w^*} = S_1(S_1(C))$.

Why?

$X \in \mathfrak{S}\mathfrak{D} \implies (B_{X^*}, w^*)$ is convexly sequential.

Ingredients:

Theorem (Bourgain)

$X \not\supset \ell_1 \implies (B_{X^*}, w^*)$ is convex block compact.

Definition

Let $K \subset X^*$ be convex w^*-compact.
A set $B \subset K$ is a **boundary** of K iff

$$\forall x \in X \quad \exists x_0^* \in B \quad \text{such that} \quad x_0^*(x) = \sup\{x^*(x) : x^* \in K\}.$$

Theorem (Efremov, Godefroy)

$X \in \mathfrak{S}\mathfrak{D} \implies K = \overline{\mathrm{conv}(B)}^{\|\cdot\|}$ for all K and B as above.

Sketch of proof of the implication:

Take $C \subset B_{X^*}$ convex.

Claim: $\overline{C}^{w^*} = S_1(S_1(C))$.

Why?

- $S_1(C)$ is a boundary of $\overline{C}^{w^*}$.

$X \in \mathfrak{S}\mathfrak{D} \implies (B_{X^*}, w^*)$ is convexly sequential.

Ingredients:

Theorem (Bourgain)

$X \not\supset \ell_1 \implies (B_{X^*}, w^*)$ is convex block compact.

Definition

Let $K \subset X^*$ be convex w^*-compact.
A set $B \subset K$ is a **boundary** of K iff

$$\forall x \in X \quad \exists x_0^* \in B \quad \text{such that} \quad x_0^*(x) = \sup\{x^*(x) : x^* \in K\}.$$

Theorem (Efremov, Godefroy)

$X \in \mathfrak{S}\mathfrak{D} \implies K = \overline{\text{conv}(B)}^{\|\cdot\|}$ for all K and B as above.

Sketch of proof of the implication:

Take $C \subset B_{X^*}$ convex.

CLAIM: $\overline{C}^{w^*} = S_1(S_1(C))$.

Why?

- $S_1(C)$ is a boundary of $\overline{C}^{w^*}$.
- $\overline{C}^{w^*} = \overline{S_1(C)}^{\|\cdot\|} \subset S_1(S_1(C))$.