Some aspects of the lattice structure of $C_0(K, X)$ and $c_0(\Gamma)$

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Let $K$ be a locally compact Hausdorff space and $X$ a Banach space. The Banach space of all continuous functions from $K$ to $X$ which vanishes at infinite is denoted by $C_0(K, X)$. The norm is the sup-norm. When $K$ is compact, we denote it by $C(K, X)$. Finally if $X = \mathbb{R}$ we write $C_0(K)$ and $C(K)$ instead of $C_0(K, X)$ and $C(K, X)$, respectively.
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Remark
If $X$ is a Banach lattice, $C_0(K, X)$ is a Banach lattice with the usual order.
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**Remark**

If $X$ is a Banach lattice, $C_0(K, X)$ is a Banach lattice with the usual order.

By a Banach lattice isomorphism we mean a linear operator $T$ such that $T$ and $T^{-1}$ are both positive operators.
A result due to Kaplansky establishes that $C(K)$ and $C(S)$ are Banach lattice isomorphic if and only if $K$ and $S$ are homeomorphic. There are examples showing that Kaplansky’s theorem does not hold for $C_0(K, X)$ spaces.

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So, we have the following question:

**Problem**

If $C_0(K, X)$ and $C_0(S, X)$ are related as Banach lattices, what can we say about $K$ and $S$?

In the first part of the talk, we showed some results answering the question above. In the second part, we posed two questions about $c_0(\Gamma)$.
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Recall that for a Banach space $X$, the Schäffer constant of $X$ is defined by
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\lambda(X) := \inf\{\max\{\|x + y\|, \|x - y\|\} : \|x\| = \|y\| = 1\}.
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The following result generalizes the classical Banach-stone theorem.
Isomorphisms between $C_0(K, X)$ spaces

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$$\lambda(X) := \inf \{ \max \{ \|x + y\|, \|x - y\| \} : \|x\| = \|y\| = 1 \}.$$ 

The following result generalizes the classical Banach-stone theorem.

**Theorem (Cidral, Galego, Rincón-Villamizar)**

*Let $X$ be a Banach space with $\lambda(X) > 1$. If $T : C_0(K, X) \to C_0(S, X)$ is an isomorphism satisfying $\|T\|\|T^{-1}\| < \lambda(X)$, then $K$ and $S$ are homeomorphic.*
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Example

Let $K = \{1\}$ and $S = \{1, 2\}$. Let $T : \ell_p \to \ell_p \oplus_{\infty} \ell_p$ be given by

$$T((x_n)) = ((x_{2n}), (x_{2n-1})).$$

It is not difficult to show that $T$ is an isomorphism with $\|T\|\|T^{-1}\| = 2^{1/p}$ and $\lambda(\ell_p) = 2^{1/p}$ if $p \geq 2$. On the other hand, $T$ induces an isomorphism from $C_0(K, \ell_p)$ onto $C_0(S, \ell_p)$. 
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What about Banach lattices?
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**Definition**

An \( f \in C(K, X) \) is called non-vanishing if \( 0 \notin f(K) \). A linear operator \( T : C(K, X) \to C(S, X) \) is called non-vanishing preserving if sends non-vanishing functions into non-vanishing functions.
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Theorem (Jin Xi Chen, Z. L. Chen, N. C. Wong)

Suppose that \( T : C(K, X) \to C(S, X) \) be a Banach lattice isomorphism such that \( T \) and \( T^{-1} \) are non-vanishing preserving. Then \( K \) and \( S \) are homeomorphic.
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**Definition**

If $X$ is a Banach lattice, we define the positive Schäffer constant $\lambda^+(X)$ by

$$\lambda^+(X) := \inf \{ \max \{ \|x + y\|, \|x - y\| \} : \|x\| = \|y\| = 1, x, y > 0 \}.$$
Properties of $\lambda^+(X)$

**Proposition**

*Let $X$ be a Banach lattice. We have*

1. $\lambda^+(X) \geq 1$.
2. $\lambda^+(X) \leq \lambda^+(X)$, but there are Banach lattices for which the inequality is strict.
3. $\lambda^+(X) = 1$ if $X$ contains a copy of $c_0$.
4. If $X$ is an $L^p$-space then $\lambda^+(X) = 2^{1/p}$.

Recall that a Banach lattice $X$ is called $L^p$-space if $\|x+y\|_p = \|x\|_p + \|y\|_p$ whenever $x, y \in X$ are disjoint.

