FUNDAMENTAL SOLUTIONS FOR FRACTIONAL DIFFERENTIAL EQUATIONS INVOLVING FRACTIONAL POWERS OF FINITE DIFFERENCES OPERATORS

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Workshop on Banach spaces and Banach lattices
ICMAT, September 2019, 9th-13th

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1. Introduction

We present the solution of fractional differential equation

$$\begin{cases} \mathbb{D}_t^{\beta} u(n,t) = Bu(n,t) + g(n,t), & n \in \mathbb{Z},\ t > 0. \\ u(n,0) = \varphi(n), \ \ u_t(n,0) = \phi(n) & n \in \mathbb{Z}, \end{cases} \tag{1}$$

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$Bf(n) = (K * f)(n)$, with $K \in l^{\infty}(\mathbb{Z})$, $f \in l^p(\mathbb{Z})$, $p \in [1, \infty]$ and $\beta \in (1, 2]$. We recall that $\mathbb{D}_t^{\beta}$ denotes the Caputo fractional derivative given by

$$\mathbb{D}_t^{\beta} v(t) = \frac{1}{\Gamma(2-\beta)} \int_0^t (t-s)^{1-\beta} v''(s) ds = (g_{2-\beta} * v'')(t),$$

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$$\mathbb{D}_t^{\beta} v(t) = \frac{1}{\Gamma(2-\beta)} \int_0^t (t-s)^{1-\beta} v''(s) ds = (g_{2-\beta} * v'')(t),$$

$$g_{\alpha}(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \text{ for } \alpha > 0.$$

1. Introduction

For $1 \leq p \leq \infty$, the Banach space $(\ell^p(\mathbb{Z}), \| \quad \|_p)$ are formed by $f = (f(n))_{n\in\mathbb{Z}} \subset \mathbb{C}$ such that

$$\begin{aligned} \|f\|_p : &= \left(\sum_{n=-\infty}^{\infty} |f(n)|^p\right)^{\frac{1}{p}} < \infty, \qquad 1 \leq p < \infty; \\ \|f\|_\infty : &= \sup_{n\in\mathbb{Z}} |f(n)| < \infty. \end{aligned}$$

$\ell^1(\mathbb{Z}) \hookrightarrow \ell^p(\mathbb{Z}) \hookrightarrow \ell^\infty(\mathbb{Z})$, $(\ell^p(\mathbb{Z}))' = \ell^{p'}(\mathbb{Z})$ with $\frac{1}{p} + \frac{1}{p'} = 1$ for $1 < p < \infty$ and $p = 1$ and $p' = \infty$.
In the case that $f \in \ell^1(\mathbb{Z})$ and $g \in \ell^p(\mathbb{Z})$, then $f * g \in \ell^p(\mathbb{Z})$ where

$$(f * g)(n) := \sum_{j=-\infty}^{\infty} f(n-j)g(j), \qquad n \in \mathbb{Z},$$

and $\|f * g\|_p \leq \|f\|_1 \|g\|_p$ for $1 \leq p \leq \infty$. Note that $(\ell^1(\mathbb{Z}), *)$ is a commutative Banach algebra with unit (we write $\delta_0 = \chi_{\{0\}}$).

1. Introduction

We apply Güelfand theory to get

$$\sigma_{\ell^1(\mathbb{Z})}(f) = \mathcal{F}(f)(\mathbb{T}), \qquad f \in \ell^1(\mathbb{Z}),$$

where

$$\mathcal{F}(f)(\theta) := \sum_{n\in\mathbb{Z}} f(n)e^{in\theta}, \quad \theta \in \mathbb{T}.$$

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$$\mathcal{F}(f)(\theta) := \sum_{n\in\mathbb{Z}} f(n)e^{in\theta}, \quad \theta \in \mathbb{T}.$$

Given $a = (a(n))_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$, define $A \in \mathcal{B}(\ell^p(\mathbb{Z}))$ by convolution,

$$A(b)(n) := (a * b)(n), \qquad n \in \mathbb{Z}, \qquad b \in \ell^p(\mathbb{Z}),$$

for all $1 \le p \le \infty$, $\|A\| = \|a\|_1$ and

$$\sigma_{\mathcal{B}(\ell^p(\mathbb{Z}))}(A) = \sigma_{\ell^1(\mathbb{Z})}(a) = \mathcal{F}(a)(\mathbb{T}) \tag{2}$$

for all $1 \le p \le \infty$, (Wiener's Lemma).

