Supports and approximation properties in Lipschitz-free spaces

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Let (M,d) and (N,ϱ) be metric spaces. A map $f : M \longrightarrow N$ is called *Lipschitz* if there exists a constant $C>0$ such that

$$\varrho(f(p),f(q)) \leq C\, d(p,q) \qquad \forall p,q \in M.$$

The Lipschitz constant of f is defined as

$$\mathrm{Lip}(f) := \sup\left\{ \frac{\varrho(f(p),f(q))}{d(p,q)} \;:\; p,q\in M,\ p\neq q \right\}.$$

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Theorem (McShane, '34)

Let $S \subseteq M$. Then every Lipschitz function $f : S \longrightarrow \mathbb{R}$ can be extended to a Lipschitz function $\widetilde{f} : M \longrightarrow \mathbb{R}$ so that $\mathrm{Lip}(\widetilde{f}) = \mathrm{Lip}(f)$.

$$\widetilde{f}(p) := \sup\{ f(q) - \mathrm{Lip}(f) d(p, q) \ : \ q \in S \}$$

Let (M, d) be a complete metric space with a base point $0 \in M$ (called a *pointed metric space*). The **Lipschitz-free space** over M, denoted $\boldsymbol{\mathcal{F}}(\mathbf{M})$, is the Banach space satisfying the following universal property:

- There exists an isometric embedding $\delta : M \longrightarrow \mathcal{F}(M)$ such that $\overline{\operatorname{span}}\, \delta(M) = \mathcal{F}(M)$ and $\delta(0) = 0$.
- For any Banach space X and any Lipschitz map $L : M \longrightarrow X$ with $L(0) = 0$ there exists a unique linear operator $\overline{L} : \mathcal{F}(M) \longrightarrow X$ such that $\|\overline{L}\| = \mathrm{Lip}(L)$ and $\overline{L}\delta = L$, i.e. the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "L"] \arrow[d, "\delta"'] & X \\ \mathcal{F}(M) \arrow[ur, "\overline{L}"'] & \end{tikzcd}$$

$$\begin{tikzcd} M \arrow[r, "L"] \arrow[d, "\delta_M"'] & N \arrow[d, "\delta_N"] \\ \mathcal{F}(M) \arrow[r, "\hat{L}"] & \mathcal{F}(N) \end{tikzcd}$$

$\forall$ M, N metric spaces, $\forall$ L Lipschitz with $L(0) = 0$
$\exists!$ $\hat{L}$ linear operator s.t. $\|\hat{L}\| = \mathrm{Lip}(L)$ and $\hat{L}\delta_M = \delta_N L$.

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Indeed, by universal property define

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Indeed, by universal property define

$$\hat{L} := \overline{\delta_N L}.$$

- If M and N are bi-Lipschitz homeomorphic, then $\mathcal{F}(M)$ and $\mathcal{F}(N)$ are linearly isomorphic.
- If M and N are isometric, then $\mathcal{F}(M)$ and $\mathcal{F}(N)$ are linearly isometric.

- $\mathcal{F}(c_0)$ is linearly isomorphic to $\mathcal{F}(C([0,1]))$. (*Dutrieux, Ferenczi, '05*)
- $\mathcal{F}(B_{\mathbb{R}^n})$ is linearly isomorphic to $\mathcal{F}(\mathbb{R}^n)$. (*Kaufmann, '15*)
- There exist $(K_\alpha)_{\alpha<\omega_1}$ homeomorphic to the Cantor space such that $\mathcal{F}(K_\alpha)$ is not linearly isomorphic to $\mathcal{F}(K_\beta)$. (*Hájek, Lancien, P, '16*)

Theorem (Godefroy, Kalton, '03)

A construction of examples of non-separable Banach spaces which are bi-Lipschitz homeomorphic but not linearly isomorphic.

Theorem (Godefroy, Kalton, '03)

If a separable Banach space X is isometric to a subset of a Banach space Y, then X is already linearly isometric to a subspace of Y.

Theorem (Godefroy, Kalton, '03)

Let X be a Banach space with the bounded approximation property. If a Banach space Y is bi-Lipschitz homeomorphic to X, then Y also has the bounded approximation property.

Let (M, d) be a complete pointed metric space with the base point $0 \in M$.

