Fredholm operators on interpolation spaces

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Interpolation functors, Fredholm operator)

Definition

A mapping $F\colon \vec{\mathcal{B}} \to \mathcal{B}$ from the category $\vec{\mathcal{B}}$ of all couples of Banach spaces into the category $\mathcal{B}$ of all Banach spaces is said to be an interpolation functor (or method) if, for any couple $\vec{X} := (X_0, X_1)$, the Banach space $F(X_0, X_1)$ is intermediate with respect to $\vec{X}$ (i.e., $X_0 \cap X_1 \subset F(X_0, X_1) \subset X_0 + X_1$), and

$$T\colon F(X_0, X_1) \to F(Y_0, Y_1) \quad \text{for all} \quad T\colon (X_0, X_1) \to (Y_0, Y_1),$$

where $T\colon (X_0, X_1) \to (Y_0, Y_1)$ means that $T\colon X_0 + X_1 \to Y_0 + Y_1$ is a linear operator such that the restrictions $T|_{X_0}\colon X_0 \to Y_0$, $T|_{X_1}\colon X_1 \to Y_1$ are bounded operators.

Remark. The space of all operators $T\colon \vec{X} \to \vec{Y}$ is a Banach space equipped with the norm

$$\|T\|_{\vec{X}\to\vec{Y}} := \max\left\{\|T|_{X_0}\|_{X_0\to Y_0},\, \|T|_{X_1}\|_{X_1\to Y_1}\right\}$$

- The real method. For $\theta \in (0,1)$ and $p \in [1,\infty]$, $(X_0, X_1)_{\theta,p}$ is defined as the Banach space of all $x \in X_0 + X_1$ equipped with the norm

$$\|x\|_{\theta,p} = \left(\int_0^\infty \left[t^{-\theta} K(t, x; \vec{X}) \right]^p \frac{dt}{t} \right)^{1/p},$$

where

$$K(t, x; \vec{X}) := \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1};\ x = x_0 + x_1\}, \qquad t > 0.$$

- The complex method. Let $S := \{z \in \mathbb{C};\ 0 < \mathrm{Re} z < 1\}$ be an open strip on the plane. For a given $\theta \in (0,1)$ and any couple $\vec{X} = (X_0, X_1)$ we denote by $\mathcal{F}(\vec{X})$ the Banach space of all bounded continuous functions $f\colon \bar{S} \to X_0 + X_1$ on the closure $\bar{S}$ that are analytic on S, and

$$\mathbb{R} \ni t \mapsto f(j+it) \in X_j, \quad j = 0, 1$$

is a bounded continuous function, and equipped with the norm

$$\|f\|_{\mathcal{F}(\vec{X})} = \max\left\{\sup_{t\in\mathbb{R}} \|f(it)\|_{X_0},\ \sup_{t\in\mathbb{R}} \|f(1+it)\|_{X_1}\right\}.$$

The (lower) complex interpolation space $[\vec{X}]_\theta := \{f(\theta);\ f \in \mathcal{F}(\vec{X})\}$ and is equipped with the norm:

$$\|x\|_\theta := \inf\left\{\|f\|_{\mathcal{F}(\vec{X})};\ f(\theta) = x\right\}.$$

- Variants of the complex method. Let $\mathbf{B}$ be the class of all Banach spaces over the complex field. A mapping $\mathcal{X}\colon \mathbf{B} \to \mathbf{B}$ is called a pseudolattice lattice (on $\mathbb{Z}$), if it satisfy the following conditions:
  (i) for every $B \in \mathbf{B}$ the space $\mathcal{X}(B)$ consists of B valued sequences $\{b_n\} = \{b_n\}_{n\in\mathbb{Z}}$ modelled on $\mathbb{Z}$;
  (ii) whenever A is a closed subspace of B it follows that $\mathcal{X}(A)$ is a closed subspace of $\mathcal{X}(B)$;
  (iii) there exists a positive constant C such that, for all $A, B \in \mathbf{B}$ and all bounded linear operators $T\colon A \to B$ and every sequence $\{a_n\} \in \mathcal{X}(A)$, the sequence $\{Ta_n\} \in \mathcal{X}(B)$ and satisfies the estimate
  $$\|\{Ta_n\}\|_{\mathcal{X}(B)} \leqslant C\|T\|_{A\to B}\|\{a_n\}\|_{\mathcal{X}(A)};$$
  (iv) $\|b_m\|_B \leqslant \|\{b_n\}\|_{\mathcal{X}(B)}$ for each $m \in \mathbb{Z}$, all $\{b_n\} \in \mathcal{X}(B)$ and all Banach spaces B.

