Fragmentation, amalgamation and twisted Hilbert spaces

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Palais' problem

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The theory of twisted Hilbert spaces grew from this problem.

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The first non-trivial twisted Hilbert was obtained by Enflo, Lindenstrauss and Pisier, giving a negative answer to Palais' problem.

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So pasting all with the ℓ_2 norm it results

$$0 \longrightarrow \ell_2(\ell_2^{n^2}) \longrightarrow \ell_2(ELP^n) \longrightarrow \ell_2(\ell_2^n) \longrightarrow 0,$$

and cannot exists a continouos proyection to the subspace.

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Given an admissible pair (X_0, X_1) of complex Banach spaces, let $\Sigma = X_0 + X_1$ endowed with the norm

$$\|x\| = \inf\{\|x_0\|_0 + \|x_1\|_1 : x = x_0 + x_1\}.$$

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This space $\mathcal{F}$ is a Banach space under the norm

$$\|f\|_{\mathcal{F}} = \sup\{\|f(it + j)\|_j : j = 0, 1; t \in \mathbb{R}\}.$$

Complex interpolation

Now, we can define the interpolated space at $0 \leq \theta \leq 1$

$$X_\theta = (X_0, X_1)_\theta = \{x \in \Sigma : x = f(\theta) \text{ for some } f \in \mathcal{F}\}$$

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Now, if $\delta_\theta : \mathcal{F} \to \Sigma$ is the evaluation map at θ, then X_θ is the quotient of $\mathcal{F}$ by $\ker \delta_\theta$,

$$0 \longrightarrow \ker \delta_\theta \longrightarrow \mathcal{F} \longrightarrow X_\theta \longrightarrow 0.$$

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If the interpolated space is a Hilbert space H we can complete the diagram doing a push-out

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker \delta & \longrightarrow & \mathcal{F} & \xrightarrow{\delta} & H & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \delta'} & & \downarrow & & \| & & \\
0 & \longrightarrow & H & \longrightarrow & PO & \longrightarrow & H & \longrightarrow & 0
\end{array}$$

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Example (Kalton-Peck derivation)

Fix the couple (ℓ_p, ℓ_q), where $\frac{1}{p} + \frac{1}{q} = 1$. When we interpolate this scale at $1/2$ appears the space ℓ_2, and the map $B(x)(z) = x^{2\left(\frac{1}{p} - \frac{1}{q}\right)(1-z)}$ is a bounded homogeneous selection for $\delta_{1/2}$,

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$$\mathcal{K}(x) = -2\left(\frac{1}{p} - \frac{1}{q}\right) x \log \frac{|x|}{\|x\|}.$$

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Given a finite set $A \subset \mathbb{N}$, we define

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Lemma

$(L(A), L^*(A))_\theta = (L, L^*)_\theta(A)$ with derivation $\Omega_A(x) = 1_A \Omega(1_A x)$.

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Let Λ be a Köthe space defined on a measure space M. Given a Banach space X one can define the vector valued Banach space $\Lambda(X)$ of measurable functions $f : M \to X$ such that the function $\hat{f}(\cdot) = \|f(\cdot)\|_X : M \to \mathbb{R}$ given by $t \mapsto \|f(t)\|_X$ is in Λ, endowed with the norm $\|\|f(\cdot)\|_X\|_\Lambda$.

Amalgamation

Theorem

Fix $0<\theta<1$. *Let* (λ_0, λ_1) *an interpolation couple of Banach lattices on the same measure space for which* $(\lambda_0, \lambda_1)_\theta = \lambda_0^{1-\theta}\lambda_1^\theta$ *with associated derivation* ω_θ. *Let* (X_0, X_1) *be an interpolation couple of Banach spaces with associated derivation* Ω_θ *at* θ. *Assume that* $\lambda_0(X_0)$ *is reflexive. Then*

$$(\lambda_0(X_0), \lambda_1(X_1))_\theta = \lambda_0^{1-\theta}\lambda_1^\theta\left((X_0, X_1)_\theta\right)$$

with associated derivation Φ_θ *defined on the dense subspace of simple functions as follows: given* $f=\sum_{n=1}^N a_n 1_{A_n}$ *then*

$$\Phi_\theta\left(f\right) = \omega_\theta\left(\widehat{f}(\cdot)\right)\sum_{n=1}^N \frac{a_n}{\|a_n\|}1_{A_n} + \sum_{n=1}^N \Omega_\theta(a_n)1_{A_n}.$$

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$$\Phi(a) = \left(\frac{2}{p} - \frac{2}{p^*}\right) \sum_{k=1}^{N} \left(a_k \log \frac{\|a_k\|}{\|a\|} - \sum_n a_k(n) \log \frac{|a_k(n)|}{\|a_k\|} e_n \right) u_k.$$

Thank you for your attention