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Space of Lipschitz functions

Then
$$\mathrm{Lip}_0(M) = \{f : M \longrightarrow \mathbb{R} \ : \ f \text{ Lipschitz}, \ f(0) = 0\}$$
with the norm $\|f\| = \mathrm{Lip}(f)$ is a Banach space.

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For $p \in M$ consider the evaluation functional $\delta(p) \in \mathrm{Lip}_0(M)^*$ defined by

$$\langle f, \delta(p)\rangle = f(p) \qquad \forall f \in \mathrm{Lip}_0(M).$$

Then the Dirac map $\delta : M \to \mathrm{Lip}_0(M)^*$ is an isometric embedding.

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Lipschitz-free space

The space
$$\mathcal{F}(M) = \overline{\mathrm{span}}^{\|\cdot\|} \ \delta(M) \subseteq \mathrm{Lip}_0(M)^*$$
with the norm inherited from $\mathrm{Lip}_0(M)^*$ is the Lipschitz-free space over M.

- $\mathcal{F}(M)^* \equiv \mathrm{Lip}_0(M)$ and for (f_γ) and f in $B_{\mathrm{Lip}_0(M)}$ we have

$$f_\gamma \xrightarrow{w^*} f \quad \Longleftrightarrow \quad (f_\gamma(p) \to f(p) \qquad \forall\, p \in M)\,.$$

Theorem (Weaver, '17)

If M has a finite diameter or it is complete and convex (e.g. Banach space) then $\mathcal{F}(M)$ is the unique predual of $\mathrm{Lip}_0(M)$.

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- **Theorem (Kadets, '85)**

  If K is a subset of M containing the base point, then $\mathcal{F}(K)$ is isometric to a subspace of $\mathcal{F}(M)$. Precisely,

$$\mathcal{F}(K) \equiv \mathcal{F}_M(K) := \overline{\mathrm{span}}\, \delta(K) \subseteq \mathcal{F}(M).$$

  If K is a Lipschitz retract of M, then $\mathcal{F}_M(K)$ is complemented in $\mathcal{F}(M)$.

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- For a closed subset $K \subseteq M$, define the *kernel* of K as

$$\mathcal{I}_M(K) = \{f \in \mathrm{Lip}_0(M) :\; f(p) = 0 \;\; \forall p \in K\}.$$

  Then $\mathcal{F}_M(K)^\perp = \mathcal{I}_M(K)$ and $\mathcal{I}_M(K)_\perp = \mathcal{F}_M(K)$.

- An *elementary molecule* in $\mathcal{F}(M)$ is

$$\frac{\delta(p)-\delta(q)}{d(p,q)} \in S_{\mathcal{F}(M)} \quad \text{where} \quad p,q \in M,\ p \neq q.$$

For every $\mu \in \mathcal{F}(M)$ and every $\varepsilon > 0$ there exists a representation

$$\mu = \sum_{n=1}^{\infty} a_n \frac{\delta(p_n)-\delta(q_n)}{d(p_n,q_n)} \quad \text{such that} \quad \sum_{n=1}^{\infty} |a_n| \leq \|\mu\| + \varepsilon.$$

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- Terminology: Arens-Eells spaces, Transportation cost spaces

Definition

Let $\beta : \mathcal{F}(X) \longrightarrow X$ be the linear extension of the identity on X (the barycentre map). A Banach space X has the **isometric Lipschitz lifting property** if there exists a linear operator $T : X \longrightarrow \mathcal{F}(X)$ such that $\|T\| = 1$ and $\beta T = \mathrm{Id}_X$.

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Theorem (Godefroy, Kalton, '03)

Every separable Banach space has the isometric Lipschitz lifting property.

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2. $\mathcal{F}\left(\mathbb{R}^2\right) \not\xrightarrow{\simeq} L_1$. (*Naor, Schechtman, '07; Kislyakov, '75*)
3. $\mathcal{F}\left(\mathbb{R}^d\right) \equiv L_1(\mathbb{R}^d, \mathbb{R}^d)/\{g \in L_1(\mathbb{R}^d, \mathbb{R}^d) : \sum_{i=1}^d \partial_i g_i = 0$ as distributions$\}$.
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Question

Is $\mathcal{F}(\mathbb{R}^2) \simeq \mathcal{F}(\mathbb{R}^3)$*?*

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4. $\mathcal{F}\left(\mathbb{R}^d\right)$ is complemented in $\mathcal{F}\left(\mathbb{R}^d\right)^{**}$. (*Cúth, Kalenda, Kaplický, '18*)

Question (Godefroy, Lancien, Zizler, '14)

Is $\mathcal{F}(\ell_1)$ complemented in $\mathcal{F}\left(\ell_1\right)^{**}$?