If $X = (\mathbb{R}^2, \|\cdot\|_\infty)$ then $\lambda^+(X) = 1$. So, the converse of 3) does not hold.

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**Theorem (E. M. Galego, M. A. Rincón-Villamizar)**

Let $X$ be a Banach lattice with $\lambda^+(X) > 1$. If $T: C_0(K, X) \to C_0(S, X)$ is a Banach lattice isomorphism satisfying $\|T\|\|T^{-1}\| < \lambda^+(X)$, then $K$ and $S$ are homeomorphic.

Remark: The above example shows that theorem is optimal since $\lambda^+(\ell^p) = \frac{1}{p}$ for all $1 \leq p \leq \infty$. 

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**Example**

Let $K$ and $S$ be two non-homeomorphic uncountable compact metric spaces such that the topological sums $K \oplus K$ and $S \oplus S$ are homeomorphic. We have the following Banach lattice isometries

$$C(K, X) \cong C(K \oplus K) \cong C(S \oplus S) \cong C(S, X),$$

where $X$ is the Banach lattice $\ell^2_\infty$. 
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**Remark**

If $C([0, \alpha], \ell^2_\infty)$ is Banach lattice isomorphic to $C([0, \beta], \ell^2_\infty)$, then $[0, \alpha]$ and $[0, \beta]$ are homeomorphic. Indeed, since $\ell^2_\infty$ is Banach lattice isometric to $C(\{1, 2\})$, we have the following Banach lattice isometries:

$$C([0, \alpha], \ell^2_\infty) \cong C([0, \alpha] \oplus [0, \alpha]) \quad \text{and}$$

$$C([0, \beta], \ell^2_\infty) \cong C([0, \beta] \oplus [0, \beta])$$

By Kaplansky’s theorem we conclude that $[0, \alpha]$ and $[0, \beta]$ are homeomorphic.
The above example illustrates next theorem

Theorem (E. M. Galego, M. A. Rincón-Villamizar) Let $X$ be a Banach lattice containing no copy of $c_0$ and suppose that for each $n \in \mathbb{N}$, there is no a Banach lattice isomorphism from $X^{n+1}$ into $X^n$.

For each infinite ordinals $\alpha$ and $\beta$ the following statements are equivalent:

1. $C([0,\alpha],X)$ and $C([0,\alpha],X)$ are Banach lattice isomorphic.
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Remark There are many Banach lattices $X$ satisfying hypothesis of the above theorem.
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There are many Banach lattices $X$ satisfying hypothesis of the above theorem.
In general if $C_0(K, X)$ and $C_0(S, X)$ are related as Banach lattices, we cannot conclude that $K$ and $S$ are homeomorphic but even so they share topological properties.
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Theorem (E. M. Galego, M. A. Rincón-Villamizar)

Let $X$ be a Banach lattice with $\lambda^+(X) > 1$. Suppose that $C(K, X)$ and $C(S, X)$ are Banach lattice isomorphic. Then $K$ is sequential (Fréchet, sequentially compact) if and only if $S$ so is;
Two questions about $c_0(\Gamma)$

If $K = \Gamma$ where $\Gamma$ is a set with discrete topology we denote $C_0(K)$ by $c_0(\Gamma)$. Also $\ell_\infty(\Gamma)$ denotes the Banach space of all bounded families indexed by $\Gamma$, endowed with the sup-norm.
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**Definition**

We say that $Y$ contains almost isometric copies of $X$ if for each $\varepsilon > 0$ there is an into isomorphism $T_\varepsilon : X \to Y$ such that $\|T_\varepsilon\|\|T_\varepsilon^{-1}\| \leq 1 + \varepsilon$. A result due to Rosenthal establishes that $X^*$ contains almost isometric copies of $c_0(\Gamma)$ if and only if $X^*$ contains isometric copies of $\ell_\infty(\Gamma)$. 

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Questions

By $B(X, Y)$ we mean the Banach space of all bounded linear operators from $X$ to $Y$.
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Under what conditions the following statements are equivalent:

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A theorem due to Lozanovskii says that a Banach lattice $X$ contains a copy of $c_0$ if and only if $X$ contains a lattice copy of $c_0$. 

**Question 2**
Is Lozanovskii's theorem valid for $c_0(\Gamma)$?
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Thank you!