Aims of the talk

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The main aim of this talk is to study the fractional differential equations in $\ell^p(\mathbb{Z})$ for $1 \leq p \leq \infty$. To do this.

(i) We apply Güelfand theory to describe convolution operators.

(ii) We calculate the kernel of the convolution fractional powers.

(iii) We solve some fractional evolution equation in $\ell^p(\mathbb{Z})$.

(iv) Finally we obtain explict solutions for fractional evolution equation for some fractional powers of finite difference operators.

2. Finite difference operators on $\ell^1(\mathbb{Z})$

Finite difference operators $A \in \mathcal{B}(\ell^p(\mathbb{Z}))$ given by

$$Af(n) := \sum_{j=-m}^{m} a(j) f(n-j), \quad a_j \in \mathbb{C},$$

for some $m \in \mathbb{N}$, i.e. $a = (a(n)_{n \in \mathbb{Z}}) \in c_c(\mathbb{Z})$ are convolution operator and the discrete Fourier Transform of a is a trigonometric polynomial

$$\mathcal{F}(a)(\theta) = \sum_{j=-m}^{m} a(j) e^{ij\theta}.$$

2. Finite difference operators on $\ell^1(\mathbb{Z})$

1. $D_+ f(n) := f(n) - f(n+1) = ((\delta_0 - \delta_{-1}) * f)(n)$;
2. $D_- f(n) := f(n) - f(n-1) = ((\delta_0 - \delta_1) * f)(n)$;
3. $\Delta_d f(n) := f(n+1) - 2f(n) + f(n-1) = ((\delta_{-1} - 2\delta_0 + \delta_1) * f)(n)$;
4. $\mathcal{D} f(n) := f(n+1) - f(n-1) = ((\delta_{-1} - \delta_1) * f)(n)$;
5. $\Delta_{++} f(n) := f(n+2) - 2f(n+1) + f(n) = ((\delta_{-2} - 2\delta_{-1} + \delta_0) * f)(n)$;
6. $\Delta_{--} f(n) := f(n) - 2f(n-1) + f(n-2) = ((\delta_0 - 2\delta_1 + \delta_2) * f)(n)$;
7. $\Delta_{dd} f(n) := f(n+2) - 2f(n) + f(n-2) = ((\delta_{-2} - 2\delta_0 + \delta_2) * f)(n)$;

for $n \in \mathbb{Z}$, [Bateman, 1943].

2. Finite difference operators on $\ell^1(\mathbb{Z})$

Proposition

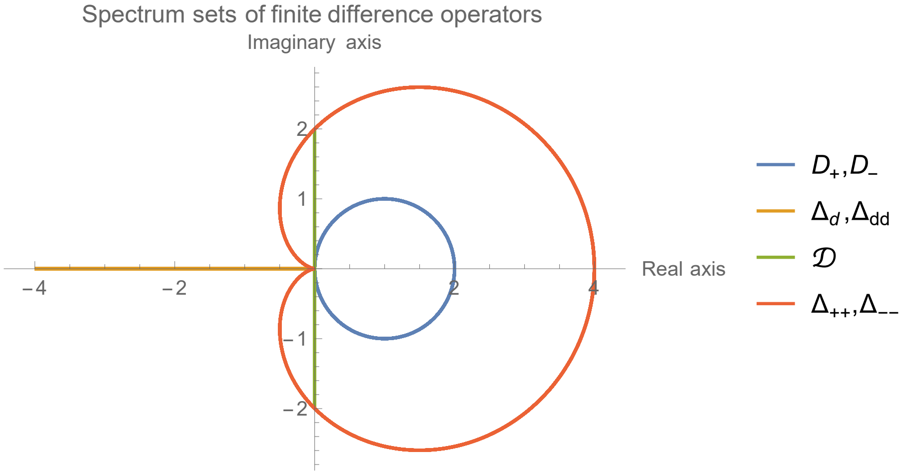
The following equalities hold:

(i)

$$\begin{aligned}
\Delta_d &= -(D_+ + D_-) = -D_+ D_-, \\
\mathcal{D} &= -(D_+ - D_-) = (-D_+ + 2I)D_- = (D_- - 2I)D_+, \\
\Delta_{dd} &= (\Delta_{++} - 2\Delta_d + \Delta_{--}) = \mathcal{D}^2 f.
\end{aligned}$$

(ii) $(D_+)' = D_-$; $(D_-)' = D_+$; $(\Delta_d)' = \Delta_d$; $(\mathcal{D})' = \mathcal{D}$; $(\Delta_{dd})' = \Delta_{dd}$.

2. Finite difference operators on $\ell^1(\mathbb{Z})$



2. Finite difference operators on $\ell^1(\mathbb{Z})$

$$e^{za} := \sum_{n=0}^{\infty} \frac{z^n a^{*n}}{n!}, \qquad z \in \mathbb{C}.$$

$$\cosh(za) := \sum_{n=0}^{\infty} \frac{z^{2n} a^{*2n}}{(2n)!}, \qquad z \in \mathbb{C}.$$

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Proposition

Let $A \in \mathcal{B}(l^p(\mathbb{Z}))$, with $Af = a * f,\ f \in \ell^p(\mathbb{Z})$ and $a \in \ell^\infty(\mathbb{Z})$. Then, the Fourier transform of the semigroup $\{e^{at}\}_{t\geq 0}$ generated by a is given by

$$\mathcal{F}(e^{at})(\theta) = e^{\mathcal{F}(a)(\theta)t}.$$

$$\mathcal{F}(cosh(at))(\theta) = cosh(\mathcal{F}(a)(\theta)t).$$

2. Finite difference operators on $\ell^1(\mathbb{Z})$

Operator	$\mathcal{F}(\cdot)(z)$	Associated semigroup
$-D_+$	$z-1$	$e^{-z}\frac{z^n}{n!}\chi_{\mathbb{N}_0}(n) =: g_{z,+}(n)$
$-D_-$	$\frac{1}{z}-1$	$e^{-z}\frac{z^{-n}}{(-n)!}\chi_{-\mathbb{N}_0}(n) =: g_{z,-}(n)$
Δ_d	$z+\frac{1}{z}-2$	$e^{-2z}I_n(2z)$
$\mathcal{D}$	$z-\frac{1}{z}$	$J_n(2z)$
$-\mathcal{D}$	$\frac{1}{z}-z$	$J_{-n}(2z)$
$-D_+ + 2$	$z+1$	$e^{z}\frac{z^n}{n!}\chi_{\mathbb{N}_0}(n)$
$-D_- + 2$	$\frac{1}{z}+1$	$e^{z}\frac{z^n}{n!}\chi_{-\mathbb{N}_0}(n)$
Δ_{++}	z^2-2z+1	$\frac{e^z(i\sqrt{2z})^{-n}H_{-n}(2iz)}{(-n)!}\chi_{\mathbb{N}_0}(n) =: h_{z,+}(n)$
Δ_{--}	$\frac{1}{z^2}-2\frac{1}{z}+1$	$\frac{e^z(i\sqrt{2z})^{n}H_{n}(2iz)}{n!}\chi_{-\mathbb{N}_0}(n) =: h_{z,-}(n)$
Δ_{dd}	$z^2-2+\frac{1}{z^2}$	$e^{-2z}I_n(2z)\chi_{2\mathbb{Z}}(n)$