- For every Banach couple $\vec{B} = (B_0, B_1)$ and every couple of pseudolattices $\vec{\mathcal{X}} = (\mathcal{X}_0, \mathcal{X}_1)\colon \vec{\mathbf{B}} \to \vec{\mathbf{B}}$, let $\mathcal{J}(\vec{\mathcal{X}}, \vec{B})$ be the Banach space of all $B_0 \cap B_1$ valued sequences $\{b_n\}_{n\in\mathbb{Z}}$ such that $\{e^{jn} b_n\}_{n\in\mathbb{Z}} \in \mathcal{X}_j(B_j)$ $(j = 0, 1)$, equipped with the norm.

$$\|\{b_n\}\|_{\mathcal{J}(\vec{\mathcal{X}},\vec{B})} = \max\left\{\|\{b_n\}\|_{\mathcal{X}_0(B_0)}, \|\{e^n b_n\}\|_{\mathcal{X}_1(B_1)}\right\}.$$

- Following Cwikel–Kalton–Milman–Rochberg (2002), for every s in the annulus $\mathbb{A} := \{z \in \mathbb{C};\, 1 < |z| < e\}$, we define the Banach space $\vec{B}_{\vec{\mathcal{X}},s}$ to consist of all elements of the form $b = \sum_{n\in\mathbb{Z}} s^n b_n$ (convergence in $B_0 + B_1$ with $\{b_n\} \in \mathcal{J}(\vec{\mathcal{X}}, \vec{B})$, equipped with the norm

$$\|b\|_{\vec{B}_{\vec{\mathcal{X}},s}} = \inf\left\{\|\{b_n\}\|_{\mathcal{J}(\vec{\mathcal{X}},\vec{B})};\ b = \sum_{n\in\mathbb{Z}} s^n b_n\right\}.$$

The map $\vec{B} \mapsto \vec{B}_{\vec{\mathcal{X}},s}$ is an interpolation method (on $\vec{\mathbf{B}}$).

- A couple $\vec{\mathcal{X}} = (\mathcal{X}_0, \mathcal{X}_1)$ of Banach pseudolattices, is said to be translation invariant if for any Banach space B,

$$\left\|\{S^k(\{b_n\}_{n\in\mathbb{Z}}\}\right\|_{\mathcal{X}_j(B)} = \left\|\{b_n\}_{n\in\mathbb{Z}}\right\|_{\mathcal{X}_j(B)}, \quad j \in \{0,1\}$$

  for all $\{b_n\}_{n\in\mathbb{Z}} \in \mathcal{X}_j(B)$, each $k \in \mathbb{Z}$, where S is the left-shift operator defined by $S\{b_n\} = \{b_{n+1}\}$.
- $\vec{\mathcal{X}} = (\mathcal{X}_0, \mathcal{X}_1)$ is said to be a rotation invariant Banach couple of pseudolattices whenever the rotation map

$$\{b_n\}_{n\in\mathbb{Z}} \mapsto \{e^{in\tau} b_n\}_{n\in\mathbb{Z}}$$

  is an isometry of $\mathcal{X}_j(B)$ onto itself for every real τ and every Banach space B.

Definition

A bounded linear operator $T\colon X \to Y$ between Banach spaces is said to be semi-Fredholm if $T(X)$ is closed in Y and at least one of the spaces $\ker T$, $Y/T(X)$ is finite-dimensional. Then the index of T is given by

$$\operatorname{ind}(T) := \dim(\ker T) - \dim(Y/T(X)).$$

If $\operatorname{ind}(T)$ is finite, T is called a Fredholm operator.

Properties: (1) If $T\colon X \to Y$ is a Fredholm operator, then the dual operator $T^*\colon Y^* \to X^*$ is also Fredholm and

$$\operatorname{ind}(T^*) = -\operatorname{ind}(T).$$

(2) If $T\colon X \to Y$ and $S\colon Y \to Z$ are Fredholm operators, then $ST\colon X \to Z$ is also a Fredholm operator with

$$\operatorname{ind}(ST) = \operatorname{ind}(T) + \operatorname{ind}(S).$$

(3) A strictly singular perturbation of a Fredholm operator remains Fredholm and has the same index, i.e., if $T: X \to Y$ is a Fredholm operator and $S: X \to Y$ is a strictly singular operator, then $T + S$ is a Fredholm operator and

$$\mathrm{ind}(T + S) = \mathrm{ind}(T).$$

(4) If X is a Banach space and $S: X \to X$ is a strictly singular (in particular a compact) operator, then $I_X - \lambda S$ is a Fredholm operator for every λ with

$$\mathrm{ind}(I_X - \lambda S) = 0.$$

(5) Every Fredholm operator $T: X \to Y$ between Banach spaces has a pseudoinverse which is also Fredholm operator, i.e., such an operator $S: Y \to X$ satisfying:

$$TST = T.$$

In particular this yields that the equation $Tx = y$ has a solution if and only if Sy is a solution of this equation.