6. $\mathcal{F}(M) \overset{\equiv}{\hookrightarrow} L_1 \iff M \overset{=}{\hookrightarrow} \mathbb{R}$-tree. (*Godard, '10*)
7. $\mathcal{F}(M) \equiv \ell_1 \iff M$ is a subset of an $\mathbb{R}$-tree with zero length measure and containing the branching points. (*Dalet, Kaufmann, Procházka, '16*)

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8. $\ell_1(\mathrm{dens}(M)) \underset{c}{\overset{\simeq}{\hookrightarrow}} \mathcal{F}(M)$. (*Hájek, Novotný, '17*)
9. If M is bounded uniformly discrete, then $\mathcal{F}(M) \simeq \ell_1$.

9. $c_0 \overset{\simeq}{\not\hookrightarrow} \mathcal{F}\left([0,1]^d\right)$. (*Cúth, Doucha, Wojtaszczyk, '16*).
10. If M is a compact subset of a superreflexive Banach space, then $c_0 \overset{\simeq}{\not\hookrightarrow} \mathcal{F}(M)$. (*Kochanek, P., '18*)

Question (Cúth, Doucha Wojtaszczyk, '16)

Is it true that $c_0 \overset{\simeq}{\hookrightarrow} \mathcal{F}(\ell_2)$*?*

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Question (Dutrieux, Ferenczi, '05)

If X *is a Banach space, does* $\mathcal{F}(c_0) \overset{\simeq}{\hookrightarrow} \mathcal{F}(X)$ *imply* $c_0 \xhookrightarrow{bi-Lip} X$*?*

11. $\mathcal{F}(c_0) \not\simeq C([0,1])$, $\mathcal{F}(c_0) \not\simeq$ Gurariĭ space. (*CDW, '16*)

Question (Cúth, Doucha Wojtaszczyk, '16)

Is $\mathcal{F}(c_0)$ *isomorphic to Holmes space or Pełczyński space?*

Joint work with R. J. Aliaga (Valencia), C. Petitjean (Paris) and A. Procházka (Besançon).

Problem

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$$u_{pq} = \frac{\delta(p) - \delta(q)}{d(p,q)} \in S_{\mathcal{F}(M)} \quad \text{where} \quad p, q \in M,\ p \neq q.$$

- The *metric segment* between points p and q in M is

$$[p,q] = \{r \in M \ : \ d(p,r) + d(q,r) = d(p,q)\}.$$

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Theorem (Aliaga, P., '18)

Let M be a complete pointed metric space and let $\mu \in \operatorname{span}(\delta(M)) \subseteq \mathcal{F}(M)$. TFAE:

1. *μ is an extreme point of $B_{\mathcal{F}(M)}$,*
2. *$\mu = u_{pq}$ for some $p, q \in M$, $p \neq q$ such that $d(p,q) < d(p,r) + d(r,q)$ for all $r \in M \setminus \{p,q\}$, i.e. $[p,q] = \{p,q\}$.*

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Let M be a complete pointed metric space and let $\mu \in \operatorname{span}(\delta(M)) \subseteq \mathcal{F}(M)$. TFAE:

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- Equivalently: Does every extreme point of $B_{\mathcal{F}(M)}$ belong to $\operatorname{span}(\delta(M))$?
- Also equivalent: Is every extreme point μ of the form $\mu = \nu + \lambda$, where ν is positive (i.e. $\langle \nu, f \rangle \geq 0$ for every $f \geq 0$) and $\lambda \in \operatorname{span}(\delta(M))$? (*Aliaga, Petitjean, Procházka, '19*)

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- The answer is YES if:
  - M is compact and $\mathcal{F}(M) \equiv \operatorname{lip}_0(M)^*$ (Weaver, '99). That is for instance if M is countable compact or compact ultrametric (*Dalet, '15*), compact Hölder space or the Cantor set (*Weaver, '99*).
  - $\mathcal{F}(M)$ has a natural predual. (*García-Lirola, Petitjean, Procházka, Rueda Zoca, '17*)
  - M is a subset of an $\mathbb{R}-$tree. (*Aliaga, Petitjean, Procházka, '19*)

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Recall, if $K \subseteq M$ closed and $\mathcal{F}_M(K) := \overline{\operatorname{span}}\, \delta(K \cup \{0\}) \subseteq \mathcal{F}(M)$, then $\mathcal{F}(K \cup \{0\}) \equiv \mathcal{F}_M(K)$.