2. Finite difference operators on $\ell^1(\mathbb{Z})$

Theorem

(i) *The Bessel function J_n has a factorization expression given by*

$$J_n(2z) = (g_{-z,+} * g_{z,-})(n), \qquad n \in \mathbb{Z},\ z \in \mathbb{C}.$$

(ii) *The Bessel function I_n admits factorization product given by*

$$\begin{aligned} e^{-2z} I_n(2z) &= (g_{z,+} * g_{z,-})(n), \\ I_n(2z) &= (j_{z,+} * j_{z,-})(n). \end{aligned}$$

(iii) *The Bessel function $e^{-2z} I_n(2z)\chi_{2\mathbb{Z}}(n)$ admits a factorization given by*

$$I_n(2z)\chi_{2\mathbb{Z}}(n) = h_{z,+}(n) * I_n(2z) * e^{-2z} I_n(2z) * h_{z,-}(n).$$

3. Fractional powers of discrete operators

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The Generalized Binomial Theorem is given by

$$(a+b)^{\alpha} = \sum_{j=0}^{\infty} \binom{\alpha}{j} a^{\alpha-j} b^{j}, \quad \alpha \in \mathbb{C}.$$

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The Generalized Binomial Theorem is given by

$$(a+b)^\alpha = \sum_{j=0}^{\infty} \binom{\alpha}{j} a^{\alpha-j} b^j, \quad \alpha \in \mathbb{C}.$$

For $\alpha > 0$, $\binom{\alpha}{j} \sim \frac{1}{j^{\alpha+1}}$ and $\|a\| \leq 1$

$$(\delta_0 + a)^\alpha = \sum_{j=0}^{\infty} \binom{\alpha}{j} a^j, \quad \alpha > 0.$$

3. Fractional powers of discrete operators

For $0 < \alpha < 1$, the Balakrishnan's formula is expressed by

$$(-A)^\alpha x = \frac{1}{\Gamma(-\alpha)} \int_0^\infty (T(t)x - x) \frac{dt}{t^{1+\alpha}}, \; x \in D(A).$$

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Theorem

Let $0 < \alpha < 1$, and $A \in \mathcal{B}(\ell^p(\mathbb{Z}))$, $1 \le p \le \infty$ a generator of a uniformly bounded semigroup, with $Af = a * f$, $f \in \ell^p(\mathbb{Z})$ and $a \in \ell^1(\mathbb{Z})$. Then the fractional powers $(-A)^\alpha$ is well-posedness and it is expressed by

$$(-A)^\alpha f = (-a)^\alpha * f,$$

where

$$(-a)^\alpha(n) := \frac{1}{2\pi} \int_0^{2\pi} (-\mathcal{F}(a)(\theta))^\alpha e^{-in\theta} d\theta.$$

3. Fractional powers of discrete operators

$$\Lambda^\alpha(m) := \binom{-\alpha-1+m}{m} = (-1)^m \binom{\alpha}{m}, \text{ for } m \in \mathbb{N}_0.$$

