

Numerical index with respect to an operator

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Workshop on Banach spaces and Banach lattices

Madrid, 10th September 2019



**UNIVERSIDAD
DE GRANADA**

- 1 Classic numerical index
- 2 Extending the concept of numerical range
- 3 Numerical index with respect to an operator

Some notation

X, Y real or complex Banach spaces

B_X closed unit ball

S_X unit sphere

X^* topological dual

$\mathcal{L}(X, Y)$ Banach space of all bounded linear operators from X to Y

$\mathcal{L}(X)$ Banach algebra of all bounded linear operators from X to X

Classic numerical index

Definitions

Numerical range for Hilbert spaces (Toeplitz, 1918)

H Hilbert space, $(\cdot \mid \cdot)$ inner product, $T \in \mathcal{L}(H)$

$$W(T) = \{(Tx \mid x) : x \in H, (x \mid x) = 1\}$$

- Convex subset of $\mathbb{K}$

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Numerical range and numerical radius (Bauer, Lumer, early 60's)

X Banach space, $T \in \mathcal{L}(X)$

$$V(T) = \{x^*(Tx) : x \in S_X, x^* \in S_{X^*}, x^*(x) = 1\}$$

$$\begin{aligned} v(T) &= \sup\{|\lambda| : \lambda \in V(T)\} \\ &= \sup\{|x^*(Tx)| : x \in S_X, x^* \in S_{X^*}, x^*(x) = 1\} \end{aligned}$$

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Obviously one has $v(T) \leq \|T\|$

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Set of values

$$\{n(X) : X \text{ complex Banach space}\} = [e^{-1}, 1]$$

$$\{n(X) : X \text{ real Banach space}\} = [0, 1]$$

Some known results

- H Hilbert space, $n(H) = 0$ in real case and $n(H) = 1/2$ in complex case.

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- Let $\{X_\lambda : \lambda \in \Lambda\}$ be an arbitrary family of Banach spaces. Then

$$n\left(\left[\oplus_{\lambda \in \Lambda} X_\lambda\right]_{c_0}\right) = n\left(\left[\oplus_{\lambda \in \Lambda} X_\lambda\right]_{\ell_1}\right) = n\left(\left[\oplus_{\lambda \in \Lambda} X_\lambda\right]_{\ell_\infty}\right) = \inf_{\lambda \in \Lambda} n(X_\lambda)$$

$$n\left(\left[\oplus_{\lambda \in \Lambda} X_\lambda\right]_{\ell_p}\right) \leq \inf_{\lambda \in \Lambda} n(X_\lambda)$$

(Martín-Payá, 2000)

Some known results

- Let X be a Banach space, K compact Hausdorff and μ positive measure. Then

$$n(C(K, X)) = n(L_1(\mu, X)) = n(X) \quad (\text{Martín-Payá, 2000})$$

$$n(L_\infty(\mu, X)) = n(X) \quad (\text{Martín-Villena, 2003})$$

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- $n(X^*) \leq n(X)$
and the inequality can be strict (Boyko-Kadets-Martín-Werner, 2007)

Extending the concept of numerical range

Intrinsic numerical range

(Bonsall-Duncan, 1971)

Let X be a Banach space. Then for every $T \in \mathcal{L}(X)$

$$\overline{\text{conv}} V(T) = \{\Phi(T) : \Phi \in \mathcal{L}(X)^*, \|\Phi\| = \Phi(\text{Id}) = 1\}.$$

Consequently, $v(T) = \max\{|\Phi(T)| : \Phi \in \mathcal{L}(X)^*, \|\Phi\| = \Phi(\text{Id}) = 1\}.$

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Intrinsic numerical range with respect to G

X, Y Banach spaces, $G \in \mathcal{L}(X, Y)$ with $\|G\| = 1$, $T \in \mathcal{L}(X, Y)$

$$\tilde{V}_G(T) = \{\Phi(T) : \Phi \in \mathcal{L}(X, Y)^*, \|\Phi\| = \Phi(G) = 1\}$$

Spatial numerical range

Spatial numerical range, Bauer–Lumer

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Approximated spatial numerical range with respect to G (Ardalani, 2014)