Key Lemma

Let $(p,q) \in M$, $p \neq q$, and suppose that $u_{pq} = \lambda\mu_1 + (1-\lambda)\mu_2$ for some $\mu_1, \mu_2 \in S_{\mathcal{F}(M)}$ and $0 < \lambda < 1$. Then

$$\mu_1, \mu_2 \in \mathcal{F}_M([p,q]_\varepsilon) \quad \forall \varepsilon > 0,$$

where $[p,q]_\varepsilon = \{r \in M \ : \ d(p,r) + d(r,q) - d(p,q) \leq \varepsilon\}$.

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$$\mu_1, \mu_2 \in \mathcal{F}_M([p,q]_\varepsilon) \quad \forall \varepsilon > 0,$$

where $[p,q]_\varepsilon = \{r \in M \ : \ d(p,r) + d(r,q) - d(p,q) \leq \varepsilon\}$.

$$\overset{?}{\Longrightarrow} \qquad \mu_1, \mu_2 \in \mathcal{F}_M([p,q])$$

For $K \subseteq M$ closed, define $\mathcal{F}_M(K) := \overline{\operatorname{span}}\, \delta(K \cup \{0\}) \subseteq \mathcal{F}(M)$.

Theorem (Aliaga, P., '18)

Let M be a complete pointed metric space and let $\{K_i : i \in I\}$ be a family of closed subsets of M. Then

$$\bigcap_{i \in I} \mathcal{F}_M(K_i) = \mathcal{F}_M\left(\bigcap_{i \in I} K_i\right).$$

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Definition

Let M be a complete pointed metric space. For a $\mu \in \mathcal{F}(M)$, we define the **support of $\boldsymbol{\mu}$** as

$$\operatorname{supp}(\mu) := \bigcap \{K \subseteq M \text{ closed} : \mu \in \mathcal{F}_M(K)\}.$$

Corollary

The support of μ is the smallest closed set $K \subseteq M$ such that $\mu \in \mathcal{F}_M(K)$, i.e. $\mu \in \mathcal{F}_M(\operatorname{supp}(\mu))$ and $\operatorname{supp}(\mu) \subseteq K$ whenever $\mu \in \mathcal{F}_M(K)$.

Proposition

Let $K \subseteq M$ closed and let $\mu \in \mathcal{F}(M)$. TFAE:

1. $\mathrm{supp}(\mu) \subseteq K$,
2. $\mu \in \mathcal{F}_M(K)$,
3. $\langle \mu, f \rangle = \langle \mu, g \rangle$ *for any* $f, g \in \mathrm{Lip}_0(M)$ *such that* $f|_K = g|_K$.

Proposition

Let $\mu \in \mathcal{F}(M)$ and $p \in M$. TFAE:

1. $p \in \mathrm{supp}(\mu)$,
2. *For every neighbourhood U of p there exists $f \in \mathrm{Lip}_0(M)$ such that* $\mathrm{supp}(f) \subseteq U$ *and* $\langle \mu, f \rangle > 0$.

Let m be a Radon measure on M. Then

$$\delta : M \longrightarrow \mathcal{F}(M) \text{ is Bochner integrable } \iff d(\cdot, 0) \in L_1(|m|).$$

In such case

$$\mu := \int_M \delta(p)\, dm(p) \in \mathcal{F}(M)$$

satisfies

$$\langle \mu, f \rangle = \int_M f(p)\, dm(p) \qquad \forall f \in \mathrm{Lip}_0(M).$$

We say that μ is induced by measure m.

Proposition

If $\mu \in \mathcal{F}(M)$ is induced by a Radon measure m on M, then the support of μ agrees with the support of m, possibly up to the base point.