Theorem

(Atkinson) For an operator $T\colon X \to Y$ between Banach spaces the following statements are equivalent:

(i) *T is Fredholm operator.*

(ii) *There exist compact (finite rank) operators $K_1\colon X \to X$ and $K_2\colon Y \to Y$ and an operator $S\colon Y \to X$ such that*

$$ST = I_X - K_1 \quad \text{and} \quad TS = I_Y - K_2.$$

Theorem

(Kato) If $T\colon X \to Y$ is a Fredholm operator between Banach spaces, then for any operator $S\colon X \to Y$ such that

$$\|T\| < \gamma(S) := \inf\{\|Sx\|_Y;\, \mathrm{dist}(x, \ker S) > 0\}.$$

Then $T + S$ is Fredholm with

$$\dim(\ker(T+S)) \leqslant \dim(\ker T), \quad \mathrm{ind}(T+S) = \mathrm{ind}(T).$$

Theorem

(*I. Ya. Shneiberg, 1974*) *Let* $T\colon (X_0, X_1) \to (Y_0, Y_1)$ *be an operator between Banach couples. Then the following statements are true*:

(i) *If* $T\colon [X_0, X_1]_{\theta_*} \to [Y_0, Y_1]_{\theta_*}$ *is invertible for some* $\theta_* \in (0,1)$, *then there exists* $\varepsilon > 0$ *such that*

$$T\colon [X_0, X_1]_{\theta_*} \to [Y_0, Y_1]_{\theta_*}$$

is invertible for all $\theta \in (\theta_* - \varepsilon, \theta_* + \varepsilon)$.

(ii) *If* $T\colon [X_0, X_1]_{\theta_*} \to [Y_0, Y_1]_{\theta_*}$ *is Fredholm for some* $\theta_* \in (0,1)$, *then there exists* $\varepsilon > 0$ *such that*

$$T\colon [X_0, X_1]_{\theta} \to [Y_0, Y_1]_{\theta}$$

is Fredholm and the index is constant for all $\theta \in (\theta_* - \varepsilon, \theta_* + \varepsilon)$.

Let $\Omega \subset \mathbb{R}^n$ $(n \geqslant 2)$ be a domain above the graph of real-valued Lipschitz function defined in $\mathbb{R}^{n-1}$ $\big($i.e., $\Omega = \{(x, \phi(x) + t);\, x \in \mathbb{R}^{n-1},\, t > 0\}$, where $\phi\colon \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function$\big)$.

Question: For which $1 < p < \infty$ the Dirichlet problem for the Laplacian:

$$\Delta u = 0 \quad \text{in} \quad \Omega \tag{$*$}$$

under the conditions $M(u) \in L^p(\partial\Omega)$ and $u|_{\partial\Omega} = f \in L^p(\partial\Omega)$ has a solution? Here, M stands for the nontangential maximal operator given by

$$M(u)(x) := \sup\{|u(y)|;\, y \in \Omega,\, |x - y| < 2\,\mathrm{dist}(y, \partial\Omega)\}, \qquad x \in \partial\Omega$$

and $u|_{\partial\Omega}$ is defined (in the sense of nontangential convergence to the boundary) by

$$u|_{\partial\Omega}(x) := \lim_{\substack{\Omega \ni y \to x \\ |x-y| < 2\,\mathrm{dist}(y, \partial\Omega)}} u(y), \qquad x \in \partial\Omega.$$

- R. Coifman, A. McIntosh and Y. Meyer (1982); G. Verechota (1984).

$$\|M(\mathcal{D}f)\|_{L^p(\partial\Omega)} \leqslant C\|f\|_{L^p(\partial\Omega)}, \quad \mathcal{D}f|_{\partial\Omega} = \Big(\frac{1}{2}I + \mathcal{K}\Big)f,$$

for every $f \in L^p(\partial\Omega)$, $1 < p < \infty$. The solution of the Dirichlet problem $(*)$ is given by

$$u(x) = \mathcal{D}\Big(\Big(\frac{1}{2}I + \mathcal{K}\Big)^{-1}f\Big)(x), \quad x \in \Omega$$

whenever the inverse $\big(\frac{1}{2}I + \mathcal{K}\big)^{-1}$ exists in $L^p(\partial\Omega)$. Here $\mathcal{D}$ is the harmonic double layer potential operator defined by