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For $G = \text{Id}$, by Bishop–Phelps–Bollobás theorem

$$V_{\text{Id}}(T) = \overline{V(T)} \quad \text{for every } T \in \mathcal{L}(X)$$

Relationship

Two possible numerical ranges

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Relationship (Martín, 2016)

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Numerical radius with respect to G

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Numerical index with respect to an operator

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X, Y Banach spaces, $G \in \mathcal{L}(X, Y)$ with $\|G\| = 1$

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Characterization

For $k \in [0, 1]$, TFAE:

- $n_G(X, Y) \geq k$,
- $\inf_{\delta > 0} \sup \{|y^*(Tx)| : y^* \in S_{Y^*}, x \in S_X, \operatorname{Re} y^*(Gx) > 1 - \delta\} \geq k\|T\| \forall T \in \mathcal{L}(X, Y)$,
- $\max_{|\theta|=1} \|G + \theta T\| \geq 1 + k\|T\| \quad \forall T \in \mathcal{L}(X, Y)$.

Examples

Spear operators (Ardalani, 2014; Kadets, Martín, Merí, Pérez, 2018)

X, Y Banach spaces, $G \in \mathcal{L}(X, Y)$.

G **spear operator** $\iff n_G(X, Y) = 1 \iff \max_{|\theta|=1} \|G + \theta T\| = 1 + \|T\| \ \forall T \in \mathcal{L}(X, Y)$.

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- Fourier transform ($\mathcal{F}: L_1(\mathbb{R}) \longrightarrow C_0(\mathbb{R})$),
- Inclusion $A(\mathbb{D}) \hookrightarrow C(\mathbb{T})$.
- Identity operator on $C(K)$, $L_1(\mu)$...

Set of values

X, Y Banach spaces

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Hilbert spaces

H Hilbert space with $\dim(H) \geq 2$, X, Y Banach spaces:

- Real case: $\mathcal{N}(\mathcal{L}(X, H)) = \mathcal{N}(\mathcal{L}(H, Y)) = \{0\}$. In particular, $\mathcal{N}(\mathcal{L}(H)) = \{0\}$.
- Complex case: $\mathcal{N}(\mathcal{L}(X, H)) \subset [0, 1/2]$ and $\mathcal{N}(\mathcal{L}(Y, H)) \subset [0, 1/2]$.

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H_1, H_2 complex Hilbert spaces with dimension greater than one:

- $\mathcal{N}(\mathcal{L}(H_1, H_2)) = \{0, 1/2\}$ if H_1 and H_2 are isometrically isomorphic.
- $\mathcal{N}(\mathcal{L}(H_1, H_2)) = \{0\}$ otherwise.

Set of values

ℓ_p -spaces

For $1 < p < \infty$, X, Y real Banach spaces, $M_p = \sup_{t \in [0,1]} \frac{|t^{p-1} - t|}{1+t^p}$,

$$\mathcal{N}(\mathcal{L}(X, \ell_p)) \subset [0, M_p] \quad \text{and} \quad \mathcal{N}(\mathcal{L}(\ell_p, Y)) \subset [0, M_p].$$

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L_∞ -spaces

μ_1, μ_2 σ -finite measures. If at least one of the spaces $L_\infty(\mu_i)$, $i = 1, 2$, has dimension at least two, $\mathcal{N}(\mathcal{L}(L_\infty(\mu_1), L_\infty(\mu_2))) = \{0, 1\}$ (real and complex case).

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L_1 -spaces

μ_1, μ_2 σ -finite measures, $\mathcal{N}(\mathcal{L}(L_1(\mu_1), L_1(\mu_2))) \subset \{0, 1\}$ (real and complex case).

★ $\mathcal{N}(\mathcal{L}(\ell_1, L_1[0, 1])) = \{0\}$.