Let (M,d) be a complete pointed metric space with the base point $0 \in M$ and let $\mathrm{Lip}_0(M) = \{f : M \longrightarrow \mathbb{R} \ : \ f \text{ Lipschitz}, \ f(0) = 0\}$ be equipped with the norm $\|f\| = \mathrm{Lip}(f)$.

Consider the isometry $\delta : M \longrightarrow \mathrm{Lip}_0(M)^*$, given by $\langle f, \delta(p)\rangle = f(p)$. The **Lipschitz-free space** over M is the space

$$\mathcal{F}(M) = \overline{\mathrm{span}}^{\|\cdot\|} \{\delta(p) \ : \ p \in M\} \subseteq \mathrm{Lip}_0(M)^*.$$

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- $$\begin{tikzcd} M \arrow[r, "L"] \arrow[d, "\delta"'] & X \\ \mathcal{F}(M) \arrow[ur, "\overline{L}"'] & \end{tikzcd}$$

  M metric sp., X Banach sp., L Lipschitz, $L(0) = 0$
  $\Rightarrow \ \exists! \ \overline{L}$ linear, $\|\overline{L}\| = \mathrm{Lip}(L), \ \overline{L}\delta = L$

  $$\begin{tikzcd} M \arrow[r, "L"] \arrow[d, "\delta_M"'] & N \arrow[d, "\delta_N"] \\ \mathcal{F}(M) \arrow[r, "\hat{L}"] & \mathcal{F}(N) \end{tikzcd}$$

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- $\mathcal{F}(M)^* \equiv \mathrm{Lip}_0(M)$ and $f_i \xrightarrow{w^*} f \iff f_i \xrightarrow{p.w.} f$ in $B_{\mathrm{Lip}_0(M)}$.

For $K \subseteq M$ closed, define $\mathcal{F}_M(K) := \overline{\operatorname{span}}\, \delta(K \cup \{0\}) \subseteq \mathcal{F}(M)$.

Theorem (Aliaga, P., '18)

Let M be a complete pointed metric space and let $\{K_i : i \in I\}$ be a family of closed subsets of M. Then

$$\bigcap_{i \in I} \mathcal{F}_M(K_i) = \mathcal{F}_M\left(\bigcap_{i \in I} K_i\right).$$

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Definition

Let M be a complete pointed metric space. For a $\mu \in \mathcal{F}(M)$, we define the **support of μ** as

$$\operatorname{supp}(\mu) := \bigcap \{K \subseteq M \text{ closed} : \mu \in \mathcal{F}_M(K)\}.$$

Corollary

The support of μ is the smallest closed set $K \subseteq M$ such that $\mu \in \mathcal{F}_M(K)$, i.e. $\mu \in \mathcal{F}_M(\operatorname{supp}(\mu))$ and $\operatorname{supp}(\mu) \subseteq K$ whenever $\mu \in \mathcal{F}_M(K)$.

The space $\mathrm{Lip}_0(M)$ has

- linear structure
- order structure:
  Say $f \leq g$ if $f(p) \leq g(p)$ for every $p \in M$.
  Define
  $$\bigwedge f_\lambda := \inf f_\lambda, \quad \bigvee f_\lambda := \sup f_\lambda.$$
  Then
  $$\mathrm{Lip}\left(\bigwedge f_\lambda\right),\ \mathrm{Lip}\left(\bigvee f_\lambda\right) \ \leq\ \sup\ \mathrm{Lip}(f_\lambda).$$
- and, if M is bounded, algebraic structure:
  $$\mathrm{Lip}(f \cdot g) \leq \mathrm{Lip}(f)\,\|g\|_\infty + \|f\|_\infty\,\mathrm{Lip}(g) \leq\ 2\,\mathrm{diam}(M)\ \mathrm{Lip}(f)\ \mathrm{Lip}(g).$$

Recall, for a closed subset $K \subseteq M$,

$$\mathcal{I}_M(K) = \{f \in \mathrm{Lip}_0(M) : \ f(p) = 0 \ \forall p \in K\}$$

and $\mathcal{F}_M(K)^{\perp} = \mathcal{I}_M(K)$, $\mathcal{I}_M(K)_{\perp} = \mathcal{F}_M(K)$.