$$\mathcal{D}f(x) := \frac{1}{\omega_n}\int_{\partial\Omega} \frac{\langle \vec{\nu}, y - x\rangle}{|x-y|^n} f(y)\, d\sigma(y), \quad x \in \Omega,$$

and $\mathcal{K}$ its principial-value boundary version given by

$$\mathcal{K}f(x) := \lim_{\varepsilon \to 0+} \frac{1}{\omega_n} \int_{y \in \partial\Omega,\, |x-y|>\varepsilon} \frac{\langle \vec{\nu}(y), y-x \rangle}{|x-y|^n} f(y)\, d\sigma(y), \quad x \in \partial\Omega,$$

where ω_n is the area of the unit sphere in $\mathbb{R}^n$, and $\vec{\nu}(y)$ is the outward unit normal defined at almost every boundary point $y \in \partial\Omega$.

- Verchota proved that $\frac{1}{2}I + \mathcal{K}$ is invertible on $L^2(\partial\Omega)$. From Shnieberg's result it follows that there exists $\varepsilon > 0$ such that for all $p \in (2-\varepsilon, 2+\varepsilon)$

  $$\frac{1}{2}I + \mathcal{K} \colon L^p(\partial\Omega) \to L^p(\partial\Omega)$$

  is invertible.

The domination property for interpolation functors (Asekritova, Kruglyak, M.)

Let F and G be interpolation functors.

Definition

- G is said to be dominated by F for invertibility whenever, for any Banach couples (X_0, X_1) and (Y_0, Y_1) and any operator $T\colon (X_0, X_1) \to (Y_0, Y_1)$, invertibility of $T\colon F(X_0, X_1) \to F(Y_0, Y_1)$ implies invertibility of

$$T\colon G(X_0, X_1) \to G(Y_0, Y_1).$$

- G is said to be dominated by F for the Fredholmness property if for any Banach couples (X_0, X_1), (Y_0, Y_1) and any bounded linear operator $T\colon (X_0, X_1) \to (Y_0, Y_1)$ the Fredholmness of $T\colon F(X_0, X_1) \to F(Y_0, Y_1)$ implies the Fredholmness of

$$T\colon G(X_0, X_1) \to G(Y_0, Y_1).$$

Theorem

Suppose that the functor G is dominated by the regular functor F for invertibility. Then, for any regular Banach couples (X_0, X_1), (Y_0, Y_1) and any operator $T\colon (X_0, X_1) \to (Y_0, Y_1)$, the Fredholmness of

$$T|_{F(X_0,X_1)}\colon F(X_0, X_1) \to F(Y_0, Y_1)$$

implies the Fredholmness of $T|_{G(X_0,X_1)}\colon G(X_0, X_1) \to G(Y_0, Y_1)$ with

$$\mathrm{ind}(T|_{G(X_0,X_1)}) = \mathrm{ind}(T|_{F(X_0,X_1)}).$$

Theorem

Let $T\colon (X_0, X_1) \to (Y_0, Y_1)$ be an operator between couples of complex Banach spaces. If $T\colon [X_0, X_1]_{\theta_*} \to [Y_0, Y_1]_{\theta_*}$ is invertible for some $\theta_* \in (0, 1)$, then

$$T\colon (X_0, X_1)_{\theta_*,q} \to (Y_0, Y_1)_{\theta_*,q}$$

is invertible for all $q \in [1, \infty]$.

Theorem

If $T\colon (X_0, X_1) \to (Y_0, Y_1)$ is such that $T\colon [X_0, X_1]_{\theta_*} \to [Y_0, Y_1]_{\theta_*}$ is Fredholm then for all $1 \leqslant q \leqslant \infty$ the operator

$$T\colon (X_0, X_1)_{\theta_*,q} \to (Y_0, Y_1)_{\theta_*,q}$$

is Fredholm and we have

$$\mathrm{ind}(T|_{(X_0,X_1)_{\theta_*,q}}) = \mathrm{ind}(T|_{[X_0,X_1]_{\theta_*}}).$$

Corollary

The real interpolation functors

$$K_{\theta,q}(\,\cdot\,) := (\,\cdot\,)_{\theta,q}$$

are dominated by the functor $C_\theta(\,\cdot\,) := [\,\cdot\,]_\theta$ for the Fredholmness property.

