Fragmentation, amalgamation and twisted Hilbert spaces

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September 12, 2019

This work was supported by project MTM2016-76958-C2-1-P



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Palais' problem

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The theory of twisted Hilbert spaces grew from this problem.

Twisted Hilbert spaces

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The first non-trivial twisted Hilbert was obtained by Enflo, Lindenstrauss and Pisier, giving a negative answer to Palais' problem.

ELP space

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in such a way that $\lim_{n\to\infty} ||P_n|| = \infty$. So pasting all with the ℓ_2 norm it results

$$0 \longrightarrow \ell_2(\ell_2^{n^2}) \longrightarrow \ell_2(ELP^n) \longrightarrow \ell_2(\ell_2^n) \longrightarrow 0,$$

and cannot exists a continuous proyection to the subspace.

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Given an admissible pair (X_0, X_1) of complex Banach spaces, let $\Sigma = X_0 + X_1$ endowed with the norm

$$||x|| = \inf\{||x_0||_0 + ||x_1||_1 : x = x_0 + x_1\}.$$

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This space ${\mathcal F}$ is a Banach space under the norm

$$||f||_{\mathcal{F}} = \sup\{||f(it+j)||_j : j = 0, 1; t \in \mathbb{R}\}.$$

Complex interpolation

Now, we can define the interpolated space at $0 \leq \theta \leq 1$

$$X_{\theta} = (X_0, X_1)_{\theta} = \{ x \in \Sigma : x = f(\theta) \text{ for some } f \in \mathcal{F} \}$$

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Now, if $\delta_{\theta} : \mathcal{F} \to \Sigma$ is the evaluation map at θ , then X_{θ} is the quotient of \mathcal{F} by ker δ_{θ} ,

$$0 \longrightarrow \ker \delta_{\theta} \longrightarrow \mathcal{F} \longrightarrow X_{\theta} \longrightarrow 0.$$

Complex interpolation

The following lemma provides the connection between complex interpolation and twisted Hilbert spaces

Lemma

 δ'_{θ} : ker $\delta_{\theta} \to X_{\theta}$ is bounded and onto for $0 < \theta < 1$.

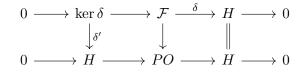
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If the interpolated space is a Hilbert space H we can complete the diagram doing a push-out



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Example (Kalton-Peck derivation)

Fix the couple (ℓ_p, ℓ_q) , where $\frac{1}{p} + \frac{1}{q} = 1$. When we interpolate this scale at 1/2 appears the space ℓ_2 , and the map $B(x)(z) = x^{2\left(\frac{1}{p} - \frac{1}{q}\right)(1-z)}$ is a bounded homogeneous selection for $\delta_{1/2}$,

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$$\mathcal{K}(x) = -2\left(\frac{1}{p} - \frac{1}{q}\right) x \log \frac{|x|}{\|x\|}.$$

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Let L be a Banach space such that there is a common unconditional basis for L and his dual L^* . Given a finite set $A \subset \mathbb{N}$, we define

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Lemma

$$(L(A), L^*(A))_{\theta} = (L, L^*)_{\theta}(A)$$
 with derivation
 $\Omega_A(x) = 1_A \Omega(1_A x).$

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Let Λ be a Köthe space defined on a measure space M. Given a Banach space X one can define the vector valued Banach space $\Lambda(X)$ of measurable functions $f: M \to X$ such that the function $\hat{f}(\cdot) = \|f(\cdot)\|_X : M \to \mathbb{R}$ given by $t \mapsto \|f(t)\|_X$ is in Λ , endowed with the norm $\|\|f(\cdot)\|_X\|_{\Lambda}$.

Amalgamation

Theorem

Fix $0 < \theta < 1$. Let (λ_0, λ_1) an interpolation couple of Banach lattices on the same measure space for which $(\lambda_0, \lambda_1)_{\theta} = \lambda_0^{1-\theta} \lambda_1^{\theta}$ with associated derivation ω_{θ} . Let (X_0, X_1) be an interpolation couple of Banach spaces with associated derivation Ω_{θ} at θ . Assume that $\lambda_0(X_0)$ is reflexive. Then

$$(\lambda_0(X_0), \lambda_1(X_1))_{\theta} = \lambda_0^{1-\theta} \lambda_1^{\theta} \left((X_0, X_1)_{\theta} \right)$$

with associated derivation Φ_{θ} defined on the dense subspace of simple functions as follows: given $f = \sum_{n=1}^{N} a_n 1_{A_n}$ then

$$\Phi_{\theta}(f) = \omega_{\theta}\left(\widehat{f}(\cdot)\right) \sum_{n=1}^{N} \frac{a_n}{\|a_n\|} \mathbf{1}_{A_n} + \sum_{n=1}^{N} \Omega_{\theta}(a_n) \mathbf{1}_{A_n}.$$

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$$\Phi(a) = \left(\frac{2}{p} - \frac{2}{p^*}\right) \sum_{k=1}^N \left(a_k \log \frac{\|a_k\|}{\|a\|} - \sum_n a_k(n) \log \frac{|a_k(n)|}{\|a_k\|} e_n\right) u_k.$$

Thank you for your attention