Lemma

If M is a complete pointed metric space and $\{K_i \ : \ i \in I\}$ are closed subsets of M, then

$$\overline{\mathrm{span}}^{w^*} \bigcup_{i \in I} \mathcal{I}_M(K_i) = \mathcal{I}_M\left(\bigcap_{i \in I} K_i\right).$$

Proof of Theorem:

$$\bigcap_{i \in I} \mathcal{F}_M(K_i) = \bigcap_{i \in I} (\mathcal{I}_M(K_i)_{\perp}) = \left(\bigcup_{i \in I} \mathcal{I}_M(K_i)\right)_{\perp} =$$

$$\left(\overline{\mathrm{span}}^{w^*} \bigcup_{i \in I} \mathcal{I}_M(K_i)\right)_{\perp} = \mathcal{I}_M\left(\bigcap_{i \in I} K_i\right)_{\perp} = \mathcal{F}_M\left(\bigcap_{i \in I} K_i\right). \quad \square$$

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Sketch of proof for bounded M:

- If $\mu \in \mathcal{F}(M)$, $g \in \mathrm{Lip}_0(M)$ and we define $(\mu \circ g)(f) = \langle \mu, f.g\rangle$ for $f \in \mathrm{Lip}_0(M)$, then $\mu \circ g \in \mathcal{F}(M)$.

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Theorem (Weaver, '95)

If $\mathcal{A}$ is a w^*-closed ideal of $\mathrm{Lip}_0(M)$, then $\mathcal{A} = \mathcal{I}_M(\mathcal{H}(\mathcal{A}))$, where

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- Finally,

$$\overline{\operatorname{span}}^{w^*} \bigcup_{i\in I} \mathcal{I}_M(K_i) = \mathcal{I}_M\left(\mathcal{H}\left(\overline{\operatorname{span}}^{w^*} \bigcup_{i\in I} \mathcal{I}_M(K_i)\right)\right) = \mathcal{I}_M\left(\bigcap_{i\in I} K_i\right).$$

Proposition

Let $A \subseteq M$ be a bounded set containing the base point and let $g : M \longrightarrow \mathbb{R}$ be a Lipschitz function with $\operatorname{supp}(g) \subseteq A$. *For every $f \in \mathrm{Lip}_0(A)$ define*

$$T_g(f)(p) = \begin{cases} g(p).f(p) & \text{if } p \in A \\ 0 & \text{if } p \notin A \end{cases}.$$

Then $T_g(f) \in \mathrm{Lip}_0(M)$ and

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In particular, if M is bounded and $A = M$, then $(T_g)_*(m) = m \circ g$.

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Sketch of proof for unbounded M:

- Let $f \in \mathcal{I}_M\left(\bigcap_{i\in I} K_i\right)$ and let U be a w^*-neighbourhood of f. We want to show that $U \cap \operatorname{span} \cup_{i\in I} \mathcal{I}_M(K_i) \neq \emptyset$.

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- Lipschitz functions with bounded supports are w^*-dense in $\mathrm{Lip}_0(M)$ and in $\mathcal{I}_M(K)$, so we may assume that f has a bounded support.
- Use operators $T_{``\chi_{\operatorname{supp}(f)}"}$ and $(T_{``\chi_{\operatorname{supp}(f)}"})_*$ to pass to/from a bounded space and apply the Lemma for the bounded case.

$\square$

Joint works with P. Hájek (Prague), G. Lancien (Besançon), R. Smith (Dublin).

A Banach space X has

- the **approximation property (AP)** if, given $K \subseteq X$ compact and $\varepsilon > 0$, there is a bounded finite-rank operator $T : X \longrightarrow X$ such that $\|Tx - x\| \leq \varepsilon$ for all $x \in K$,
- the **$\boldsymbol{\lambda}-$bounded approximation property ($\boldsymbol{\lambda}-$BAP)**, $1 \leq \lambda < \infty$, if moreover $\|T\| \leq \lambda$
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If X is separable, then X has the BAP if and only if there exists a bounded sequence of finite-rank operators $T_n : X \longrightarrow X$ such that $\lim_{n\to\infty} \|T_n x - x\| = 0$ for all $x \in X$.
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Problem

For which metric spaces M does $\mathcal{F}(M)$ have the BAP?