Theorem

Let $\vec{\mathcal{X}} = (\mathcal{X}_0, \mathcal{X}_1)$ be a couple of translation and rotation invariant pseudolattices and let $T\colon \vec{X} \to \vec{Y}$. Assume that $T_{\theta_*}\colon \vec{X}_{\vec{\mathcal{X}},e^{\theta_*}} \to \vec{Y}_{\vec{\mathcal{X}},e^{\theta_*}}$ is invertible for some $\theta_* \in (0,1)$. Then $T_\theta\colon \vec{X}_{\vec{\mathcal{X}},e^{\theta}} \to \vec{Y}_{\vec{\mathcal{X}},e^{\theta}}$ is invertible for all θ in an open neighbourhood $I = \{\theta \in (0,1);\ |\theta - \theta_*| < \varepsilon\}$ of θ_* with

$$\varepsilon = \big[2e\eta(\theta_*)\big(1 + \|T\|_{\vec{X}\to\vec{Y}}\|T^{-1}\|_{\vec{Y}_{\vec{\mathcal{X}},e^{\theta_*}} \to \vec{X}_{\vec{\mathcal{X}},e^{\theta_*}}}\big)\big]^{-1},$$

where $\eta(\theta_*) = \max\big\{(e^{\theta_*} - 1)^{-1}, (e - e^{\theta_*})^{-1}\big\}$. Moreover T_θ^{-1} agrees with $T_{\theta_*}^{-1}$ on $Y_0 \cap Y_1$ and

$$\|T_\theta^{-1}\|_{\vec{Y}_{\vec{\mathcal{X}},e^{\theta}} \to \vec{X}_{\vec{\mathcal{X}},e^{\theta}}} \leqslant 2\|T_{\theta_*}^{-1}\|_{\vec{Y}_{\vec{\mathcal{X}},e^{\theta_*}} \to \vec{X}_{\vec{\mathcal{X}},e^{\theta_*}}}, \quad \theta \in I.$$

Theorem

Let $\vec{\mathcal{X}} = (\mathcal{X}_0, \mathcal{X}_1)$ *be a couple of rotation and translation invariant pseudolattices and let* $\{F_\theta\}_{\theta\in(0,1)}$ *be a family of interpolation functors given by* $F_\theta(X_0, X_1) := (X_0, X_1)_{\vec{\mathcal{X}}, e^\theta}$ *for any Banach couple* (X_0, X_1)*. Suppose that* F_θ *is regular functor and* $F_\theta(X_0, X_1) = F_\theta(X_0^\circ, X_1^\circ)$ *for any Banach couple* (X_0, X_1)*. If* $T\colon (X_0, X_1) \to (Y_0, Y_1)$ *is such that the operator*

$$T|_{F_{\theta_*}(X_0,X_1)}\colon F_{\theta_*}(X_0, X_1) \to F_{\theta_*}(Y_0, Y_1) \quad \text{is Fredholm.}$$

Then there exists $\varepsilon = \varepsilon(\theta_*, \vec{\mathcal{X}}) > 0$ *such that for any* $\theta \in (\theta_* - \varepsilon, \theta_* + \varepsilon)$ *the operator*

$$T|_{F_\theta(X_0,X_1)}\colon F_\theta(X_0, X_1) \to F_\theta(Y_0, Y_1)$$

is also Fredholm and $\operatorname{ind}(T|_{F_\theta(X_0,X_1)}) = \operatorname{ind}(T|_{F_{\theta_*}(X_0,X_1)})$.

The uniqueness of inverses of interpolated operators

Lemma

Let (A_0, A_1) and (B_0, B_1) be Banach couples and let $T\colon (A_0, A_1) \to (B_0, B_1)$ be such that $T|_{A_0}$ and $T|_{A_1}$ are invertible operators. Then, the following conditions are equivalent:

(i) $(T|_{A_0})^{-1} b = (T|_{A_1})^{-1} b, \qquad b \in B_0 \cap B_1$;

(ii) $T\colon A_0 + A_1 \to B_0 + B_1$ *is invertible*;

(iii) *For any interpolation functor F,*

$$T|_{F(A_0,A_1)}\colon F(A_0, A_1) \to F(B_0, B_1) \quad \textit{is invertible.}$$

- Remark. Let $\vec{X} = (X_0, X_1)$ be a Banach couple and $T\colon (X_0, X_1) \to (X_0, X_1)$ be an operator. If $0 \leqslant \alpha < \beta \leqslant 1$ and $T_\alpha := T|_{[\vec{X}]_\alpha}$ and $T_\beta := T|_{[\vec{X}]_\beta}$ are invertible, then the inverses T_α^{-1} and T_β^{-1} do not coincide on $X_0 \cap X_1$ in general.