Fractional power	Kernel	Explicit expression
D_+^α	K_+^α	$\Lambda^\alpha(n)\chi_{\mathbb{N}_0}$
D_-^α	K_-^α	$\Lambda^\alpha(n)\chi_{-\mathbb{N}_0}$
$(-\Delta_d)^\alpha$	K_d^α	$\frac{(-1)^n\Gamma(2\alpha+1)}{\Gamma(1+\alpha+n)\Gamma(1+\alpha-n)}$
$\mathcal{D}^\alpha$	$K_{\mathcal{D}_+}^\alpha$	$\frac{i^n}{2}\frac{\Gamma(\alpha+1)}{\Gamma(\frac{\alpha}{2}+\frac{n}{2}+1)\Gamma(\frac{\alpha}{2}-\frac{n}{2}+1)}$
$(-\mathcal{D})^\alpha$	$K_{\mathcal{D}_-}^\alpha$	$\frac{(-i)^n}{2}\frac{\Gamma(\alpha+1)}{\Gamma(\frac{\alpha}{2}+\frac{n}{2}+1)\Gamma(\frac{\alpha}{2}-\frac{n}{2}+1)}$
$(-D_+ + 2I)^\alpha$	$K_{(-D_++2I)}^\alpha$	$(-1)^m\Lambda^\alpha(n)\chi_{\mathbb{N}_0}$
$(-D_- + 2I)^\alpha$	$K_{(-D_-+2I)}^\alpha$	$(-1)^m\Lambda^\alpha(n)\chi_{-\mathbb{N}_0}$
Δ_{++}^α	$K_{D_{++}}^\alpha$	$\Lambda^{2\alpha}(n)\chi_{\mathbb{N}_0}$
Δ_{--}^α	$K_{D_{--}}^\alpha$	$\Lambda^{2\alpha}(n)\chi_{-\mathbb{N}_0}$
$(-\Delta_{dd})^\alpha$	K_{dd}^α	$\frac{(-i)^n}{2}\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+\frac{n}{2}+1)\Gamma(\alpha-\frac{n}{2}+1)}$

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Theorem

Let $\alpha \in (0,1)$. These kernels admit following factorization equalities.

1. $K_d^\alpha = K_-^\alpha * K_+^\alpha$,
2. $K_{\mathcal{D}_+}^\alpha = K_{-D_-+2}^\alpha * K_+^\alpha$,
3. $K_{\mathcal{D}_-}^\alpha = K_{-D_++2}^\alpha * K_-^\alpha$,
4. $K_{++}^\alpha = K_{\mathcal{D}_+}^\alpha * K_{\mathcal{D}_+}^\alpha$,
5. $K_{--}^\alpha = K_{\mathcal{D}_-}^\alpha * K_{\mathcal{D}_-}^\alpha$,
6. $K_{dd}^\alpha = K_{\mathcal{D}_-}^\alpha * K_{\mathcal{D}_+}^\alpha$.

4. Fundamental solutions of fractional evolution equations

We consider the operator $Bf(n) = (b * f)(n)$, with $b \in \ell^\infty(\mathbb{Z})$, $f \in \ell^p(\mathbb{Z})$, $p \in [1, \infty]$. We obtain an explicit representation of solutions of

$$\begin{cases} \mathbb{D}_t^\beta u(n,t) = Bu(n,t) + g(n,t), & n \in \mathbb{Z},\ t > 0. \\ u(n,0) = \varphi(n), \quad u_t(n,0) = \phi(n) & n \in \mathbb{Z}. \end{cases}$$

Here $\beta \in (1, 2]$ is real number. We recall that $\mathbb{D}_t^\beta$ denotes the Caputo fractional derivative given by

$$\mathbb{D}_t^\beta v(t) = \frac{1}{\Gamma(2-\beta)} \int_0^t (t-s)^{1-\beta} v''(s)ds = (g_{2-\beta} * v'')(t).$$

4. Fundamental solutions of fractional evolution equations

For $\beta, \gamma > 0$, and $b \in \ell^\infty(\mathbb{Z})$ we define

$$B_{\beta,\gamma}(n,t) := \sum_{j=0}^{\infty} b^{*j}(n) g_{j\beta+\gamma}(t) = \sum_{j=0}^{\infty} b^{*j}(n) \frac{t^{j\beta+\gamma-1}}{\Gamma(j\beta+\gamma)}, \ \ n \in \mathbb{Z}, \ \ t > 0.$$

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Lemma

If $b \in \ell^1(\mathbb{Z})$ then $B_{\beta,\gamma}(\cdot, t) \in \ell^1(\mathbb{Z})$, for $t > 0$ and $\beta, \gamma > 0$.

