Some aspects of the lattice structure of $C_0(K, X)$ and $c_0(\Gamma)$

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Preliminaries

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Let K be a locally compact Hausdorff space and X a Banach space. The Banach space of all continuous functions from K to X which vanishes at infinite is denoted by $C_0(K, X)$. The norm is the sup-norm. When K is compact, we denote it by $C(K, X)$. Finally if $X = \mathbb{R}$ we write $C_0(K)$ and $C(K)$ instead of $C_0(K, X)$ and $C(K, X)$, respectively.

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If X is a Banach lattice, $C_0(K, X)$ is a Banach lattice with the usual order.

By a Banach lattice isomorphism we mean a linear operator T such that T and T^{-1} are both positive operators.

A result due to Kaplansky establishes that $C(K)$ and $C(S)$ are Banach lattice isomorphic if and only if K and S are homeomorphic. There are examples showing that Kaplansky's theorem does not hold for $C_0(K, X)$ spaces.

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In first part of talk we show some results answering question above. In second one, we posed two questions about $c_0(\Gamma)$.

Isomorphisms between $C_0(K, X)$ spaces

Recall that for a Banach space X, the Schäffer constant of X is defined by

$$\lambda(X) := \inf\{\max\{\|x + y\|, \|x - y\|\} : \|x\| = \|y\| = 1\}.$$

The following result generalizes the classical Banach-stone theorem.

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The following result generalizes the classical Banach-stone theorem.

Theorem (Cidral, Galego, Rincón-Villamizar)

Let X be a Banach space with $\lambda(X) > 1$. If $T : C_0(K, X) \to C_0(S, X)$ is an isomorphism satisfying $\|T\|\|T^{-1}\| < \lambda(X)$, then K and S are homeomorphic.

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Example

Let $K = \{1\}$ and $S = \{1, 2\}$. Let $T\colon \ell_p \to \ell_p \oplus_\infty \ell_p$ be given by

$$T((x_n)) = ((x_{2n}), (x_{2n-1})).$$

It is not difficult to show that T is an isomorphism with $\|T\|\|T^{-1}\| = 2^{1/p}$ and $\lambda(\ell_p) = 2^{1/p}$ if $p \geq 2$. On the other hand, T induces an isomorphism from $C_0(K, \ell_p)$ onto $C_0(S, \ell_p)$.

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What about Banach lattices?

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Definition

An $f \in C(K, X)$ is called non-vanishing if $0 \notin f(K)$. A linear operator $T\colon C(K, X) \to C(S, X)$ is called non-vanishing preserving if sends non-vanishing functions into non-vanishing functions.

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Theorem (Jin Xi Chen, Z. L. Chen, N. C. Wong)

Suppose that $T\colon C(K, X) \to C(S, X)$ be a Banach lattice isomorphism such that T and T^{-1} are non-vanishing preserving. Then K and S are homeomorphic.

Schäffer constant in Banach lattices

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If X is a Banach lattice, we define the positive Schäffer constant $\lambda^+(X)$ by

$$\lambda^+(X) := \inf\{\max\{\|x+y\|, \|x-y\|\} : \|x\| = \|y\| = 1, x, y > 0\}.$$

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Recall that a Banach lattice X is called L_p-space if $\|x+y\|^p = \|x\|^p + \|y\|^p$ whenever $x, y \in X$ are disjoint.

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If $X = (\mathbb{R}^2, \|\cdot\|_\infty)$ then $\lambda^+(X) = 1$. So, converse of 3) does not hold.

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Remark

The above example shows that theorem is optimal since $\lambda^+(\ell_p) = 2^{1/p}$ for all $1 \leq p \leq \infty$.

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Example

Let K and S be two non-homeomorphic uncountable compact metric spaces such that the topological sums $K \oplus K$ and $S \oplus S$ are homeomorphic. We have the following Banach lattice isometries

$$C(K, X) \cong C(K \oplus K) \cong C(S \oplus S) \cong C(S, X),$$

where X is the Banach lattice ℓ^2_∞.

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The above Phenomenon does not occur for countable compact metric spaces as we see below.

If α is an ordinal, $[0, \alpha]$ denotes the set of all ordinals less or equal than α, endowed with the order topology.

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Remark

If $C([0,\alpha], \ell_\infty^2)$ is Banach lattice isomorphic to $C([0,\beta], \ell_\infty^2)$, then $[0,\alpha]$ and $[0,\beta]$ are homeomorphic. Indeed, since ℓ_∞^2 is Banach lattice isometric to $C(\{1,2\})$, we have the following Banach lattice isometries

$$C([0,\alpha], \ell_\infty^2) \cong C([0,\alpha] \oplus [0,\alpha]) \quad \text{and}$$
$$C([0,\beta], \ell_\infty^2) \cong C([0,\beta] \oplus [0,\beta])$$

By Kaplansky's theorem we conclude that $[0,\alpha]$ and $[0,\beta]$ are homeomorphic.

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Theorem (E. M. Galego, M. A. Rincón-Villamizar)

Let X be a Banach lattice containing no copy of c_0 and suppose that for each $n \in \mathbb{N}$, there is no a Banach lattice isomorphism from X^{n+1} into X^n. For each infinite ordinals α and β the following statements are equivalent:

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1. *$C([0, \alpha], X)$ and $C([0, \alpha], X)$ are Banach lattice isomorphic.*

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Remark

There are many Banach lattices X satisfying hypothesis of the above theorem.

In general if $C_0(K, X)$ and $C_0(S, X)$ are related as Banach lattices, we cannot conclude that K and S are homeomorphic but even so they share topological properties.

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Theorem (E. M. Galego, M. A. Rincón-Villamizar)

Let X be a Banach lattice with $\lambda^+(X) > 1$. Suppose that $C(K, X)$ and $C(S, X)$ are Banach lattice isomorphic. Then K is sequential (Fréchet, sequentially compact) if and only if S so is;

Two questions about $c_0(\Gamma)$

If $K = \Gamma$ where Γ is a set with discrete topology we denote $C_0(K)$ by $c_0(\Gamma)$. Also $\ell_\infty(\Gamma)$ denotes the Banach space of all bounded families indexed by Γ, endowed with the sup-norm.

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Definition

We say that Y contains almost isometric copies of X if for each $\varepsilon > 0$ there is an into isomorphism $T_\varepsilon \colon X \to Y$ such that $\|T_\varepsilon\| \|T_\varepsilon^{-1}\| \leq 1 + \varepsilon$.

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A result due to Rosenthal establishes that X^* contains almost isometric copies of $c_0(\Gamma)$ if and only if X^* contains isometric copies of $\ell_\infty(\Gamma)$.

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Question 2

Is Lozanovskii's theorem valid for $c_0(\Gamma)$?

References

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Thank you!