- Example. The dilatation operator D_a ($a > 0$, $a \neq 1$) given by $D_a f(t) = f(at)$, $t > 0$ is bounded on $L^p = L^p(\mathbb{R}_+)$ for every $1 < p < \infty$ and

$$\sigma(D_a, L^p) = \left\{\lambda \in \mathbb{C};\ |\lambda| = a^{-1/p}\right\}.$$

If $|\lambda| = a^{-1/p}$, $p_0 < p < p_1$, then the operator $T = \lambda I - D_a$ is invertible on

$$L^{p_0} = [L^1, L^\infty]_\alpha, \quad L^{p_1} = [L^1, L^\infty]_\beta$$

with $\alpha = 1 - 1/p_0$ and $\beta = 1 - 1/p_1$ but T is not invertible on L^p.

- M. Zafran (1980) An operator $T\colon (X_0, X_1) \to (X_0, X_1)$ is said to have the uniqueness-of-resolvent property if

$$(T_\alpha - \lambda I)^{-1}|_{X_0 \cap X_1} = (T_\beta - \lambda I)^{-1}|_{X_0 \cap X_1}$$

for all α, $\beta \in [0,1]$ and $\lambda \notin \sigma(T_\alpha) \cup \sigma(T_\beta)$.

- T. Ransford (1986) An operator $T\colon (X_0, X_1) \to (X_0, X_1)$ satisfies the local uniqueness-of-resolvent condition, if for all $\alpha \in (0,1)$ and $\lambda \notin \sigma(T_\alpha)$, there exists a neighbourhood $U \subset (0,1)$ of α such that $(T_\theta - \lambda I)^{-1}$ exists and

$$(T_\theta - \lambda I)^{-1} = (T_\alpha - \lambda I)^{-1}|_{X_0 \cap X_1}, \quad \theta \in U.$$

Theorem

(E. Albrecht and V. Müller) If (X_0, X_1) is a complex Banach couple and an operator $T\colon (X_0, X_1) \to (X_0, X_1)$ is such that

$$T_\alpha\colon [X_0, X_1]_\alpha \to [X_0, X_1]_\alpha$$

is invertible for some $\alpha \in (0,1)$, then there exists a neighbourhood $U \subset (0,1)$ of α such that T_θ is invertible and T_θ^{-1} agrees with T_α^{-1} on $X_0 \cap X_1$ for any $\theta \in U$.

The uniqueness of inverses on intersection of interpolated Banach spaces

Definition

A family $\{F_\theta\}_{\theta\in(0,1)}$ of interpolation functors is said to be stable if for any Banach couples $\vec{A} = (A_0, A_1)$ and $\vec{B} = (B_0, B_1)$ and for every operator $S\colon \vec{A} \to \vec{B}$ such that the restriction S_{θ_*} of S to $F_{\theta_*}(\vec{A})$ is invertible for some $\theta_* \in (0,1)$, there exists $\varepsilon > 0$ such that, for any $\theta \in I(\theta_*) = (\theta_* - \varepsilon, \theta_* + \varepsilon)$, we have

(i) $S_\theta\colon F_\theta(\vec{A}) \to F_\theta(\vec{B})$ is invertible operator;

(ii) $S_\theta^{-1}\colon F_\theta(\vec{B}) \to F_\theta(\vec{A})$ agrees with $S_{\theta_*}^{-1}\colon F_{\theta_*}(\vec{B}) \to F_{\theta_*}(\vec{A})$ on $B_0 \cap B_1$, i.e., $S_\theta^{-1} y = S_{\theta_*}^{-1} y$ for all $y \in B_0 \cap B_1$;

(iii) $\sup_{\theta\in I(\theta_*)} \|S_\theta^{-1}\|_{F_\theta(\vec{B})\to F_\theta(\vec{A})} \leqslant C \|S_{\theta_*}^{-1}\|_{F_{\theta_*}(\vec{B})\to F_{\theta_*}(\vec{A})}$ for some $C = C(\theta_*)$.

Theorem

If $\vec{\mathcal{X}} = (\mathcal{X}_0, \mathcal{X}_1)$ is a Banach couple of translation and rotation invariant pseudolattices, then the family of interpolation functors $\{F_\theta\}_{\theta\in(0,1)}$ is stable, where $F_\theta(A_0, A_1) \cong (A_0, A_1)_{\vec{\mathcal{X}}, e^\theta}$ for any Banach couple (A_0, A_1).

Remark. Let $\{F_\theta\}_{\theta\in(0,1)}$ be a stable family of interpolation functors and let $T\colon (X_0, X_1) \to (Y_0, Y_1)$. Then the set of all $\theta \in (0,1)$ for which

$$T\colon F_\theta(X_0, X_1) \to F_\theta(Y_0, Y_1)$$

is invertible, is open, so it is a union of open disjoint intervals. These intervals we will call intervals of invertibility of T with respect to the family $\{F_\theta\}_{\theta\in(0,1)}$.