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Lemma

If $b \in \ell^1(\mathbb{Z})$ then $B_{\beta,\gamma}(\cdot, t) \in \ell^1(\mathbb{Z})$, for $t > 0$ and $\beta, \gamma > 0$.

$$\begin{aligned} B_{1,1}(n,t) &= \sum_{j=0}^{\infty} b^{*j}(n) \frac{t^j}{j!} = e^{tb}(n); \\ B_{2,1}(n,t) &= \sum_{j=0}^{\infty} b^{*j}(n) \frac{t^{2j}}{(2j)!} = \cosh(tb)(n); \\ (B_{\beta,\gamma}(n,\cdot) * g_\alpha)(t) &= B_{\beta,\gamma+\alpha}(n,t), \quad \alpha, t > 0. \end{aligned}$$

4. Fundamental solutions of fractional evolution equations

$$\begin{cases} \mathbb{D}_t^{\beta} u(n,t) = Bu(n,t) + g(n,t), & n \in \mathbb{Z}, \ t > 0. \\ u(n,0) = \varphi(n), \ \ u_t(n,0) = \phi(n) & n \in \mathbb{Z}. \end{cases}$$

Theorem

Let $b \in \ell^1(\mathbb{Z})$, $\varphi, \phi, g(\cdot, t) \in \ell^p(\mathbb{Z})$ for $t > 0$ and $\sup_{s \in [0,t]} ||g(\cdot, s)||_p < \infty$. Then the function

$$u(n,t) = \sum_{m \in \mathbb{Z}} B_{\beta,1}(n-m,t)\varphi(m) + \sum_{m \in \mathbb{Z}} B_{\beta,2}(n-m,t)\phi(m) + \sum_{m \in \mathbb{Z}} \int_0^t B_{\beta,\beta}(n-m,t-s)g(m,s)ds,$$

is the unique solution of the initial value problem on $\ell^p(\mathbb{Z})$ for $1 \le p \le \infty$.

5. Explicit solutions for fractional powers

Given $\varphi, \phi, g(\cdot, t) \in \ell^p(\mathbb{Z})$, $a \in \ell^1(\mathbb{Z})$ and

$$Af = a * f, \qquad f \in \ell^p(\mathbb{Z}).$$

Now we consider the following evolution problem

$$\begin{cases} \mathbb{D}_t^{\beta} u(n,t) = \pm(\pm A)^{\alpha} u(n,t) + g(n,t), & n \in \mathbb{Z},\ t > 0. \\ u(n,0) = \varphi(n), \ \ u_t(n,0) = \phi(n), & n \in \mathbb{Z}. \end{cases}$$

5. Explicit solutions for fractional powers

1. $D_+ f(n) := f(n) - f(n+1) = ((\delta_0 - \delta_{-1}) * f)(n)$;
2. $D_- f(n) := f(n) - f(n-1) = ((\delta_0 - \delta_1) * f)(n)$;
3. $\Delta_d f(n) := f(n+1) - 2f(n) + f(n-1) = ((\delta_{-1} - 2\delta_0 + \delta_1) * f)(n)$;
4. $\mathcal{D} f(n) := f(n+1) - f(n-1) = ((\delta_{-1} - \delta_1) * f)(n)$;
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6. $\Delta_{--} f(n) := f(n) - 2f(n-1) + f(n-2) = ((\delta_0 - 2\delta_1 + \delta_2) * f)(n)$;
7. $\Delta_{dd} f(n) := f(n+2) - 2f(n) + f(n-2) = ((\delta_{-2} - 2\delta_0 + \delta_2) * f)(n)$;

for $n \in \mathbb{Z}$.