Theorem (Godefroy, Kalton, '03)

If X is a finite-dimensional Banach space, then $\mathcal{F}(X)$ has the MAP.

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- The space $\mathcal{F}(\mathbb{U})$, where $\mathbb{U}$ is the Urysohn universal space, has the MAP. *(Fonf, Wojtaszczyk, '08)*
- If $M \subseteq \ell_2^N$, then $\mathcal{F}(M)$ has $C\sqrt{N}$-BAP. *(Lancien, P., '13)*
- If M is a countable proper metric space, then $\mathcal{F}(M)$ has the MAP. *(Dalet, '14)*

Definition

Let X be a Banach space and $\beta : \mathcal{F}(X) \longrightarrow X$ be the linear extension of the identity on X (the barycentre map). We say that X has the **isometric Lipschitz lifting property** if there exists a linear operator $T : X \longrightarrow \mathcal{F}(X)$ such that $\|T\| = 1$ and $\beta T = \mathrm{Id}_X$.

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Corollary (Godefroy, Kalton, '03)

The BAP is stable under bi-Lipschitz homeomorphisms between Banach spaces.

A refinement of the construction of the lifting leads to:

Theorem (Godefroy, Ozawa, '14)

If X is a separable Banach space and K is a closed convex subset containing 0 such that $\overline{\operatorname{span}}\, K = X$, then X is isometric to a 1-complemented subspace of $\mathcal{F}(K)$.
In particular, there exists a compact set K such that $\mathcal{F}(K)$ fails AP.

Theorem (Hájek, Lancien, P., '16)

If X is a separable Banach space, then there exists a compact set $K \subset X$ homeomoprhic to the Cantor space such that X is isomorphic to a complemented subspace of $\mathcal{F}(K)$.
In particular, there exists a compact metric space K homeomoprhic to the Cantor space such that $\mathcal{F}(K)$ fails the AP.

Proposition

Let M be a separable pointed metric space. Suppose there exist finite subsets M_n of M containing the base point such that $M_1 \subseteq M_2 \subseteq M_3 \subseteq \dots$ and $\bigcup_{n=1}^{\infty} M_n = M$, and linear operators $E_n : \mathrm{Lip}_0(M_n) \longrightarrow \mathrm{Lip}_0(M)$ such that:

- *$\|E_n\| \leq \lambda$ for some $\lambda \geq 1$ (uniformly bounded),*
- *$|E_n(f)(p) - f(p)| \leq \alpha_n \,\mathrm{Lip}(f)$ for all $p \in M_n, f \in \mathrm{Lip}_0(M_n)$, where $\alpha_n \searrow 0$ (near-extension operators).*

Then $\mathcal{F}(M)$ has the $\lambda-$BAP.

Proof: Define operators $L_n : \mathrm{Lip}_0(M) \longrightarrow \mathrm{Lip}_0(M)$ by

$$L_n(f) := E_n(f|_{M_n}) \qquad \text{for all } f \in \mathrm{Lip}_0(M).$$

Then L_n are $\lambda-$bounded $w^* - w^*$ continuous finite-rank operators such that $\lim_{n\to\infty} L_n(f)(p) = f(p)$ for all $f \in \mathrm{Lip}_0(M)$ and all $p \in M$. Hence, their preadjoint operators

$$(L_n)_* : \mathcal{F}(M) \longrightarrow \mathcal{F}(M)$$

yield the $\lambda-$BAP for $\mathcal{F}(M)$. $\square$

Doubling spaces

A metric space M is called *doubling* if there exists a constant $D > 0$ such that any open ball in M with radius R can be covered with at most D many open balls of radius $\frac{R}{2}$.

Theorem (Lancien, P, '13)

If M is a doubling metric space, then $\mathcal{F}(M)$ has the $C(\log(D))$-BAP.

Proof:
Extension operators due to Lee and Naor, '05/ Brudnyi and Brudnyi, '06.