Question. Let $I \subset (0,1)$ be any interval of invertibility of T. Is it true that for any θ, $\theta' \in I$ the inverses T_θ^{-1} and $T_{\theta'}^{-1}$ agree on

$$F_\theta(Y_0, Y_1) \cap F_{\theta'}(Y_0, Y_1)\,?$$

Theorem

Let $1 \leqslant q \leqslant \infty$ and let $T\colon (X_0, X_1) \to (Y_0, Y_1)$ and let $I \subset (0,1)$ be an interval of invertibility of T with respect to the family $\{(\cdot)_{\theta,q}\}_{\theta\in(0,1)}$ of real interpolation functors. Then for any θ_0, $\theta_1 \in I$,

$$T_{\theta_0}^{-1}(y) = T_{\theta_1}^{-1}(y), \quad y \in (Y_0, Y_1)_{\theta_0,q} \cap (Y_0, Y_1)_{\theta_1,q}.$$

Theorem

Let $T\colon (X_0, X_1) \to (Y_0, Y_1)$ be an operator between couples of complex Banach spaces and let $I \subset (0,1)$ be an interval of invertibility of T with respect to the family $\{[\,\cdot\,]_\theta\}_{\theta\in(0,1)}$. Then for any θ_0, $\theta_1 \in I$,

$$T_{\theta_0}^{-1}(y) = T_{\theta_1}^{-1}(y), \quad y \in [Y_0, Y_1]_{\theta_0} \cap [Y_0, Y_1]_{\theta_1}.$$

Definition

A family of interpolation functors $\{F_\theta\}_{\theta\in(0,1)}$ satisfies the (Δ)-condition if for any Banach couple $\vec{A}=(A_0,A_1)$ and for any θ_0, θ_1 with $0<\theta_0<\theta_1<1$, we have continuous inclusions

$$F_{\theta_0}(\vec{A})\cap F_{\theta_1}(\vec{A}) \hookrightarrow \bigcap_{\theta_0<\theta<\theta_1} F_\theta(\vec{A}) \hookrightarrow (F_{\theta_0}(\vec{A}))^c \cap (F_{\theta_1}(\vec{A}))^c,$$

where the norm in $\bigcap_{\theta_0<\theta<\theta_1} F_\theta(\vec{A})$ is given by

$$\|a\|_{\bigcap_{\theta_0<\theta<\theta_1} F_\theta(\vec{A})} = \sup_{\theta_0<\theta<\theta_1} \|a\|_{F_\theta(\vec{A})}.$$

and the Gagliardo completion $(F_{\theta_i}(\vec{A}))^c$, $j\in\{0,1\}$ is taken with respect to the sum $F_{\theta_0}(\vec{A})+F_{\theta_1}(\vec{A})$.

Definition

. A family of interpolation functors $\{F_\theta\}_{\theta\in(0,1)}$ satisfies the reiteration condition if for any Banach couple $\vec{A}=(A_0,A_1)$ and for any θ_0, θ_1, $\lambda\in(0,1)$,
we have

$$F_\lambda(F_{\theta_0}(\vec{A}),F_{\theta_1}(\vec{A}))=F_{(1-\lambda)\theta_0+\lambda\theta_1}(\vec{A}).$$

Theorem

Let $T\colon(X_0,X_1)\to(Y_0,Y_1)$ and let $I\subset(0,1)$ be an interval of invertibility of T with respect to the stable family of interpolation functors $\{F_\theta\}_{\theta\in(0,1)}$. Assume that $\{F_\theta\}_{\theta\in(0,1)}$ satisfy both the (Δ) and the reiteration condition. Then for any $\theta_0,\theta_1\in I$, the inverse operators $T_{\theta_0}^{-1}$ and $T_{\theta_1}^{-1}$ agree on $F_{\theta_0}(\vec{Y})\cap F_{\theta_1}(\vec{Y})$, i.e.,

$$T_{\theta_0}^{-1}(y)=T_{\theta_1}^{-1}(y),\quad y\in F_{\theta_0}(\vec{Y})\cap F_{\theta_1}(\vec{Y}).$$

The surjection modulus of operators on complex spaces

- Let $T\colon E \to F$ be a linear operator between Banach spaces. The surjection modulus of T is given by

$$q(T) := \sup\{\tau > 0;\ T(B_E) \supset \tau B_F\}.$$

An operator T is called a surjection if $q(T) > 0$, which is equivalent to $T(E) = F$. If $\|T\| = q(T) = 1$, then T is said to be a metric surjection (i.e., T maps the open unit ball of E onto the open unit ball of F).