5. Explicit solutions for fractional powers

Fractional power	Kernel	Explicit expression
D_+^α	K_+^α	$\Lambda^\alpha(n)\chi_{\mathbb{N}_0}$
D_-^α	K_-^α	$\Lambda^\alpha(n)\chi_{-\mathbb{N}_0}$
$(-\Delta_d)^\alpha$	K_d^α	$\frac{(-1)^n\Gamma(2\alpha+1)}{\Gamma(1+\alpha+n)\Gamma(1+\alpha-n)}$
$\mathcal{D}^\alpha$	$K_{\mathcal{D}_+}^\alpha$	$\frac{i^n}{2}\frac{\Gamma(\alpha+1)}{\Gamma(\frac{\alpha}{2}+\frac{n}{2}+1)\Gamma(\frac{\alpha}{2}-\frac{n}{2}+1)}$
$(-\mathcal{D})^\alpha$	$K_{\mathcal{D}_-}^\alpha$	$\frac{(-i)^n}{2}\frac{\Gamma(\alpha+1)}{\Gamma(\frac{\alpha}{2}+\frac{n}{2}+1)\Gamma(\frac{\alpha}{2}-\frac{n}{2}+1)}$
$(-D_+ + 2I)^\alpha$	$K_{(-D_++2I)}^\alpha$	$(-1)^m\Lambda^\alpha(n)\chi_{\mathbb{N}_0}$
$(-D_- + 2I)^\alpha$	$K_{(-D_-+2I)}^\alpha$	$(-1)^m\Lambda^\alpha(n)\chi_{-\mathbb{N}_0}$
Δ_{++}^α	$K_{D_{++}}^\alpha$	$\Lambda^{2\alpha}(n)\chi_{\mathbb{N}_0}$
Δ_{--}^α	$K_{D_{--}}^\alpha$	$\Lambda^{2\alpha}(n)\chi_{-\mathbb{N}_0}$
$(-\Delta_{dd})^\alpha$	K_{dd}^α	$\frac{(-i)^n}{2}\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+\frac{n}{2}+1)\Gamma(\alpha-\frac{n}{2}+1)}$

5. Explicit solutions for fractional powers

A^α	$B^\alpha_{\beta,\gamma}$
D^α_+	$(-1)^n \sum_{j=0}^{\infty} \binom{\alpha j}{n} g_{j\beta+\gamma}(t)\chi_{\mathbb{N}_0}(n)$
D^α_-	$(-1)^n \sum_{j=0}^{\infty} \binom{\alpha j}{-n} g_{j\beta+\gamma}(t)\chi_{-\mathbb{N}_0}(n)$
Δ^α_{++}	$(-1)^n \sum_{j=0}^{\infty} \binom{2\alpha j}{n} g_{j\beta+\gamma}(t)\chi_{\mathbb{N}_0}(n)$
Δ^α_{--}	$(-1)^n \sum_{j=0}^{\infty} \binom{2\alpha j}{-n} g_{j\beta+\gamma}(t)\chi_{-\mathbb{N}_0}(n)$
$\mathcal{D}^\alpha$	$\frac{i^n}{2} \sum_{j=0}^{\infty} \frac{\Gamma(\alpha j+1)}{\Gamma(\frac{\alpha j}{2}+\frac{n}{2}+1)\Gamma(\frac{\alpha j}{2}-\frac{n}{2}+1)} g_{j\beta+\gamma}(t)$
$(-\mathcal{D})^\alpha$	$\frac{(-i)^n}{2} \sum_{j=0}^{\infty} \frac{\Gamma(\alpha j+1)}{\Gamma(\frac{\alpha j}{2}+\frac{n}{2}+1)\Gamma(\frac{\alpha j}{2}-\frac{n}{2}+1)} g_{j\beta+\gamma}(t)$

5. Explicit solutions for fractional powers

$-(-A)^\alpha$	$B^\alpha_{\beta,\gamma}$
$-(-\Delta_d)^\alpha$	$(-1)^n \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(2\alpha j+1)}{\Gamma(\alpha j+n+1)\Gamma(\alpha j-n+1)} g_{\beta j+2}(t)$
$-(-\Delta_{dd})^\alpha$	$\frac{(-i)^n}{2} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(2\alpha j+1)}{\Gamma(\alpha j+\frac{n}{2}+1)\Gamma(\alpha j-\frac{n}{2}+1)} g_{\beta j+\beta}(t)$

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