Sums of Banach spaces

Proposition

Let $\{X_\lambda: \lambda \in \Lambda\}$, $\{Y_\lambda: \lambda \in \Lambda\}$ be two families of Banach spaces and let $G_\lambda \in \mathcal{L}(X_\lambda, Y_\lambda)$ with $\|G_\lambda\| = 1$ for every $\lambda \in \Lambda$. Let E be one of the Banach spaces c_0 , ℓ_∞ or ℓ_1 , let $X = [\oplus_{\lambda \in \Lambda} X_\lambda]_E$ and $Y = [\oplus_{\lambda \in \Lambda} Y_\lambda]_E$ and define the operator $G: X \rightarrow Y$ by

$$G[(x_\lambda)_{\lambda \in \Lambda}] = (G_\lambda x_\lambda)_{\lambda \in \Lambda}$$

for every $(x_\lambda)_{\lambda \in \Lambda} \in [\oplus_{\lambda \in \Lambda} X_\lambda]_E$. Then

$$n_G(X, Y) = \inf_{\lambda} n_{G_\lambda}(X_\lambda, Y_\lambda).$$

Sums of Banach spaces

Proposition

Let $\{X_\lambda : \lambda \in \Lambda\}$, $\{Y_\lambda : \lambda \in \Lambda\}$ be two families of Banach spaces and let $G_\lambda \in \mathcal{L}(X_\lambda, Y_\lambda)$ with $\|G_\lambda\| = 1$ for every $\lambda \in \Lambda$. Let E be one of the Banach spaces c_0 , ℓ_∞ or ℓ_1 , let $X = [\oplus_{\lambda \in \Lambda} X_\lambda]_E$ and $Y = [\oplus_{\lambda \in \Lambda} Y_\lambda]_E$ and define the operator $G: X \rightarrow Y$ by

$$G[(x_\lambda)_{\lambda \in \Lambda}] = (G_\lambda x_\lambda)_{\lambda \in \Lambda}$$

for every $(x_\lambda)_{\lambda \in \Lambda} \in [\oplus_{\lambda \in \Lambda} X_\lambda]_E$. Then

$$n_G(X, Y) = \inf_{\lambda} n_{G_\lambda}(X_\lambda, Y_\lambda).$$

Moreover, for $1 < p < \infty$

$$n_G\left([\oplus_{\lambda \in \Lambda} X_\lambda]_{\ell_p}, [\oplus_{\lambda \in \Lambda} Y_\lambda]_{\ell_p}\right) \leq \inf_{\lambda} n_{G_\lambda}(X_\lambda, Y_\lambda).$$

Vector-valued function spaces

Theorem

Let X, Y be Banach spaces and $G \in \mathcal{L}(X, Y)$ with $\|G\| = 1$.

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Let X, Y be Banach spaces and $G \in \mathcal{L}(X, Y)$ with $\|G\| = 1$.

- K compact, consider $\tilde{G}: C(K, X) \rightarrow C(K, Y)$, $\tilde{G}(f) = G \circ f$, then:

$$n_{\tilde{G}}(C(K, X), C(K, Y)) = n_G(X, Y).$$

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Moreover, for vector-valued L_p -spaces

$$n_{\tilde{G}}(L_p(\mu, X), L_p(\mu, Y)) \leq n_G(X, Y)$$

for $1 < p < \infty$, with $\tilde{G}$ analogously defined.

Adjoint operators

Numerical index with respect to adjoint operators

X, Y Banach spaces, $G \in \mathcal{L}(X, Y)$ with $\|G\| = 1$,

$$n_{G^*}(Y^*, X^*) \leq n_G(X, Y)$$

and this inequality can be strict.

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L -embedded space

A Banach space Y is **L -embedded** if $Y^{**} = J_Y(Y) \oplus_1 Y_s$ for suitable closed subspace Y_s of Y^{**} (J_Y is the natural isometric inclusion of Y into its bidual).

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Proposition

X, Y Banach spaces, $G \in \mathcal{L}(X, Y)$ rank-one operator of norm 1. Then $n_{G^*}(Y^*, X^*) = n_G(X, Y)$ and so, same happens to all the successive adjoints of G .

References



M. A. ARDALANI, Numerical index with respect to an operator, *Studia Math.* **225** (2014), 165–171.



M. MARTÍN, On different definitions of numerical range, *J. Math. Anal. Appl.* **433** (2016), 877–886.



V. KADETS, M. MARTÍN, J. MERÍ AND A. PÉREZ, *Spear operators between Banach spaces*, Lecture Notes in Mathematics, **2205**, Springer, 2018.



V. KADETS, M. MARTÍN, J. MERÍ, A. PÉREZ AND A. QUERO, On the numerical index with respect to an operator, *Dissertationes Mathematicae* (to appear, available in arxiv.org with code 1905.12257).