Definition

Let $\mathcal{G}(\vec{X})$ the Banach space of all continuous functions $g\colon \bar{S} \to X_0 + X_1$ that are analytic on the strip S and grow no faster than $C(1 + |z|)$ for some $C > 0$. We endow $\mathcal{G}(\vec{X})$ with the norm

$$\|g\|_{\mathcal{G}} := \max_{j=0,1} \left\{ \sup_{s \neq t} \frac{\|g(j + is) - g(j + it)\|_{X_j}}{|s - t|} \right\}.$$

The upper complex interpolation space is defined by

$$[\vec{X}]^{\theta} := \{g'(\theta);\, g \in \mathcal{G}\}$$

and equipped with the quotient norm.

Theorem

Let $\vec{X} = (X_0, X_1)$ and $\vec{Y} = (Y_0, Y_1)$ be complex Banach couples, and let $T\colon \vec{X} \to \vec{Y}$ be an operator with $M' = \|T'\|_{\vec{Y}' \to \vec{X}'}$. Then for all θ_0, $\theta \in (0,1)$,

$$q_\theta(T) \geqslant M' \max\left\{\frac{q_{\theta_0}(T) - q(\theta,\theta_0)M'}{M' - q(\theta,\theta_0)q_{\theta_0}(T)}, 0\right\},$$

where $q_\theta(T) = q\big(T\colon [\vec{X}]^\theta \to [\vec{Y}]^\theta\big)$,

$$q(\lambda, z) = \left|\frac{d(\lambda) - d(z)}{1 - \overline{d(z)}d(\lambda)}\right|, \quad \lambda, z \in \mathbb{D}$$

and d is a conformal map of the open strip S onto the open disc $\mathbb{D}$ of the complex plane $\mathbb{C}$.

- Example. The map $z \mapsto \operatorname{tg} z$ is a conformal map of the open strip $\{z \in \mathbb{C};\ -\frac{\pi}{4} < \mathrm{Re} z < \frac{\pi}{4}\}$ onto the disc $\mathbb{D}$. Thus φ defined by

$$\varphi(z) = \operatorname{tg}\left(z - \frac{1}{2}\right)\frac{\pi}{2}, \quad z \in S$$

is a conformal map of S onto $\mathbb{D}$ and so q is given by

$$q(\lambda, z) = \left| \frac{\operatorname{tg}(\lambda - \frac{1}{2})\frac{\pi}{2} - \operatorname{tg}(z - \frac{1}{2})\frac{\pi}{2}}{1 - \operatorname{tg}(\lambda - \frac{1}{2})\frac{\pi}{2}\ \operatorname{tg}(\bar{z} - \frac{1}{2})\frac{\pi}{2}} \right|, \quad \lambda, z \in \mathbb{D}.$$

Let (X_0, X_1) be a Banach couple of complex Banach funtion lattices on a σ-finite measure space (Ω, Σ, μ). The Calderón product $X_0^{1-\theta} X_1^{\theta}$ $(0 < \theta < 1)$ is defined to be the space of all $f \in L^0(\mu)$ such that

$$|f| \leqslant \lambda\, |f_0|^{1-\theta} |f_1|^{\theta}, \quad \mu - a.e.$$

for some $\lambda > 0$ and $f_j \in X_j$ with $\|f_j\|_{X_j} \leqslant 1$, $j = 0, 1$. The Calderón product is a Banach function lattice on (Ω, Σ, μ) equipped with the norm

$$\|f\| = \inf \left\{\lambda > 0 \colon\, |f| \leqslant \lambda\, |f_0|^{1-\theta} |f_1|^{\theta},\, f_0 \in B_{X_0},\, f_1 \in B_{X_1}\right\}.$$

Theorem

Let (X_0, X_1), $\vec{Y} = (Y_0, Y_1)$ be couples of Banach lattices with the Fatou property. Assume that $T\colon (X_0, X_1) \to (Y_0, Y_1)$ is such that $T\colon X_0^{1-\theta_*} X_1^{\theta_*} \to Y_0^{1-\theta_*} Y_1^{\theta_*}$ is an invertible operator for some $\theta_* \in (0,1)$. Then there exists $\delta > 0$ such that

$$T\colon X_0^{1-\theta} X_1^{\theta} \to Y_0^{1-\theta} Y_1^{\theta}$$

is an invertible operator whenever $|\theta - \theta_*| < \delta$.