Proposition

Let M be a separable pointed metric space. Suppose there exist finite subsets M_n of M containing the base point such that $M_1 \subseteq M_2 \subseteq M_3 \subseteq \ldots$ and $\overline{\bigcup_{n=1}^{\infty} M_n} = M$, and linear operators $E_n : \mathrm{Lip}_0(M_n) \longrightarrow \mathrm{Lip}_0(M)$. Consider the following conditions:

1. $\|E_n\| \leq \lambda$ *for some* $\lambda \geq 1$ *(uniformly bounded),*
2. $E_n(f)\big|_{M_n} = f$ *for all* $f \in \mathrm{Lip}_0(M_n)$ *(extension),*
3. $E_n(E_m(f|_{M_m})|_{M_n}) = E_m(E_n(f|_{M_n})|_{M_m}) = E_m(f|_{M_m})$ *for all* $m \leq n \in \mathbb{N}$ *and all* $f \in \mathrm{Lip}_0(M)$ *(commuting operators),*
4. $|M_{n+1}| = |M_n| + 1$ *(with rank n).*

If the conditions (1)–(3) are satisfied, then $\mathcal{F}(M)$ has a finite dimensional Schauder decomposition.
If the conditions (1)–(4) are satisfied, then $\mathcal{F}(M)$ has a Schauder basis.

Proof: Commuting extension operators on $\mathrm{Lip}_0(M)$ induce commuting projections on $\mathcal{F}(M)$.

ℓ_1

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Recall that for $M \subset \mathbb{R}^N$ with nonempty interior, $\mathcal{F}(M) \simeq \mathcal{F}\left(\mathbb{R}^N\right)$ (*Kaufmann, '15*). In particular, $\mathcal{F}(M)$ has a basis.

Finite dimensional spaces

Theorem (P, Smith, '14)

If K is a compact convex subset of a finite-dimensional Banach space, then $\mathcal{F}(K)$ has the MAP.

Proof: For any norm on $\mathbb{R}^N$, if f is a uniformly differentiable function, then the coordinate-wise affine interpolation of f on cubes almost preserves the Lipschitz constant. Therefore we first uniformly approximate a Lipschitz function by a smooth function using convolution with a smooth mollifier.

Retractions

If there exist uniformly bounded (commuting) Lipschitz retractions

$$r_n : M \longrightarrow M_n,$$

then operators $E_n : \mathrm{Lip}_0(M_n) \longrightarrow \mathrm{Lip}_0(M)$ defined by

$$E_n(f) := f \circ r_n \qquad \text{for all } f \in \mathrm{Lip}_0(M_n)$$

satisfy the hypothesis of the Proposition.

Theorem (Godefroy, Ozawa, '14)

If K is a "small" Cantor set, then $\mathcal{F}(K)$ has the MAP.

Theorem (Cúth, Doucha, '14)

If M is a separable ultrametric space, then $\mathcal{F}(M)$ has a monotone Schauder basis.

Theorem (Hájek, Novotný, '17)

$\mathcal{F}(\mathcal{N})$ has a Schauder basis for a net $\mathcal{N}$ in $C(K)$ for K metrizable compact.

Theorem (Godefroy, '15; Ambrosio, Puglisi, '16)

For a separable metric space M and a sequence of finite subsets $M_1 \subseteq M_2 \subseteq M_3 \subseteq \ldots \subseteq M$ such that $\overline{\bigcup_{n=1}^{\infty} M_n} = M$, TFAE:

- *$\mathcal{F}(M)$ has the $\lambda-$BAP.*
- *There exist $\lambda-$bounded linear near-extension operators for Lipschitz functions from sets M_n to the whole space M.*
- *There exist $\lambda-$bounded linear near-extension operators for Banach space-valued Lipschitz functions from sets M_n to the whole space M.*
- *There exist $\lambda-$Lipschitz uniform near-extensions for Banach space-valued 1-Lipschitz functions from sets M_n to the whole space M.*

Questions

- *A direct construction of a compact space K such that $\mathcal{F}(K)$ fails the AP. (Godefroy)*
- *Does $\mathcal{F}(M)$ have the MAP for any $M \subset \mathbb{R}^N$? (Godefroy)*
- *Does $\mathcal{F}(\ell_2)$ have a Schauder basis or a finite dimensional decomposition?*
- *Let M be a uniformly discrete metric space. Does $\mathcal{F}(M)$ have the BAP? (Kalton)*
- *Are some analogues of results due to Godefroy and Godefroy and Kalton true for finite dimensional decompositions or bases? Can these properties for free spaces be characterized by the existence of extension operators for Lipschitz functions? Are they equivalent for a Banach space and its free space?*
- ...

Thank you for your attention!